Inequalities of Jensen’s Type for $K$-Bounded Modulus Convex Complex Functions

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Abstract. Let $D \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$. We say that the function $f : D \subset \mathbb{C} \to \mathbb{C}$ is called $K$-bounded modulus convex, for the given $K > 0$, if it satisfies the condition

$$|(1 - \lambda) f(x) + \lambda f(y) - f((1 - \lambda)x + \lambda y)| \leq \frac{1}{2} K \lambda (1 - \lambda) |x - y|^2$$

for any $x, y \in D$ and $\lambda \in [0, 1]$.

In this paper we establish some new Jensen’s type inequalities for the complex integral on $\gamma$, a smooth path from $C$ and $K$-bounded modulus convex functions. Some examples for the complex exponential and complex logarithm are also given.

1. Introduction

Let $(X; \| \cdot \|_X)$ and $(Y; \| \cdot \|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$. Let $C$ be a convex set in $X$. In the recent paper [3] we introduced the following class of functions:

Definition 1. A mapping $F : C \subset X \to Y$ is called $K$-bounded norm convex, for some given $K > 0$, if it satisfies the condition

$$(1.1) \quad \|(1 - \lambda) F(x) + \lambda F(y) - F((1 - \lambda)x + \lambda y)\|_Y \leq \frac{1}{2} K \lambda (1 - \lambda) \|x - y\|_X^2$$

for any $x, y \in C$ and $\lambda \in [0, 1]$. For simplicity, we denote this by $F \in \mathcal{BN}_K(C)$.

We have from (1.1) for $\lambda = \frac{1}{2}$ the Jensen’s inequality

$$\left\| \frac{F(x) + F(y)}{2} - F\left(\frac{x + y}{2}\right) \right\|_Y \leq \frac{1}{8} K \|x - y\|_X^2$$

for any $x, y \in C$.

We observe that $\mathcal{BN}_K(C)$ is a convex subset in the linear space of all functions defined on $C$ and with values in $Y$.

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In the same paper [3], we obtained the following result which provides a large class of examples of such functions.

**Theorem 1.** Let \((X, \|\cdot\|_X)\) and \((Y, \|\cdot\|_Y)\) be two normed linear spaces, \(C\) an open convex subset of \(X\) and \(F : C \to Y\) a twice-differentiable mapping on \(C\). Then for any \(x, y \in C\) and \(\lambda \in [0, 1]\) we have

\[
\| (1 - \lambda) F(x) + \lambda F(y) - F((1 - \lambda)x + \lambda y) \|_Y \leq \frac{1}{2} K \lambda (1 - \lambda) \| y - x \|_X^2 ,
\]

where

\[
K_{F''} := \sup_{z \in C} \| F''(z) \|_{\mathcal{L}(X^2; Y)} ,
\]

is assumed to be finite, namely \(F \in BN_{K_{F''}}(C)\).

We have the following inequalities of Hermite-Hadamard type [3]:

**Theorem 2.** Let \((X, \|\cdot\|_X)\) and \((Y, \|\cdot\|_Y)\) be two normed linear spaces over the complex number field \(C\) with \(Y\) complete. Assume that the mapping \(F : C \subset X \to Y\) is continuous on the convex set \(C\) in the norm topology. If \(F \in BN_K(C)\) for some \(K > 0\), then we have

\[
\left\| \frac{F(x) + F(y)}{2} - \int_0^1 F((1 - \lambda)x + \lambda y) \, d\lambda \right\|_Y \leq \frac{1}{12} K \| x - y \|_X^2
\]

and

\[
\left\| \int_0^1 F((1 - \lambda)x + \lambda y) \, d\lambda - F\left(\frac{x + y}{2}\right) \right\|_Y \leq \frac{1}{24} K \| x - y \|_X^2
\]

for any \(x, y \in C\).

The constants \(\frac{1}{12}\) and \(\frac{1}{24}\) are best possible.

