The valuation tree for $n^2 + 7$

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Abstract. The 2-adic valuation of an integer $x$ is the highest power of 2 which divides $x$. It is denoted by $\nu_2(x)$. The goal of the present work is to describe the sequence $\{\nu_2(n^2 + a)\}$ for $1 \leq a \leq 7$. The first six cases are elementary. The last case considered here, namely $a = 7$, presents distinct challenges. It is shown here how to represent this family of valuations in the form of an infinite binary tree, with two symmetric infinite branches.

1. Introduction

For a prime $p$ and an integer $x$, the $p$-adic valuation $\nu_p(x)$ is the highest power of $p$ which divides $x$. The problems considered here deal with $p$-adic valuations of sequences generated by a polynomial. In detail, we consider the sequence

\[ V_p(f) = \{\nu_p(f(n)) : n \in \mathbb{N}\} \]

for a polynomial $f$ with integer coefficients. An important ingredient in the analysis of $V_p(f)$ is the ring of $p$-adic integers $\mathbb{Z}_p$. One description of this ring is via series:

\[ x \in \mathbb{Z}_p \text{ if and only if } x = \sum_{k=k_0}^{\infty} c_k p^k \]

where $k_0 \geq 0$ and $0 \leq c_k \leq p - 1$ are integers. The series is convergent in the $p$-adic norm

\[ |x|_p = p^{-k_0} \]

and $x \in \mathbb{Z}_p$ is invertible in $\mathbb{Z}_p$ if and only if $|x|_p = 1$; that is, $k_0 = 0$.

The next result appears in [?].

**Theorem 1.1.** Let $p$ be a prime and $f \in \mathbb{Z}[x]$ be a polynomial, irreducible over $\mathbb{Z}$. Then $V_p(f)$ is either periodic or unbounded. Moreover, $V_p(f)$ is periodic if and only if $f$ has no zeros in $\mathbb{Z}_p$. In the periodic case, the minimal periodic length is a power of $p$. 

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Some elementary examples

This section contains the description of \( \{ \nu_2(n^2 + a) \} \) for some small values of \( a \), where the analysis is elementary.

**Proposition 2.1.** The 2-adic valuation of \( n^2 + 1 \) is given by

\[
\nu_2(n^2 + 1) = \begin{cases} 
0 & \text{if } n \equiv 0 \mod 2 \\
1 & \text{if } n \equiv 1 \mod 2.
\end{cases}
\]

**Proof.** If \( n \) is even, then \( n^2 + 1 \) is odd and so \( \nu_2(n^2 + 1) = 0 \). On the other hand, if \( n \) is odd, say \( n = 2m + 1 \), then \( n^2 + 1 = 2(2m^2 + m + 1) \). Therefore \( n^2 + 1 \) is twice an odd number and so \( \nu_2(n^2 + 1) = 1 \).

\[ \square \]

![Figure 1. The complete tree for \( \nu_2(n^2 + 1) \)](image)

Figure 1 contains a tree representation of the set of valuations \( \{ \nu_2(n^2 + 1) \} \). The process creates a collections of levels, formed by vertices with a set of indices associated to them, denoted by \( I(v) \).

A vertex \( v \) is split to a next level below if the set of values \( \{ \nu_2(n^2 + 1) \} \), for indices \( n \in I(v) \), does not reduce to a singleton; that is, there are indices \( n_1, n_2 \in I(v) \), such that \( \nu_2(n_1^2 + 1) \neq \nu_2(n_2^2 + 1) \). In the case a vertex splits, then the two new descendants are at one level higher and if \( I(v) \) has the form \( \{ n \equiv j \mod 2^a \} \), then the index sets of the descendants are \( \{ n \equiv j \mod 2^{a+1} \} \) (for the left-one) and \( \{ n \equiv j + 2^a \mod 2^{a+1} \} \) (for the right-one). A vertex which does not split is called terminal. Starting with the root of the tree, all the vertices that split from the \( k^{th} \) level form the \( (k+1)^{st} \) level.

