SCIENTIA
Series A: *Mathematical Sciences*, Vol. 30 (2020), 79–89
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

A natural way to write non-Natural numbers

Gurdial Arora, Valerio De Angelis, and Sindhu Unnithan

Abstract. In the early 20th century, continued fractions used to be a regular ingredient of the school curriculum, but they have now all but disappeared. In this article we present an elementary treatment based on the Stern-Brocot tree, with emphasis on the fact that they provide a more natural way to represent irrational numbers than the usual decimal expansion.

This article tells a story whose plot is easy to summarize: the way we write real numbers using decimal expansions is not natural. There is instead a natural way to write them using *continued fractions*, that was quite popular in the schools about 100 years ago, but unfortunately has now just about disappeared from the curriculum.

A main ingredient we will use in the development of our story is the Stern-Brocot tree, whose definition we will soon give, but can be briefly described as an elegant and efficient way to generate all the rational numbers without any repetitions and always producing fractions in lowest terms.

There are many excellent articles, books and web sites that treat continued fractions, ranging from the elementary to the very advanced [**6, 7, 3, 4, 8**]. In many such treatments, the connection between continued fractions and the Stern-Brocot tree is also discussed, by showing how the digits of a continued fraction expansion have an interpretation as paths on the tree [**1, 11**].

In this article, we propose a different point of view: we start with the Stern-Brocot tree and use it to *define* continued fractions. We think this approach is elementary and intuitive, and accessible to anybody with only basic mathematical knowledge. Our purpose is to outline this alternative treatment, and we will mostly omit the proofs of the theorems. The results in this article whose proof does not directly follow from the definitions are well established results that can be found in the references on continued fractions cited in the bibliography.

As a disclaimer, we do not expect there is any new result in this article, and we recognize that much of the discussion about what is "natural" and what is not in the way we represent numbers is rather subjective and should be viewed as personal opinion.

2000 *Mathematics Subject Classification.* Primary 11-01, Secondary 11A55.
Key words and phrases. Continued fractions, Stern-Brocot tree, Elementary number theory.

1. Numbers and how to write them

Why do we write numbers using ten digits? The answer, of course, is: because we have ten fingers. In fact the word “digit” is from the Latin “digitus” for finger. The *Hindu-Arabic system* used in much of today's world uses the idea that a digit can be placed in different slots, or positions, and then its value depends on the position. When a digit is moved from one slot to the next, its value gets multiplied by a fixed number b.

Using three slots, and choosing ten for b, one is $\boxed{}\,\boxed{}\,\boxed{1}$, ten is $\boxed{}\,\boxed{1}\,\boxed{}$ one hundred is $\boxed{1}\,\boxed{}\,\boxed{}$. The brilliant idea that the empty slot should be considered a digit itself led to the invention of zero. But this positional system leaves a question unanswered: *How should we choose b?* The choice $b = 10$ is just because people have 10 fingers. This is not very natural. For example, why not include toes and use $b = 20$?

Numbers were not created by people. As the German mathematician Leopold Kronecker said: *God made the integers; all else is the work of man.* If there are intelligent beings on another planet who do not have ten fingers, they will not use ten as a base. Is there a more natural choice for b?

Any number greater than 1 could be used. If we use the smallest possible number, $b = 2$, we get the binary system, whose only numerals are 0 and 1, and 2020 written in binary is 11111100100. Is this a natural way to write Natural numbers?

If we use $b = 62$, with numerals $0, 1, \ldots, 8, 9, a, b, \ldots, y, z, A, B, \ldots, Y, Z$ then 2020 becomes vz. Clearly the larger the base, the smaller the number of digits. But what is the right compromise?

Consider three and fifty-seven, why does 3 have its own numeral and 57 needs to be made from two numerals? If we use 62 as base then 57 is U. The reality is:

- There is no natural base to write the Natural numbers.
- The only natural way to represent a Natural number would be to give each its own numeral.
- In other words, the Natural numbers are already natural.

What about a non-Natural real number, such as $\frac{11}{7}$ or e? Their decimal expansions are

$$\frac{11}{7} = 1.5714285714285714285714285714285714 28\ldots$$

$$e = 2.71898189845904523536028747135266249775\ldots$$

A number such as e is not Natural, but it is quite natural, and once again we have to rely on the ten digits that originate from our ten fingers. But now the story is different. Continued fractions give us a natural way to represent any real number. Unfortunately, they are currently out of fashion in our schools.