Following [1, p. 59], let \((X, \|\cdot\|_X)\) and \((Y, \|\cdot\|_Y)\) be two normed linear spaces, \(\Omega\) an open subset of \(X\) and \(F : \Omega \to Y\). If \(a \in \Omega\), \(u \in X \setminus \{0\}\) and if the limit

\[
\lim_{t \to 0} \frac{1}{t} [F(a + tu) - F(a)]
\]

exists, then we denote this derivative \(\partial_u F(a)\). It is called the directional derivative of \(F\) at \(a\) in the direction \(u\). If the directional derivative is defined in all directions and there is a continuous linear mapping \(\Phi\) from \(X\) into \(Y\) such that for all \(u \in X\)

\[
\partial_u F(a) = \Phi(u),
\]

then we say that \(F\) is Gâteaux-differentiable at \(a\) and that \(\Phi\) is the Gâteaux differential of \(F\) at \(a\). If a mapping \(F\) is differentiable at a point \(a\), then clearly all its directional derivatives exist and we have

\[
\partial_u F(a) = F'(a) u, \ u \in X.
\]

Thus \(F\) is Gâteaux-differentiable at \(a\). However, the Gâteaux differential may exist without the differential existing. The existence of directional derivatives at a point
does not imply that the mapping is Gâteaux-differentiable. To distinguish the differential from the Gâteaux differential, the differential is often referred as the Fréchet differential.

In an earlier and more comprehensive version of [3], see [2], we also obtained the following Jensen’s type discrete inequality:

**Theorem 3.** Let \( (X; \| \cdot \|_X) \) and \( (Y; \| \cdot \|_Y) \) be two normed linear spaces over the complex number field \( \mathbb{C} \). Assume that the mapping \( F : C \subset X \rightarrow Y \) is defined on the open convex set \( C \) and \( F \in BN_K (C) \) for some \( K > 0 \). If \( x_k \in C, \ p_k \geq 0 \) for \( k \in \{1, \ldots, n\} \) with \( \sum_{k=1}^{n} p_k = 1 \) and \( F \) is Gâteaux-differentiable at \( \sum_{k=1}^{n} p_k x_k \in C \), then for any \( y_j \in C \) and \( q_j \geq 0 \) for \( j \in \{1, \ldots, m\} \) with \( \sum_{j=1}^{m} q_j = 1 \) and \( \sum_{j=1}^{m} q_j y_j = \sum_{k=1}^{n} p_k x_k \) we have

\[
(1.6) \quad \left\| \sum_{j=1}^{m} q_j F(y_j) - F \left( \sum_{k=1}^{n} p_k x_k \right) \right\|_Y \leq \frac{1}{2} K \sum_{j=1}^{m} q_j \left\| y_j - \sum_{k=1}^{n} p_k x_k \right\|^2_X.
\]

In particular, we have

\[
(1.7) \quad \left\| \sum_{j=1}^{n} p_j F(x_j) - F \left( \sum_{k=1}^{n} p_k x_k \right) \right\|_Y \leq \frac{1}{2} K \sum_{j=1}^{n} p_j \left\| x_j - \sum_{k=1}^{n} p_k x_k \right\|^2_X.
\]

If \( (X; \langle \cdot, \cdot \rangle) \) is an inner product space, then

\[
\sum_{j=1}^{n} p_j \left\| x_j - \sum_{k=1}^{n} p_k x_k \right\|^2_X = \sum_{j=1}^{n} p_j \left\| x_j \right\|^2_X - \sum_{k=1}^{n} \left\| p_k x_k \right\|^2_X
\]

and by (1.7) we have

\[
(1.8) \quad \left\| \sum_{j=1}^{n} p_j F(x_j) - F \left( \sum_{k=1}^{n} p_k x_k \right) \right\|_Y \leq \frac{1}{2} K \left[ \sum_{j=1}^{n} p_j \left\| x_j \right\|^2_X - \sum_{k=1}^{n} \left\| p_k x_k \right\|^2_X \right].
\]