The process begins with the root \( v_0 \) with \( I(v_0) = \mathbb{N} \). The data \( \nu_2(1^2 + 1) = 1 \) and \( \nu_2(2^2 + 1) = 0 \), shows that \( v_0 \) splits. The vertices at the second level are \( \{ n \equiv 0 \mod 2 \} \) (for the left-one) and \( \{ n \equiv 1 \mod 2 \} \) (for the right-one). Now consider the vertex \( v_1 \) with \( I(v_1) = \{ n \equiv 0 \mod 2 \} \) and \( v_2 \) with \( I(v_2) = \{ n \equiv 1 \mod 2 \} \). For each \( n \in I(v_1) \), \( n^2 + 1 = (2m)^2 + 1 \equiv 1 \mod 2 \) and \( \{ \nu_2(n^2 + 1) \} \) reduces to \( \{ 0 \} \), showing that \( v_1 \) is a terminal vertex, with assigned value 0. Similarly, for the vertex \( v_2 \), with \( I(v_2) = \{ n \equiv 1 \mod 2 \} \), one sees directly that \( \{ \nu_2(n^2 + 1) \} \equiv 1 \), showing that \( v_2 \) is also terminal, with assigned value 1. Therefore, the set of valuations \( \{ \nu_2(n^2 + 1) \} \) is represented by a finite tree, containing two levels.

**Proposition 2.2.** The 2-adic valuation of \( n^2 + 2 \) is given by

\[
\nu_2(n^2 + 2) = \begin{cases} 
1 & \text{if } n \equiv 0 \mod 2 \\
0 & \text{if } n \equiv 1 \mod 2.
\end{cases}
\]
Proof. If $n$ is odd, then $n^2 + 2$ is also odd and then $\nu_2(n^2 + 2) = 0$. On the other hand, if $n$ is even, say $n = 2m$, then $n^2 + 2 = 2(2m^2 + 1)$, twice an odd number and it follows that $\nu_2(n^2 + 2) = 1$. \hfill \Box

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{tree1.png}
\caption{The complete tree for $\nu_2(n^2 + 2)$}
\end{figure}

Proposition 2.3. The 2-adic valuation of $n^2 + 3$ is given by
\begin{equation}
\nu_2(n^2 + 3) = \begin{cases}
0 & \text{if } n \equiv 0 \mod 2 \\
2 & \text{if } n \equiv 1 \mod 2.
\end{cases}
\end{equation}

Proof. For $n$ even, $n^2 + 3$ is odd, so $\nu_2(n^2 + 3) = 0$. On the other hand, if $n = 2m + 1$, then $n^2 + 3 = 4\left([m(m+1)] + 1\right)$ and this is 4 times an odd number. \hfill \Box

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{tree2.png}
\caption{The complete tree for $\nu_2(n^2 + 3)$}
\end{figure}

The next result gives the valuations of $n^2 + 4$. The proof is similar to the one given in the previous examples, so it is omitted.

Proposition 2.4. The 2-adic valuation of $n^2 + 4$ is given by
\begin{equation}
\nu_2(n^2 + 4) = \begin{cases}
0 & \text{if } n \equiv 1 \mod 2 \\
2 & \text{if } n \equiv 0 \mod 4 \\
3 & \text{if } n \equiv 2 \mod 4.
\end{cases}
\end{equation}

In the tree, the presence of a vertex $v$ inside a square and without a numerical label, as in the second level of Figure 5, indicates that the valuation of $n^2 + 4$ for indices $n$ in $I(v)$ is not constant, and so the vertex must be split to the next level.

The reader is invited to draw trees for the sequences $\{\nu_2(n^2 + 5)\}$ and $\{\nu_2(n^2 + 6)\}$. 
3. The labeling of the classes

Given a polynomial $f(x)$ with integer coefficients, the sequence $\{\nu_2(f(n)) : n \in \mathbb{N}\}$ has been described via a tree. This is called the valuation tree attached to $f$. The vertices correspond to some selected classes

\[ C_{m,j} = \{2^mi + j : i \in \mathbb{N}\}, \]

starting with the root vertex $v_0$ for $C_{0,0} = \mathbb{N}$. The procedure to select the classes is explained below in the example $f(x) = x^2 + 16$. Some notation for the vertices of the tree is introduced next.