2. Continued fractions via the Stern-Brocot tree

We first describe how to list all the rational numbers in a simple and efficient way. Define F-addition: If $\frac{a}{b}$ and $\frac{c}{d}$ are two rational numbers in lowest terms, we define

F-addition $\oplus$ as:

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}.$$

Begin with the two fractions $\frac{0}{1}$ and $\frac{1}{0}$. These are the generation -1 entries of an infinite tree, called the Stern-Brocot tree after the German mathematician Moritz Abraham Stern (1807–1894) and the French clockmaker and amateur mathematician Achille Brocot (1817–1878). We can think of these fractions as the First and Last of all numbers, or as the universal ancestors of every other number.

To find generation 0, we perform F-addition:

$$\frac{0}{1} \oplus \frac{1}{0} = \frac{0+1}{1+0} = \frac{1}{1} = 1.$$

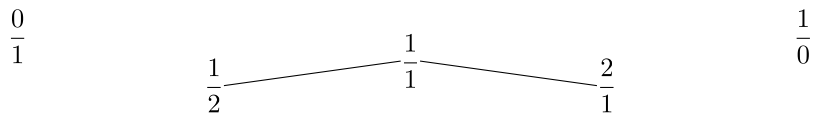
and place the result in the middle

$$\frac{0}{1} \qquad\qquad \frac{1}{1} \qquad\qquad \frac{1}{0}$$

The number 1 is the root of the Stern-Brocot tree, and it is an ancestor of every other number except 0.

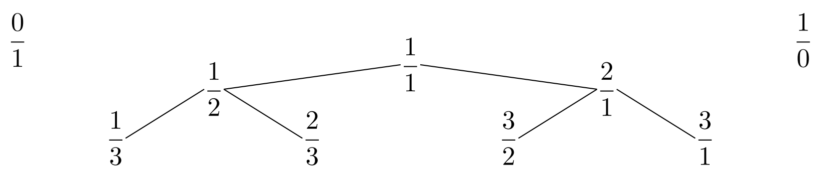
To find generation 1, we perform F-addition between generation 0 and generation -1:

$$\frac{0}{1} \oplus \frac{1}{1} = \frac{0+1}{1+1} = \frac{1}{2}, \qquad \frac{1}{0} \oplus \frac{1}{1} = \frac{1+1}{0+1} = \frac{2}{1}.$$

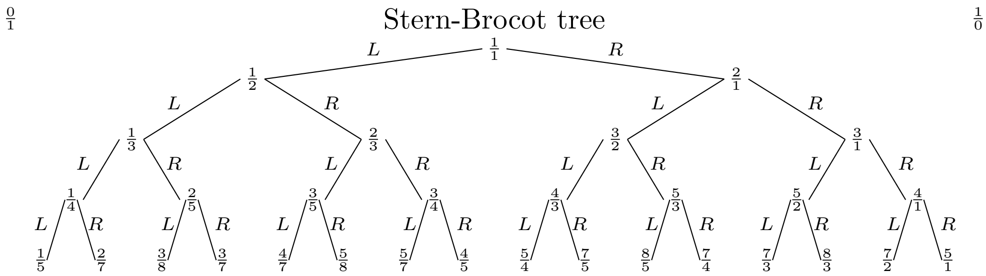


To find generation 2, we perform F-addition between each entry in generation 1 and its immediate neighbors in a previous generation:

$$\frac{0}{1} \oplus \frac{1}{2} = \frac{1}{3}, \quad \frac{1}{2} \oplus \frac{1}{1} = \frac{2}{3}, \quad \frac{1}{1} \oplus \frac{2}{1} = \frac{3}{2}, \quad \frac{2}{1} \oplus \frac{1}{0} = \frac{3}{1}.$$



And so on. We also label each edge with L or R, according to if the edge goes left or right.



The following list gives some definitions and properties of the Stern-Brocot tree. See [**1**, p.115–123] for the proofs.

Properties of the Stern-Brocot tree

(1) Each entry on the tree has a nearest left ancestor and a nearest right ancestor. For example, $4/7$ has left ancestor $1/2$ and right ancestor $3/5$.
(2) One of them is in the preceding generation (called *parent*), and one in at least two earlier generations (called *nearest distant ancestor*). So $4/7$ has $3/5$ as parent and $1/2$ as nearest distant ancestor.
(3) Each entry is in lowest terms.
(4) If $u = \dfrac{a}{b}$ is the parent or nearest distant ancestor of $\dfrac{c}{d}$, then $|ad - bc| = 1$.
(5) Every positive rational number appears exactly once as an entry on the tree.