**Corollary 1.** Let \( (X; \| \cdot \|_X) \) and \( (Y; \| \cdot \|_Y) \) be two normed linear spaces, \( C \) an open convex subset of \( X \) and \( F : C \rightarrow Y \) a twice-differentiable mapping on \( C \). If \( x_k \in C, \ p_k \geq 0 \) for \( k \in \{1, \ldots, n\} \) with \( \sum_{k=1}^{n} p_k = 1 \), then

\[
(1.9) \quad \left\| \sum_{j=1}^{n} p_j F(x_j) - F \left( \sum_{k=1}^{n} p_k x_k \right) \right\|_Y \leq \frac{1}{2} \sup_{z \in C} \| F''(z) \|_{\mathcal{L}(X; Y)} \left[ \sum_{j=1}^{n} p_j \left\| x_j - \sum_{k=1}^{n} p_k x_k \right\|^2_X \right].
\]

Let \( D \subset \mathbb{C} \) be a convex domain of complex numbers and \( K > 0 \). Following Definition 1, we say that the function \( F : D \subset \mathbb{C} \rightarrow \mathbb{C} \) is called \( K \)-bounded modulus convex, for the given \( K > 0 \), if it satisfies the condition

\[
(1.10) \quad |(1 - \lambda) F(x) + \lambda F(y) - F ((1 - \lambda) x + \lambda y)| \leq \frac{1}{2} K \lambda (1 - \lambda) \| x - y \|^2
\]
for any $x, y \in D$ and $\lambda \in [0, 1]$. For simplicity, we denote this by $F \in \mathcal{B}M_K(D)$.

All the above results can be translated for complex functions defined on convex subsets $D \subset \mathbb{C}$.

In the following, in order to obtain several inequalities for the complex integral, we need the following facts.

Suppose $\gamma$ is a smooth path from $\mathbb{C}$ parametrized by $z(t), t \in [a, b]$ and $f$ is a complex function which is continuous on $\gamma$. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of $f$ on $\gamma_{u, w} = \gamma$ as

$$\int_{\gamma_{u, w}} f(z) \, dz = \int_{\gamma_{u, w}} f(z) \, dz := \int_a^b f(z(t)) \, z'(t) \, dt.$$ We observe that that the actual choice of parametrization of $\gamma$ does not matter. This definition immediately extends to paths that are piecewise smooth. Suppose $\gamma$ is parametrized by $z(t), t \in [a, b]$, which is differentiable on the intervals $[a, c]$ and $[c, b]$, then assuming that $f$ is continuous on $\gamma$ we define

$$\int_{\gamma_{u, w}} f(z) \, dz := \int_{\gamma_{u, w}} f(z) \, dz + \int_{\gamma_{w, v}} f(z) \, dz$$ where $v := zz$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u, w}} f(z) \, |dz| := \int_a^b f(z(t)) \, |z'(t)| \, dt$$ and the length of the curve $\gamma$ is then

$$\ell(\gamma) = \int_{\gamma_{u, w}} |dz| = \int_a^b |z'(t)| \, dt.$$ Let $f$ and $g$ be holomorphic in $D$, and open domain and suppose $\gamma \subset D$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then we have the integration by parts formula

$$\int_{\gamma_{u, w}} f(z) g'(z) \, dz = f(w) g(w) - f(u) g(u) - \int_{\gamma_{u, w}} f'(z) g(z) \, dz.$$ We recall also the triangle inequality for the complex integral, namely

$$\left| \int_{\gamma_{u, w}} f(z) \, dz \right| \leq \int_{\gamma_{u, w}} |f(z)| \, |dz| \leq \|f\|_{\gamma, \infty} \ell(\gamma)$$ where $\|f\|_{\gamma, \infty} := \sup_{z \in \gamma} |f(z)|$.