Definition 3.1. A residue class $C_{m,j}$ is called terminal for the tree attached to $f$, if the valuation $\nu_2(f(2^mi + j))$ is independent of the index $i \in \mathbb{N}$. Otherwise it is called non-terminal. The same terminology is given to vertices. In the tree, terminal vertices are marked by their constant valuation and non-terminal vertices are marked with a star.

Example 3.2. The construction of the tree for $\nu_2(n^2 + 16)$ starts with the fact that $\nu_2(1^2 + 16) = 0$ and $\nu_2(2^2 + 16) = 2 \neq 0$, showing that the root node $v_0$ is non-terminal. This node is split into two vertices that form the first level. These correspond to $C_{1,0} = \{2i : i \in \mathbb{N}\}$ and $C_{1,1} = \{2i + 1 : i \in \mathbb{N}\}$. For the class $C_{1,0}$, the valuation

\[ \nu_2((2i)^2 + 16) = 2 + \nu_2(i^2 + 4) \]

depends on $i$, so $C_{1,0}$ is non-terminal. For the class $C_{1,1}$,

\[ \nu_2((2i + 1)^2 + 16) = \nu_2(4i^2 + 4i + 17) = 0, \]
showing that $C_{1,1}$ is a terminal class with valuation 0. Figure 6 shows the root and the first level of the tree associated to $\nu_2(n^2 + 16)$.

The class $C_{1,0}$ is now split into $C_{2,0} = \{4i : i \in \mathbb{N}\}$ and $C_{2,2} = \{4i + 2 : i \in \mathbb{N}\}$. These two classes form the second level. For $C_{2,0}$, the valuation

$$\nu_2((4i)^2 + 16) = 4 + \nu_2(i^2 + 1)$$

shows that this class is non-terminal. In the class $C_{2,2}$,

$$\nu_2((4i + 2)^2 + 16) = \nu_2(16i^2 + 16i + 20) = 2 + \nu_2(4i^2 + 4i + 5) = 2.$$

Therefore $C_{2,2}$ is a terminal class with valuation 2.

The third level contains the two classes $C_{3,0}$ and $C_{3,4}$ descending from $C_{2,0}$. The first class is terminal with valuation 4, since

$$\nu_2((8i)^2 + 16) = 4 + \nu_2(4i^2 + 1) = 4.$$

The second class is also terminal, with valuation 5, since

$$\nu_2((8i + 4)^2 + 16) = \nu_2(64i^2 + 64i + 32) = 5 + \nu_2(2i^2 + 2i + 1) = 5.$$

Therefore, every class in the third level is terminal and the tree is complete.
It is apriori surprising that the valuations of $\nu_2(n^2 + 7)$ present a more erratic behavior than the cases considered before. The goal of this section is to describe the set \{ $\nu_2(n^2 + 7)$ \} via its valuation tree.

Construction of the tree. The values $\nu_2(1^2 + 7) = 3 \neq \nu_2(2^2 + 7) = 0$, show that the root vertex $v_0$ has to be split into two classes to form the first level:

$$C_{1,0} = \{2n : n \in \mathbb{N}\} \text{ and } C_{1,1} = \{2n + 1 : n \in \mathbb{N}\}.$$  

It is easy to check that the class $C_{1,0}$ is terminal since $\nu_2((2n)^2 + 7) = 0$. Figure 9 shows the root vertex and the first level of the tree for $\nu_2(n^2 + 7)$.

**Lemma 4.1.** Assume $n \equiv 1 \mod 2$, so that $n \in C_{1,1}$, then $\nu_2(n^2 + 7) \geq 3$.

**Proof.** Write $n = 2n_1 + 1$, then $n^2 + 7 = 4(n_1^2 + n_1 + 2) = 8 \left(\frac{n_1(n_1+1)}{2} + 1\right)$. This gives the result since $n_1(n_1 + 1)$ is even. 