We will identify a path on the tree (that is, a sequence of L's and R's) with the rational number that the path leads to. So we will write

$$\frac{13}{18} = LR^2LRL, \quad \frac{17}{5} = R^3L^2R,$$

and so on. If x is a rational number, we will denote by x' its parent on the Stern-Brocot tree, and by x^* its nearest distant ancestor. It can be seen from the way we constructed the tree that the path for x' is obtained from the path of x by deleting the last letter, and x^* is obtained by deleting the last string consisting of the same letter, plus one more letter. So for example for $x = \dfrac{14}{9} = RLRL^3$ we find $x' = RLRL^2 = \dfrac{11}{7}$ and $x^* = RL = \dfrac{3}{2}$. It also follows by construction that $x = x' \oplus x^*$ (for example, $\dfrac{14}{9} = \dfrac{11}{7} \oplus \dfrac{3}{2}$).

2.1. The continued fraction expansion for rational numbers. Given a path on the tree, we can summarize it by counting the number of successive blocks of the same letter. Writing L^2 for LL, R^3 for RRR, etc, this means listing the exponents.

So for example the path leading to $11/7$ is $RLRL^2$, corresponding to exponents $1, 1, 1, 2$. If a path begins with L, then we list 0 as the first exponent. So the path LRL^2RL leading to $11/19$ corresponds to the sequence of exponents $0, 1, 1, 2, 1, 1$. Equivalently, we assume that the list of exponents is always referring to a string that begins with R. It follows that the path corresponding to the sequence of exponents $a_0, a_1, \ldots, a_m$ will end with L if m is odd, and with R if m is even.

We are now ready to define the continued fraction expansion of a rational number.

Definition 1. *Let $a_0, a_1, \ldots, a_m$ be the sequence of exponents corresponding to the path on the Stern-Brocot tree leading to the rational number x. We say that $a_0, a_1, \ldots, a_{m-1}, a_m + 1$ is the* continued fraction expansion *of x, and we write*

$$x = [a_0; a_1, \ldots, a_{m-1}, a_m + 1].$$

So according to our definition the last digit of a continued fraction expansion is always at least 2. But it is convenient to extend our notation, and we define

$$[a_0, ; a_1, a_2, \ldots, a_m, 1] = [a_0; a_1, a_2, \ldots, a_m + 1]$$

The reason for the terminology is now found in the following theorem.

Theorem 1. *Let a_0 be a non-negative integer, and $a_1, \ldots, a_m$ positive integers. Then*

$$[a_0; a_1, a_2, \ldots, a_m] = a_0 + \frac{1}{[a_1; a_2, \ldots, a_m]}.$$

It follows from the theorem that

$$[a_0] = a_0, \quad [a_0; a_1] = a_0 + \frac{1}{a_1}, \quad [a_0; a_1, a_2] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}}$$

and so on. The identity $\cfrac{1}{a + \cfrac{1}{1}} = \dfrac{1}{a+1}$ corresponds to $[0; a, 1] = [0; a+1]$ and explains our extension of the notation. But for the purpose of this article, the continued fraction expansion of a (non-integer) rational number is the unique sequence of integers $[a_0, a_1, \ldots, a_m]$ derived from the path on the Stern-Brocot tree and that terminates with an integer $a_m \geqslant 2$.

Example 1.

- $\dfrac{13}{18} = LR^2LRL = [0; 1, 2, 1, 1, 2] = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}}}$
- $\dfrac{17}{5} = R^3L^2R = [3; 2, 2] = 3 + \cfrac{1}{2 + \cfrac{1}{2}}$

Our definition of continued fractions is useful to visualize the connection with the Stern-Brocot tree. But it is not practical for computations when numerator or denominator are even moderately large, and we will soon describe a more efficient procedure (for an online Stern-Brocot calculator, see for example [**11**]).

The continued fraction expansion is similar to the representation of a rational number as a decimal, but with one difference: there is no arbitrary choice of a base.

Every integer may be used in the continued fraction expansion of a rational number. This will be especially significant later in our discussion of irrational numbers. Even if we adopt the "natural" point of view that each integer should have its own symbol, the representation of an irrational number as a ratio of symbols is no longer possible, so we are forced to arbitrarily choose a base such as 10 for the expansion. But with continued fractions we do not have to make any such arbitrary choice.

We now re-visit some basic results in elementary number theory.