We also define the $p$-norm with $p \geq 1$ by

$$\|f\|_{\gamma, p} := \left( \int_{\gamma_{u, w}} |f(z)|^p \, |dz| \right)^{1/p}.$$ For $p = 1$ we have

$$\|f\|_{\gamma, 1} := \int_{\gamma_{u, w}} |f(z)| \, |dz|.$$
If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder’s inequality we have
$$\|f\|_{\gamma,1} \leq [\ell(\gamma)]^{1/q} \|f\|_{\gamma,p}.$$ In the recent paper [5] we obtained the following results:

**Theorem 4.** Let $D \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$. Assume that $f$ is holomorphic on $D$ and $f \in \mathcal{BM}_K(D)$. If $\gamma \subset D$ parametrized by $z(t), t \in [a,b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ and $v \in D$, then

$$\int_{\gamma} f(z)\, dz - \left[ f(v) + f'(v) \left( \frac{w+u}{2} - v \right) \right] (w-u) \leq \frac{1}{2} \int_{\gamma} |z-v|^2 |dz|$$

and

$$\frac{1}{2} |f(w) (w-v) + f(u) (v-u) + f(v) (w-u)| - \int_{\gamma} f(z)\, dz \leq \frac{1}{4} K \int_{\gamma} |z-v|^2 |dz|.$$

Motivated by the above results, in this paper we establish some new Jensen’s type inequalities for the complex integral on $\gamma$, a smooth path from $\mathbb{C}$ and $K$-bounded modulus convex functions. Some examples for the complex exponential and complex logarithm are also given.

## 2. General Integral Inequalities

We have:

**Theorem 5.** Let $G \subset \mathbb{C}$ be a convex domain of complex numbers and $K > 0$ and that $F$ is holomorphic on $G$ with $F \in \mathcal{BM}_K(G)$. Assume also that $f : D \to G$ is continuous on $D$, $\gamma \subset D$ parametrized by $z(t), t \in [a,b]$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$ with $w \neq u$ and $\frac{1}{w-u} \int_{\gamma} f(z)\, dz \in G$, then

$$\frac{1}{w-u} \int_{\gamma} (F \circ f)(v)\, dv - F\left( \frac{1}{w-u} \int_{\gamma} f(z)\, dz \right) \leq \frac{1}{2} K \left| \frac{1}{w-u} \int_{\gamma} f(v) - \frac{1}{w-u} \int_{\gamma} f(z)\, dz \right|^2 |dv|.$$

**Proof.** Let $x, y \in G$. Since $F \in \mathcal{BM}_K(G)$, then we have

$$|F((1-\lambda)x + \lambda y) - F(x) + \lambda |F(x) - F(y)|| \leq \frac{1}{2} K \lambda (1-\lambda)|x-y|^2$$

that implies that

$$\left| \frac{F(x+y(x-y)) - F(x)}{\lambda} + F(x) - F(y) \right| \leq \frac{1}{2} K (1-\lambda)|x-y|^2$$

for $\lambda \in (0,1)$.

Since $F$ is holomorphic on $G$, then by letting $\lambda \to 0+$, we get

$$|F'(x)(y-x) + F(x) - F(y)| \leq \frac{1}{2} K |x-y|^2$$
that is equivalent to

\[(2.2) \quad |F(y) - F(x) - F'(x)(y - x)| \leq \frac{1}{2} K |y - x|^2 \]

for all \(x, y \in G\).

If we take in (2.2) \(x = \frac{1}{w - u} \int_\gamma f(z) dz\), then we get

\[(2.3) \quad \left| F(y) - F\left(\frac{1}{w - u} \int_\gamma f(z) dz\right) - F'(\frac{1}{w - u} \int_\gamma f(z) dz) \left( y - \frac{1}{w - u} \int_\gamma f(z) dz \right) \right| \leq \frac{1}{2} K \left| y - \frac{1}{w - u} \int_\gamma f(z) dz \right|^2 \]

for all \(y \in G\).

If we take in this inequality \(y = f(v), v \in \gamma\), then we get

\[(2.4) \quad \left| (F \circ f)(v) - F\left(\frac{1}{w - u} \int_\gamma f(z) dz\right) - F'(\frac{1}{w - u} \int_\gamma f(z) dz) \left( f(v) - \frac{1}{w - u} \int_\gamma f(z) dz \right) \right| \leq \frac{1}{2} K \left| f(v) - \frac{1}{w - u} \int_\gamma f(z) dz \right|^2 \]

for all \(v \in \gamma\).