The class $C_{1,1}$ is non-terminal, since $1, 3 \in C_{1,1}$ and $\nu_2(1^2+7) = 3 \neq \nu_2(3^2+7) = 4$. Therefore the vertex corresponding to $C_{1,1}$ is split into the classes $C_{2,1}$ and $C_{2,3}$, to form the third level. Recall that $C_{2,1} = \{4n \} \text{ and } C_{2,3} = \{4n + 3\}$. Neither of
these vertices is terminal. For instance, every number in $C_{2,1}$ has valuation at least 3 and

\[(4.2) \quad (4n + 1)^2 + 7 = 16n^2 + 8n + 8 \equiv 0 \mod 2^4 \]

provided $n$ is odd. A similar argument shows that $C_{2,3}$ is not terminal. The rest of this section is devoted to the proof of the next result. A similar discussion has been presented in [?].

**Theorem 4.2.** Let $v$ be a non-terminating node at the $k$-th level for the valuation tree of $\nu_2(n^2 + 7)$. Then $v$ splits into two vertices at the $(k + 1)$-level. Exactly one of them terminates, say with constant valuation $\nu_k$. The second one has valuation at least $\nu_k + 1$.

**Proof.** Start with the class $C_{1,1} = \{2n + 1\}$. As described above, this is non-terminal vertex, splitting into the classes $C_{2,1} = \{4n + 1\}$ and $C_{2,3} = \{4n + 3\}$. The class $C_{2,1}$ is non-terminal since $1, 5 \in C_{2,1}$ and $\nu_2(1) = 3$ and $\nu_2(5) = 5$. Similarly $3, 7 \in C_{2,3}$ and $\nu_2(3) = 4$ and $\nu_2(7) = 3$. Figure 10 shows the tree up to the first four levels.

![Figure 10. First four levels of the tree for $\nu_2(n^2 + 7)$](image)

The vertex $C_{2,1}$ now splits into $C_{3,1} = \{8n + 1\}$ and $C_{3,5} = \{8n + 5\}$. For the class $C_{3,1}$ observe that

\[(4.3) \quad \nu_2((8n + 1)^2 + 7) = \nu_2(8(8n^2 + 2n) + 1) = \nu_2(8) + \nu_2((8n^2 + 2n) + 1) = 3.\]

This is shown in the left-most branch in the figure. Therefore $C_{3,1}$ is a terminal class. For the other one, $C_{2,3}$, it is easily seen that it splits into $C_{3,3}$ and $C_{3,7}$. The class $C_{3,7}$ is terminal, with value 3, as shown on the right-most branch and the class $C_{3,3}$ is not, so it continues.

At this point, the part of the tree that has not been described yet, splits into two branches:

\[(4.4) \quad B_1 = \{8n + 3\} \quad \text{and} \quad B_2 = \{8n + 5\}.\]
In view of an apparent symmetry, only the case $B_1$ is discussed in detail. Note 4.4 presents information on this symmetry.

4.1. The 2-valuation of $(8n+3)^2+7$. The class $C_{3,3} = \{8n+3\}$ has valuation at least 3. Now observe first that this class is not terminal because $\nu_2((8 \times 1 + 3)^2 + 7) = 7$ and $\nu_2((8 \times 2 + 3)^2 + 7) = 4$. At the next level, the classes have the form $\{8(2n+j)+3\}$, where $j = 0$ or 1. Now

\[(4.5) \quad [8(2n+j)+3]^2 + 7 = 2^4[2(2n+j)^2 + 3j + 1],\]

shows that $\nu_2 \left( [8(2n+j)+3]^2 + 7 \right) \geq 4$ for any $n \in \mathbb{N}$ and any $j \in \{0,1\}$. Moreover,

\[(4.6) \quad [8(2n+j)+3]^2 + 7 \equiv 2^4(3j + 1) \mod 2^5.\]