3. Euclidean division

Given two integers $a > b > 0$, there are unique integers q and r such that $a = qb+r$ and $0 \leqslant r < b$. In fact,

$$q = \left\lfloor \frac{a}{b} \right\rfloor, \quad r = b\left\{ \frac{a}{b} \right\},$$

where $\lfloor \cdot \rfloor$ and $\{\cdot\}$ are the greatest integer and fractional part functions. One aspect of this important result is that it tells us how to decompose a rational number into the sum of an integer and a rational number strictly less than 1. We now re-visit it from the point of view of continued fractions.

3.1. From rationals to integers via continued fractions. Suppose we are given the continued fraction expansion (or equivalently its path on the Stern-Brocot tree) of a rational number. So the natural question is: how do we recover numerator and denominator (in lowest terms)? The answer is given by the following theorem.

THEOREM 2. *Let $x = [a_0; a_1, a_2, \dots, a_m]$ be a rational number. Then the representation $x = \dfrac{a}{b}$ as a fraction in lowest terms is given by*

$$b = \sqrt{\left| \frac{x^* - x'}{(x^* - x)(x - x')} \right|}, \quad a = xb,$$

and moreover

$$\lfloor x \rfloor = a_0, \quad \{x\} = [0; a_1, a_2, \dots, a_m].$$

EXAMPLE 2. *If $x = [2;3] = RRLL = 7/3$, then $x' = RRL = 5/2$, $x^* = R = 2$ and*

$$\sqrt{\left| \frac{x^* - x'}{(x^* - x)(x - x')} \right|} = \sqrt{\frac{1/2}{(1/3)(1/6)}} = 3.$$

But the previous theorem is of no computational interest unless we know how to do arithmetic operations with continued fractions. In other words, how should we compute $x^* - x$ other than by converting both to quotients of integers? The problem of doing arithmetic with continued fractions was discussed by Gosper [**5**], but the topic is not simple and outside the scope of this article.

One drawback of the representation of a rational number as the ratio of two integers a/b is that it does not easily allow the computation of approximations. The decimal expansion is of course useful in that respect, but it has the drawback of having to use an arbitrary choice for the base. On the other hand, the continued fraction expansion $[a_0; a_1, a_2, \ldots, a_m]$ makes no use of arbitrary choices, and truncating the sequence of digits provides very good approximations (as will be discussed later).

3.2. Euclidean algorithm for relatively prime integers. We recall a fundamental result [**1**, p.103–104, 303–304]

THEOREM 3. *Let a and b be relatively prime, positive integers. Then there are integers x and y such that $ax + by = 1$.*

REMARK 1. *The integers x and y in the previous theorem are far from unique, as can be seen by replacing x with $x - kb$ and y with $y + ka$ for any integer k. But there is a unique pair (x, y) of such integers that minimizes the distance $\sqrt{x^2 + y^2}$ from the origin.*

We now describe the classical Euclidean algorithm that produces the integers x, y of minimal distance from the origin. Start with two relatively prime integers $a > b > 0$. Define

$$r_{-1} = a, \quad r_0 = b, \quad a_0 = \lfloor r_{-1}/r_0 \rfloor, \quad r_1 = r_{-1} - a_0 r_0.$$

If a_0, a_1, …, a_{i-1}, r_{-1}, r_0, …, r_i have been defined and $r_i > 0$, define

$$a_i = \lfloor r_{i-1}/r_i \rfloor, \quad r_{i+1} = r_{i-1} - a_i r_i.$$

The sequence r_i is strictly decreasing, so let m be the last index i for which $r_i > 0$. Define sequences s_j, t_j by

$$\begin{array}{rclrcl} s_{m-2} & = & 1, & t_{m-1} & = & -a_{m-1} \\ s_{j-2} & = & t_j, & t_{j-1} & = & s_{j-1} - a_{j-1} t_j, \quad m \geqslant j \geqslant 1. \end{array}$$

THEOREM 4.

(1) $r_m = \gcd(a, b) = 1$.

(2) $\dfrac{a}{b} = [a_0; a_1, a_2, \ldots, a_m]$

(3) *s_{-1} and t_0 are the integers of minimal distance from the origin such that $as_{-1} + bt_0 = 1$.*

The Euclidean algorithm provides a recursive procedure to find the linear combination $ax + by = 1$. We will now look at the same problem from the continued fractions point of view [**6**, Ch1], [**4**].