We have

\[(2.5) \quad \frac{1}{w - u} \int_\gamma \left[ (F \circ f)(v) - F\left(\frac{1}{w - u} \int_\gamma f(z) dz\right) - F'(\frac{1}{w - u} \int_\gamma f(z) dz) \left( f(v) - \frac{1}{w - u} \int_\gamma f(z) dz \right) \right] dv

= \frac{1}{w - u} \int_\gamma (F \circ f)(v) dv - F\left(\frac{1}{w - u} \int_\gamma f(z) dz\right)

- F'(\frac{1}{w - u} \int_\gamma f(z) dz) \left( \frac{1}{w - u} \int_\gamma f(v) dv - \frac{1}{w - u} \int_\gamma f(z) dz \right) = \frac{1}{w - u} \int_\gamma (F \circ f)(v) dv - F\left(\frac{1}{w - u} \int_\gamma f(z) dz\right).\]
By using (2.4) and (2.5) we get
\[
\left| \frac{1}{w-u} \int_\gamma (F \circ f)(v) \, dv - F\left( \frac{1}{w-u} \int_\gamma f(z) \, dz \right) \right| 
\leq \frac{1}{|w-u|} \int_\gamma \left| (F \circ f)(v) - F\left( \frac{1}{w-u} \int_\gamma f(z) \, dz \right) \right| \, |dv| 
- F'(\frac{1}{w-u} \int_\gamma f(z) \, dz) \left( f(v) - \frac{1}{w-u} \int_\gamma f(z) \, dz \right) |dv| 
\leq \frac{1}{2} K \frac{1}{|w-u|} \int_\gamma \left| f(v) - \frac{1}{w-u} \int_\gamma f(z) \, dz \right|^2 \, |dv|,
\]
which proves the inequality (2.1).

\[\square\]

**Corollary 2.** With the assumptions of Theorem 5 and if
\[\|F''\|_{G, \infty} := \sup_{z \in G} |F''(z)| < \infty,\]
then
\[
(F \circ f)(v) \, dv - F\left( \frac{1}{w-u} \int_\gamma f(z) \, dz \right) \right| 
\leq \frac{1}{2} \|F''\|_{G, \infty} \frac{1}{|w-u|} \int_\gamma \left| f(v) - \frac{1}{w-u} \int_\gamma f(z) \, dz \right|^2 \, |dv|.
\]

**Remark 1.** If we take \(D = G, \gamma \subset G\) and \(f(z) = z\), then by (2.6) we get the Hermite-Hadamard type inequality (see also [5])
\[
\left| \frac{1}{w-u} \int_\gamma F(v) \, dv - F\left( \frac{w+u}{2} \right) \right| 
\leq \frac{1}{2} \|F''\|_{G, \infty} \frac{1}{|w-u|} \int_\gamma \left| v - \frac{w+u}{2} \right|^2 \, |dv|,
\]
provided \(F\) is holomorphic on \(G\) and \(\|F''\|_{G, \infty} := \sup_{z \in G} |F''(z)| < \infty.\)

We also have:

**Theorem 6.** Let \(G \subset \mathbb{C}\) be a convex domain of complex numbers and \(K > 0\) and that \(F\) is holomorphic on \(G\) with \(F \in \mathcal{B}M_K(G)\). Assume also that \(f : D \to G\) is continuous on \(D, \gamma \subset D\) parametrized by \(z(t), t \in [a,b]\) is a piecewise smooth path from \(z(a) = u\) to \(z(b) = w\) with \(w \neq u,\)
\[
\int_\gamma (F' \circ f)(v) \, dv \neq 0 \quad \text{and} \quad \frac{\int_\gamma (F' \circ f)(v) f(v) \, dv}{\int_\gamma (F' \circ f)(v) \, dv} \in G,
\]
then
\[
\left| F\left(\int_{\gamma} (F' \circ f) (v) f (v) dv\right) - \frac{1}{w-u} \int_{\gamma} (F \circ f) (z) dz \right|
\leq \frac{1}{2} K \frac{1}{|w-u|} \int_{\gamma} \left| \int_{\gamma} (F' \circ f) (v) f (v) dv \right| dz
\]

**Proof.** From (2.2) we get
\[
|F (y) - F (f (v)) - F' (f (v)) (y - f (v))| \leq \frac{1}{2} K |y - f (v)|^2
\]
for any \( y \in G \) and for \( v \in D \).