In particular, when $j = 0$, it follows that $[8(2n+j)+3]^2 + 7 \equiv 2^4 \neq 0 \mod 2^5$. This proves that, for $j = 0$, one has $\nu_2 \left( [8(2n+j)+3]^2 + 7 \right) = 4$, independent of $n$. On the other hand, for $j = 1$, the congruence (4.6) shows that $\nu_2 \left( [8(2n+j)+3]^2 + 7 \right) \geq 5$.

\[\begin{array}{c}
0 \\
\equiv 4
\end{array} 
\quad \begin{array}{c}
\geq 3
\end{array} 
\quad 1 
\quad \begin{array}{c}
\geq 5
\end{array}
\]

**Figure 11.** The root and the fourth level of the tree for $\nu_2(n^2 + 7)$

The main step of the proof is given next.

**Proposition 4.3.** Assume $C_{\alpha,\beta} = \{2^n n + \beta\}$ is a non-terminal class at level $\alpha$, where the valuation is $\geq \alpha + 1$. Then, at the next level, this class splits into two classes

\[(4.7) \quad C_{\alpha+1,\beta} = \{2^n(2n+0) + \beta\} \quad \text{and} \quad C_{\alpha+1,2^n+\beta} = \{2^n(2n+1) + \beta\}\]

one of which is terminal with valuation $\equiv \alpha + 1$ and the second one is non-terminating with valuation $\geq \alpha + 2$.

**Proof.** The starting point is the class $C_{4,11} = \{2^4n + 11\}$. It has been established that

\[(4.8) \quad \nu_2((16n + 11)^2 + 7) \geq 5.\]

Indeed, $(16n + 11)^2 + 7 = 2^5(8n^2 + 11n + 4)$, so (4.8) holds. Moreover, for $n$ odd, the valuation is $\equiv 5$ and for $n$ even, this valuation is $\geq 6$. This shows the class $C_{4,11}$ splits into two classes with the stated properties. The conclusion of the proposition is valid at the initial step.

Now take a non-terminal class $C_{\alpha,\beta}$ with valuation $\geq \alpha + 1$; that is,

\[(4.9) \quad \nu_2 \left( (2^n n + \beta)^2 + 7 \right) \geq \alpha + 1.\]
Observe that this implies \((2^an + \beta)^2 + 7 \equiv 0 \text{ mod } 2^{a+1}\). In particular, this implies \(\beta^2 + 7 \equiv 0 \text{ mod } 2^{a+1}\) and so \(\beta\) is odd. Write \(\beta^2 + 7 = 2^{a+1}\gamma\).

This class splits into the two classes
\[
C_{a+1,2^a j+\beta} = \{2^a(2n + j) + \beta : n \in \mathbb{N}\}.
\]

Now
\[
[2^a(2n + j) + \beta]^2 + 7 \equiv 2^{a+1}\beta j + \beta^2 + 7 \text{ mod } 2^{a+2}
\]
\[
\equiv 2^{a+1}(\beta j + \gamma) \text{ mod } 2^{a+2}.
\]

Since \(\beta\) is odd, the congruence \(\beta j + \gamma \equiv 1 \text{ mod } 2\) has a unique solution (either 0 or 1) in the variable \(j\). For that value of \(j\),
\[
[2^a(2n + j) + \beta]^2 + 7 \not\equiv 0 \text{ mod } 2^{a+2}
\]
and this proves \(\nu_2[2^a(2n + j) + \beta]^2 + 7] = \alpha + 1\), independent of \(n\). For the other value of \(j\),
\[
[2^a(2n + j) + \beta]^2 + 7 \equiv 0 \text{ mod } 2^{a+2}
\]
and this proves \(\nu_2[2^a(2n + j) + \beta]^2 + 7] \geq \alpha + 2\), independent of \(n\). This concludes the proof. \(\square\)

The proof of Theorem 4.2 is complete.

Note 4.4. It remains to verify the symmetry of the branches
\[
B_1 = \{8n + 3\} \quad \text{and} \quad B_2 = \{8n + 5\}.
\]
An informal description is presented here.