3.3. The nearest distant ancestor's role. Let $x = [a_0; a_1, a_2, \ldots, a_m]$ be a rational number. Recall that by definition we assume $a_m \geqslant 2$. It follows directly from the definition of parent and nearest distant ancestor that

$$x' = [a_0; a_1, a_2, \ldots, a_m - 1], \quad x^* = [a_0; a_1, a_2, \ldots, a_{m-1}].$$

THEOREM 5. *Define sequences p_i, q_i recursively as*

$$\begin{array}{ll} p_0 = a_0, & q_0 = 1 \\ p_1 = a_0 a_1 + 1, & q_1 = a_1 \\ p_i = a_i p_{i-1} + p_{i-2}, & q_i = a_i q_{i-1} + q_{i-2} \end{array}$$

Then

$$x = \frac{p_m}{q_m}, \quad x^* = \frac{p_{m-1}}{q_{m-1}}, x' = \frac{p_m - p_{m-1}}{q_m - q_{m-1}},$$

and (p_{m-1}, q_{m-1}) *is the unique pair of positive integers at minimal distance from the origin such that*

$$q_m p_{m-1} - p_m q_{m-1} = (-1)^m.$$

So the previous theorem tells us that the nearest distant ancestor of a rational number a/b on the Stern-Brocot tree provides the linear combination $ax + by = 1$ of minimal size.

Example 3. *Let* $x = RLRL^2R^2 = [1; 1, 1, 2, 3] = \frac{27}{17}$. *Then* $x^* = RLRL = [1; 1, 1, 2] = \frac{8}{5}$ *and* $(17)(8) - (27)(5) = 1$

It follows from the last theorem that truncating a continued fraction expansion gives a good approximation:

$$|[a_0; a_1, \ldots, a_m] - [a_0; a_1, \ldots, a_{m-1}]| = \left| \frac{p_m}{q_m} - \frac{p_{m-1}}{q_{m-1}} \right| = \frac{1}{q_m q_{m-1}}.$$

4. Infinite paths

We now consider infinite paths on the tree. See [**9**, Ch 10] for a thorough discussion of this material. Denote by L^∞ the infinite string $LLLL\ldots$, and similarly for R^∞.

4.1. Properties of infinite paths.

(1) Every positive, rational number is also represented by two infinite paths, obtained by attaching the strings RL^∞ and LR^∞ to the finite path. So for example,

$$\frac{11}{7} = RLRL^2 = RLRL^3R^\infty = RLRL^2RL^\infty$$

(2) Every positive, irrational number is represented by a unique infinite path that does not terminate with L^∞ or R^∞.
(3) An infinite path on the tree that does not terminate in L^∞ or R^∞ gives rise to an infinite continued fraction $[a_0; a_1, a_2, \ldots]$ obtained by counting the number of repeated letters, as for the finite case.
(4) Given an infinite continued fraction $[a_0; a_1, a_2, \ldots]$, let p_m/q_m be the rational number corresponding to the truncated continued fraction $[a_0; a_1, \ldots, a_m]$, and obtained through the recursive procedure described in the previous section. Then the sequence p_m/q_m converges to the irrational number represented by $[a_0; a_1, a_2 \ldots]$.

The approximation of irrational numbers by their truncated continued fraction can be shown to be very efficient. We will give an example in the next section.

4.2. From numbers to continued fractions. In this section we show how to construct the continued fraction expansion of a real number by using the floor and reciprocal operations.

The process is best illustrated by example. First we use a rational number:

$$\frac{30}{13} = 2.3\cdots = 2+0.3\cdots = 2+\frac{1}{3.25} = 2+\frac{1}{3+0.25\cdots} = 2+\cfrac{1}{3+\cfrac{1}{4}}$$

So $\frac{30}{13} = [2; 3, 4]$.

For an irrational number the process is the same, but it does not terminate:

$$\begin{aligned}\pi &= 3.141592653\ldots = 3+0.141592653\ldots = 3+\frac{1}{7.06251\ldots}\\ &= 3+\frac{1}{7+0.06251\ldots} = 3+\cfrac{1}{7+\cfrac{1}{15.9966\ldots}} = 3+\cfrac{1}{7+\cfrac{1}{15+0.9966\ldots}}.\end{aligned}$$

Continuing this way, we find

$$\pi = [3; 7, 15, 1, 292\ldots]$$

We now compute the truncated approximations for the continued fraction expansion of π.

$$[3;7] = \frac{22}{7} = 3.14\ldots \qquad [3;7,15] = \frac{333}{106} = 3.1415\ldots$$

$$[3;7,15,1] = \frac{355}{113} = 3.141592\ldots \qquad [3;7,15,1,292] = \frac{103993}{33102} = 3.141592653\ldots$$

Note that the last fraction with a denominator of only 5 digits gives an approximation whose first 10 decimal digits are accurate.