Taking the integral in (2.10) we get
\[
\frac{1}{|w-u|} \int_{\gamma} |F (y) - F (f (v)) - F' (f (v)) (y - f (v))| dv \leq \frac{1}{2} K \frac{1}{|w-u|} \int_{\gamma} |y - f (v)|^2 |dv|
\]
for \( y \in G \).

Using the properties of integral and modulus, we also have
\[
\frac{1}{|w-u|} \int_{\gamma} |F (y) - F (f (w)) - F' (f (w)) (y - f (w))| dw \leq \frac{1}{|w-u|} \int_{\gamma} |F (y) - F (f (w)) - F' (f (w)) (y - f (w))| |dw|
\]
for \( y \in G \).

Now, observe that
\[
\frac{1}{w-u} \int_{\gamma} |F (y) - F (f (v)) - F' (f (v)) (y - f (v))| dv
= F (y) - \frac{1}{w-u} \int_{\gamma} (F \circ f) (v) dv
- y \frac{1}{w-u} \int_{\gamma} (F' \circ f) (v) dv + \frac{1}{w-u} \int_{\gamma} (F' \circ f) (v) f (v) dv
\]
and by (2.11) and (2.12) we get the following inequality of interest
\[
\left| F (y) - \frac{1}{w-u} \int_{\gamma} (F \circ f) (v) dv
- y \frac{1}{w-u} \int_{\gamma} (F' \circ f) (v) dv + \frac{1}{w-u} \int_{\gamma} (F' \circ f) (v) f (v) dv \right|
\leq \frac{1}{2} K \frac{1}{|w-u|} \int_{\gamma} |y - f (z)|^2 |dz|
\]
for \( y \in G \).
If we take in (2.13)

\[ y = \frac{\int_{\gamma} (F' \circ f) (v) f(v) \, dv}{\int_{\gamma} (F' \circ f) (v) \, dv} \in G, \]

then we get the desired result (2.9). \qed

**Corollary 3.** With the assumptions of Corollary 2 and Theorem 6 we have

\begin{equation}
(2.14) \quad \left| \frac{F(w) w - F(u) u - \int_{\gamma} F(v) \, dv}{F(w) - F(u)} \right| \leq \frac{1}{2} \|F'' \|_{G, \infty} \frac{1}{|w - u|} \int_{\gamma} \left| \frac{\int_{\gamma} (F' \circ f) (v) f(v) \, dv}{\int_{\gamma} (F' \circ f) (v) \, dv} - f(z) \right|^2 |dz|.
\end{equation}

We have by the integration by parts formula (1.11) that

\[ \int_{\gamma} F' (v) \, dv = F(w) w - F(u) u - \int_{\gamma} F(v) \, dv \]

and

\[ \int_{\gamma} F' (v) \, dv = F(w) - F(u). \]

Therefore we can state the following result as well:

**Remark 2.** Let \( G \subset \mathbb{C} \) be a convex domain of complex numbers and that \( F \) is holomorphic on \( G \) with \( \|F'' \|_{G, \infty} := \sup_{z \in G} |F''(z)| < \infty \). Assume also that \( \gamma \subset D \) parametrized by \( z(t), t \in [a, b] \) is a piecewise smooth path from \( z(a) = u \) to \( z(b) = w \) with \( w \neq u, F(w) \neq F(u) \) and

\begin{equation}
(2.15) \quad \frac{F(w) w - F(u) u - \int_{\gamma} F(v) \, dv}{F(w) - F(u)} \in G,
\end{equation}

then by (2.14) we get

\begin{equation}
(2.16) \quad \left| \frac{F(w) w - F(u) u - \int_{\gamma} F(v) \, dv}{F(w) - F(u)} - \frac{1}{w - u} \int_{\gamma} F(z) \, dz \right| \leq \frac{1}{2} \|F'' \|_{G, \infty} \frac{1}{|w - u|} \int_{\gamma} \left| \frac{F(w) w - F(u) u - \int_{\gamma} F(v) \, dv}{F(w) - F(u)} - z \right|^2 |dz|.
\end{equation}