Every index in the branch \(B_1\) has the form \(8n + 3\). Then
\[
(8n + 3)^2 + 7 = 2^4(4n^2 + 3n + 1)
\]
shows that \(\nu_2((8n+3)^2+7) \geq 4\). To move to the next level in the tree, write \(n = 2m + j\), with \(m \in \mathbb{N}\) and \(j \in \{0, 1\}\). Then
\[
(8n + 3)^2 + 7 = [8(2m + j) + 3]^2 + 7 \equiv 2^4(3j + 1) \text{ mod } 2^5.
\]
Therefore, for \(j = 0\), then \((8n + 3)^2 + 7 \not\equiv 0 \text{ mod } 2^5\) and thus its valuation is always 4, independently of \(n\). This is the terminal vertex. On the other hand, for \(j = 1\), it follows that \((8n + 3)^2 + 7 \equiv 0 \text{ mod } 2^5\) and its valuation is at least 5. The identity
\[
\left([8(2m + j) + 3]^2 + 7\right) - \left([8(2m + 1 - j) + 5]^2 + 7\right) = 2^5(8j - 5)(2m + 1)
\]
proves that
\[
[8(2m + j) + 3]^2 + 7 \equiv [8(2m + 1 - j) + 5]^2 + 7 \text{ mod } 2^5.
\]
Therefore the roles of 0, 1 for the index \(j\) in the branch \(B_1\) are interchanged in the branch \(B_2\). This phenomena persists at all levels: if there is a movement to the left, in the branch \(B_1\) to advance to the next level; then there is a movement to the right on \(B_2\). This produces the symmetry of the two branches mentioned above. Some details are given below.

The equation
\[
x^2 + 7 \equiv 0 \text{ mod } 2^k
\]
has exactly 4 non-congruent solutions in the set \{1, 2, \ldots, 2^k - 1\} for \( k \geq 3 \). If \( r_k \) and \( r_{k_2} \) are the solutions yielding odd multiples of \( 2^k \), then the four non-congruent solutions to \( x^2 + 7 \equiv 0 \text{ mod } 2^{k+1} \) are \( r_k \pm 2^{k-1} \) and \( r_{k_2} \pm 2^{k-1} \). Exactly two yield odd multiples of \( 2^{k+1} \) and two yield even multiples. We compute the valuation for one of the odd multiples (call it \( r_k \)):

\[
\nu_2((2^k t \pm r_k)^2 + 7) = \begin{cases} 
  k & \text{if } c \text{ odd} \\
  k + 1 & \text{if } c \text{ even}
\end{cases}
\]

and

\[
\nu_2((2^k t \pm (r_k + 2^{k-1}))^2 + 7) = \begin{cases} 
  k & \text{if } c \text{ even} \\
  k + 1 & \text{if } c \text{ odd}
\end{cases}
\]

since \( r_k \) must be odd.

Recalling the proposed form of the four branches

\[
2^k t + r_k \text{ mod } 2^k,
2^k t + r_k + 2^{k-1} \text{ mod } 2^k,
2^k t + 2^k - r_k \text{ mod } 2^k,
2^k t + 2^k - (r_k + 2^{k-1}) \text{ mod } 2^k,
\]

it follows that either:

1. The branches \( 2^k t \pm r_k \) terminate with valuation equal to \( k \) and the branches \( 2^k t + r_k + 2^{k-1} \) and \( 2^k t + 2^k - (r_k + 2^{k-1}) \) continue with valuation greater than or equal to \( k + 1 \), or

2. The branches \( 2^k t \pm r_k \) continue with valuation greater than or equal to \( k + 1 \) and the branches \( 2^k t + r_k + 2^{k-1} \) and \( 2^k t + 2^k - (r_k + 2^{k-1}) \) terminate with valuation equal to \( k \).

This fact explains the symmetry in the 2-adic valuation tree of \( x^2 + 7 \), since the \( 2^k t \pm r_k \) branches lie on the outside (far left and far right) of each level and the \( 2^k t \pm (r_k + 2^{k-1}) \) branches lie on the inside.