The recursive procedure illustrated in the examples above is easily defined using the floor and reciprocal functions. We let

$$\begin{aligned} a_0 &= \lfloor x \rfloor, & t_1 &= x - a_0\\ a_i &= \lfloor 1/t_i \rfloor, & t_{i+1} &= \frac{1}{t_i} - a_i, \quad i \geqslant 1.\end{aligned}$$

5. Comparing decimal vs. continued fractions expansions

- The decimal expansion of a rational number either terminates, or it is periodic. The continued fraction expansion of a rational number always terminates.
- The decimal expansion of any irrational number is infinite and non-periodic. The continued fraction expansion of quadratic irrational numbers (that is, numbers that are solutions of a quadratic polynomial with integer coefficients) is periodic.
- The continued fraction expansion of e and some other irrational numbers constructed from e have easily discernible patterns.

Some examples are given below.

	Decimal	Continued Fractions
$\frac{11}{7}$	$1.571428571428\ldots$	$[1;1,1,3]$
$\sqrt{2}$	$1.414213562373\ldots$	$[1;2,2,2,2,2,2,2,2,2,2,2,2,2,2,\ldots]$
$\sqrt{3}$	$1.732050807568\ldots$	$[1;1,2,1,2,1,2,1,2,1,2,1,2,\ldots]$
$\frac{1+\sqrt{5}}{2}$	$1.618033988749\ldots$	$[1;1,1,1,1,1,1,1,1,1,1,1,1,1,\ldots]$
e	$2.718981898459\ldots$	$[2;1,2,1,1,4,1,1,6,1,1,8,1,1,10,\ldots]$
$\tanh(1) = \frac{e-e^{-1}}{e+e^{-1}}$	$0.761594155955\ldots$	$[0;1,3,5,7,9,11,13,15,17,19,21,23,\ldots]$
$\tanh\left(\frac{1}{2}\right) = \frac{e-1}{e+1}$	$0.462117157260\ldots$	$[0;2,6,10,14,18,22,26,30,34,38,42,46,\ldots]$

It should be evident from the previous table that while the decimal expansion is almost always a seemingly random sequence of digits, the continued fraction expansion often has a clearly discernible pattern.

We conclude by suggesting a problem for the reader. As mentioned earlier, the problem of doing direct arithmetic with continued fractions does not seem to have a simple solution. But we can experiment with the simplest case: given two continued fractions with just two digits each: $x = [a_0; a_1]$, $y = [b_0; b_1]$, what is the continued fraction expansion of $x + y$?

References

[1] D. Knuth, R. Graham, and O. Patashnik. Concrete Mathematics. Addison-Wesley Longman Publishing, 2nd ed., 1994.

[2] www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fractions/fareySB.html?#sbcalc(Retrieved: 07/19/2020) (online Stern-Brocot tree calculator).

[3] N. Calkin and H. S. Wilf, Recounting the Rationals, The American Mathematical Monthly, Vol. 107, No. 4 (Apr., 2000), pp. 360–363.

[4] J. Borwein, A. van der Poorten, J. Shallit and W. Zudilin, Neverending Fractions: An Introduction to Continued Fractions, Cambridge University Press 2014.

[5] R.W. Gosper, Continued fraction arithmetic. HAKMEM Item 101B, MIT Artificial Intelligence Memo 239, 1972.

[6] C.D. Olds, Continued Fractions, MAA, 1963.

[7] C. Kraaikamp and M. Iosifescu, Metrical Theory of Continued Fractions, Springer-Science and Business media, 2002.

[8] M. B. Muñoz, J. J. Escribano Benito, Sucesiones de Brocot,Talleres Gràficos de Editorial Ochoa, 1998.

[9] V. Moll, Numbers and Functions, AMS Student Mathematical Library V. 65, 2012.

[10] I. Stewart. The story of mathematics. Quercus, 2008.

[11] http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fractions/fareySB.html(Retrieved: 07/19/2020)

Mathematics Department ,
Xavier University of Louisiana,
1 Drexel Drive, New Orleans, LA 70125
U.S.A.
E-mail address: gdial@xula.edu
E-mail address: vdeangel@xula.edu
E-mail address: sunnitha@xula.edu

Received 07 06 2020 revised 15 07 2020