**3. Some Examples**

If we consider the function \( F(z) = \exp z, z \in \mathbb{C} \) and \( \gamma \subset \mathbb{C} \) parametrized by \( z(t), t \in [a, b] \) is a piecewise smooth path from \( z(a) = u \) to \( z(b) = w \) with \( w \neq u \), then by
we have for continuous function \( f : \gamma \to \mathbb{C} \)

\[
\begin{align*}
(3.1) \quad & \left| \frac{1}{w-u} \int_{\gamma} (\exp \circ f) (v) \, dv - \exp \left( \frac{1}{w-u} \int_{\gamma} f(z) \, dz \right) \right| \\
& \leq \frac{1}{2} \| \exp \|_{G, \infty} \frac{1}{|w-u|} \int_{\gamma} \left| f(v) - \frac{1}{w-u} \int_{\gamma} f(z) \, dz \right|^2 \, |dv|,
\end{align*}
\]

while from (2.6) we obtain

\[
(3.2) \quad \left| \frac{\exp w - \exp u}{w-u} - \exp \left( \frac{w+u}{2} \right) \right| \\
\leq \frac{1}{2} \| \exp \|_{G, \infty} \frac{1}{|w-u|} \int_{\gamma} \left| v - \frac{w+u}{2} \right|^2 \, |dv|.
\]

From (2.14) we get

\[
(3.3) \quad \left| \exp \left( \frac{1}{f_{\gamma} \left( \exp \circ f \right) (v) (f(v) \, dv)} \int_{\gamma} f_{\gamma} (\exp \circ f) (z) \, dz \right) \right| \\
\leq \frac{1}{2} \| \exp \|_{G, \infty} \frac{1}{|w-u|} \int_{\gamma} \left| f_{\gamma} (\exp \circ f) (v) f(v) \, dv \right| - f(z)^2 \, |dz|,
\]

while from (2.15) we get

\[
(3.4) \quad \left| \exp \left( \frac{(w-1) \exp w - (u-1) \exp u}{\exp w - \exp u} \right) \right| - \frac{\exp w - \exp u}{w-u} \\
\leq \frac{1}{2} \| \exp \|_{G, \infty} \frac{1}{|w-u|} \int_{\gamma} \left| (w-1) \exp w - (u-1) \exp u \right| - z^2 \, |dz|.
\]

Consider the function \( F(z) = \log(z) \) where \( \log(z) = \ln|z| + i \text{Arg}(z) \) and \( \text{Arg}(z) \) is such that \(-\pi < \text{Arg}(z) \leq \pi\). \( \log \) is called the "principal branch" of the complex logarithmic function. \( F \) is analytic on all of \( \mathbb{L} := \mathbb{C} \setminus \{x + iy : x \leq 0, y = 0\} \) and \( F'(z) = \frac{1}{z} \) on this set.

If we consider \( g : D \to \mathbb{C}, g(z) = \frac{1}{z} \) where \( D \subset \mathbb{L} \), then \( F \) is a primitive of \( g \) on \( D \) and if \( \gamma \subset D \) parametrized by \( z(t), t \in [a,b] \) is a piecewise smooth path from \( z(a) = u \) to \( z(b) = w \), then

\[
\int_{\gamma} \frac{dz}{z} = \log(w) - \log(u).
\]

Also, the function \( G : \mathbb{L} \to \mathbb{C}, G(z) = z \log(z) - z \) is analytic on \( \mathbb{L} \) and \( G'(z) = \log(z), z \in \mathbb{L} \).