5. The range of the valuation \( \nu_2(n^2 + 7) \)

The proof of Theorem 4.2 shows that, given \( k \geq 4 \), there is a class such that \( \nu_2(n^2 + 7) = k \) for all indices \( n \) in the class. As a matter of fact, these classes appear one by level. This gives the main part of the next statement.

**Theorem 5.1.** The range of \( \nu_2(n^2 + 7) \) is \( \mathbb{N} \setminus \{1, 2\} \).

Small values of the range of \( \nu_2(n^2 + 7) \) admit easy characterization. An example is discussed next.

**Lemma 5.2.** The equation \( \nu_2(n^2 + 7) = 0 \) is equivalent to \( n \equiv 0 \text{ mod } 2 \).

**Proof.** The valuation is 0 if and only if \( n^2 + 7 \equiv 1 \text{ mod } 2 \). In turn, this is equivalent to \( n \equiv 0 \text{ mod } 2 \). \(\Box\)

There are two elements missing from this range.
Lemma 5.3. The equations $\nu_2(n^2 + 7) = 1$ or $2$ have no solutions.

Proof. Any solution satisfies $n^2 + 7 = 2t$. Therefore $n^2 + 7 = 2t$ and this implies $n$ is odd, say $n = 2m + 1$. Then

$$n^2 + 7 = 4m^2 + 4m + 8 = 4(m^2 + m + 2) = 8 \left( \frac{m(m+1)}{2} + 1 \right).$$

This implies $\nu_2(n^2 + 7) \geq 3$ and establishes the result. □

The next result continues describing the appearance of small values in the range of $\nu_2(n^2 + 7)$.

Lemma 5.4. The equation $\nu_2(n^2 + 7) = 3$ is equivalent to $n \equiv \pm 1 \mod 2^3$.

Proof. If $n = 8t + 1$, then $n^2 + 7 = (8t + 1)^2 + 7 = 8(8t^2 + 2t + 1)$. Thus, $\nu_2(n^2 + 7) = 3$. The case of $n = 8t + 7$ is similar. Conversely, if $\nu_2(n^2 + 7) = 3$, then $n^2 + 7 = 2^3s$, with $s$ odd. Therefore $n^2 \equiv 1 \mod 8$ and this implies $n \equiv \pm 1 \mod 8$, as claimed. □

The position of indices with valuation 4, 5 are established in a similar manner.

Lemma 5.5. The equation $\nu_2(n^2 + 7) = 4$ is equivalent to $n \equiv \pm 3 \mod 2^4$. Similarly, $\nu_2(n^2 + 7) = 5$ is equivalent to $n \equiv \pm 5 \mod 2^5$.

The previous results suggest a clear pattern.

Problem 5.6. Given $k \in \mathbb{N}$ at least 3, prove there is $\alpha_k \in \mathbb{N}$ such that $\nu_2(n^2 + 7) = k$ if and only if $n \equiv \pm \alpha_k \mod 2^k$.

A characterization of the sequence $\{\alpha_k\}$ would be desirable.

Finally, the next table shows the smallest index $n$ for which $\nu_2(n^2 + 7) = i$. This index is called $\lambda_i$.

<table>
<thead>
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<th>i</th>
<th>3</th>
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<td>$\lambda_i$</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>21</td>
<td>11</td>
<td>53</td>
<td>331</td>
<td>843</td>
<td>1867</td>
<td>3915</td>
<td>8011</td>
<td>181</td>
<td>16565</td>
<td></td>
</tr>
</tbody>
</table>

Figure 12. The minimum index $n$ for which $\nu_2(n^2 + 7) = i$

Conclusions

The sequence of valuations $\{\nu_2(n^2 + a)\}$, for $1 \leq a \leq 6$ have been represented in terms of a finite tree. This corresponds to a closed-form expression for this valuation. The case $a = 7$ produces an infinite tree, with two branches. The experimental behavior of the tree makes it unlikely that such a closed-form expression exists for $\nu_2(n^2 + 7)$.

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