Assume also that \( f : D \to \mathbb{L} \) is continuous on \( D \), \( \gamma \subset D \) parametrized by \( z(t), t \in [a,b] \) is a piecewise smooth path from \( z(a) = u \) to \( z(b) = w \) with \( w \neq u \) and
\[
\frac{1}{w-u} \int_{\gamma} f(z) \, dz \in \mathbb{L}, \text{ then from (2.1) for } F(z) = \log z, \text{ we get }
\]

(3.5) \[\left| \frac{1}{w-u} \int_{\gamma} (\log f)(v) \, dv - \log \left( \frac{1}{w-u} \int_{\gamma} f(z) \, dz \right) \right| \leq \frac{1}{2} d_{\gamma}^2 \left| w-u \right| \int_{\gamma} \left| f(v) - \frac{1}{w-u} \int_{\gamma} f(z) \, dz \right|^2 \, |dv|,
\]

where \(d_{\gamma} := \inf_{z \in \gamma} |z|\) is assumed to be positive and finite.

For \(\gamma \subset \mathbb{L}\) and \(f(z) = z\), we get from (3.5) that

(3.6) \[\left| \frac{w \log (w) - u \log (u)}{w - u} - \log \left( \frac{w + u}{2} \right) - 1 \right| \leq \frac{1}{2} d_{\gamma}^2 \left| w-u \right| \int_{\gamma} \left| v - \frac{w + u}{2} \right|^2 \, |dv|,
\]

where \(d_{\gamma} := \inf_{z \in \gamma} |z|\) is assumed to be positive and finite.

Further, for \(F(z) = \log z\) we have

\[
\frac{w \log w - u \log u - \int_{\gamma} \log z \, dz}{\log w - \log u} = \frac{w \log w - u \log u - w \log (w) + w + u \log (u) - u}{\log w - \log u} = \frac{w - u}{\log w - \log u}.
\]

So, if \(\log w \neq \log u\) and \(\frac{w - u}{\log w - \log u} \in \mathbb{L}\),

then by (2.16) we get

(3.7) \[\left| \log \left( \frac{w - u}{\log w - \log u} \right) - \frac{w \log (w) - u \log (u)}{w - u} + 1 \right| \leq \frac{1}{2} d_{\gamma}^2 \left| w-u \right| \int_{\gamma} \left| \frac{w - u}{\log w - \log u} - z \right|^2 \, |dz|.
\]

Assume also that \(f: D \to \mathbb{L}\) is continuous on \(D\), \(\gamma \subset D\) parametrized by \(z(t)\), \(t \in [a,b]\) is a piecewise smooth path from \(z(a) = u\) to \(z(b) = w\) with \(w \neq u\) and \(\frac{1}{w-u} \int_{\gamma} f(z) \, dz \in \mathbb{L}\), then from (2.1) for \(F(z) = z^{-1}\), we get

(3.8) \[\left| \frac{1}{w-u} \int_{\gamma} [f(v)]^{-1} \, dv - \left( \frac{1}{w-u} \int_{\gamma} f(z) \, dz \right)^{-1} \right| \leq \frac{1}{d_{\gamma}^2 |w-u|} \int_{\gamma} \left| f(v) - \frac{1}{w-u} \int_{\gamma} f(z) \, dz \right|^2 \, |dv|,
\]

where \(d_{\gamma} := \inf_{z \in \gamma} |z|\) is assumed to be positive and finite.
For $\gamma \subset \mathbb{L}$ and $f(z) = z$, we get from (3.8) that

$$\left| \frac{\log(w) - \log(u)}{w - u} - \left( \frac{w + u}{2} \right)^{-1} \right| \leq \frac{1}{d_3^3} \int_\gamma \left| v - \frac{w + u}{2} \right|^2 |dv|.$$

Further, for $F(z) = z^{-1}$ we have

$$\frac{F(w) - F(u)}{w - u} = \frac{- \log(w) + \log(u)}{w - u} = \frac{\log(w) - \log(u)}{wu}$$

for $w \neq u$ and $u, w \in \mathbb{L}$.

If $w \neq u$ and $u, w \in \mathbb{L}$ with

$$- \log(w) + \log(u) \in \mathbb{L},$$

then by (2.16) we get

$$\left| \frac{(\log(w) - \log(u))}{w - u} \right|^{-1} \leq \frac{1}{d_3^3} \int_\gamma \left| \log(w) - \log(u) \right|^2 |dz|.$$

References


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