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Evaluation of Gaussian integrals by Adomian decomposition

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Abstract. In this paper, we present a new adaptation of the Adomian decomposition method, which permits evaluation of the Gaussian integrals as a convenient convergent series.

1. Introduction

The Adomian decomposition method (ADM) [**1**]-[**6**] can be an effective procedure for the evaluation of certain difficult integrals such as the Gaussian integrals. To tackle this problem, we begin by considering the general heat equation for an infinite rod with an arbitrary initial condition:

$$\left\{\begin{array}{rcl} \frac{\partial u}{\partial t} & = & k\frac{\partial^2 u}{\partial x^2},\ -\infty < x < \infty,\ k > 0,\ t > 0, \\ u(x,0) & = & f(x), \\ \mid u(x,t) \mid & \leqslant & M \text{ for all } x \in \mathbb{R},\ t > 0. \end{array}\right. \tag{1.1}$$

The explicit representation of the solution to this initial-value problem is given by

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} f(\xi) e^{-\frac{(\xi-x)^2}{4kt}} d\xi,\ -\infty < x < \infty,\ t > 0. \tag{1.2}$$

The function $G(x,t) = \frac{1}{\sqrt{4k\pi t}} e^{-\frac{x^2}{4kt}}$ in the integrand is called the fundamental solution of the heat equation. For a proof, we can refer to any book on partial differential equations, e.g., (pp. 45-48 [**7**]).

It may happen in this case that the integral $\int_{-\infty}^{\infty} f(\xi) e^{-\frac{(\xi-x)^2}{4kt}} d\xi$ in the RHS of (1.2) is not expressible in terms of elementary functions nor adequately tabulated. In this note, we produce evaluations of this type of integrals involving the error function using the Adomian decomposition method (ADM) [**1**]-[**6**].

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2. Analysis

Let us consider this example using the Adomian decomposition method (ADM) [**1**]-[**6**]. First, we define the linear operator L, and the linear remainder operator R as

$$Lu = \frac{\partial u}{\partial t} \text{ and } Ru = \frac{\partial^2 u}{\partial x^2}. \tag{2.1}$$

We rewrite the original equation (1.1) in Adomian's operator-theoretic notation as

$$Lu = kRu. \tag{2.2}$$

Define the inverse operator as

$$L^{-1}(u) = \int_0^t \frac{\partial u}{\partial t} dt. \tag{2.3}$$

Thus,

$$L^{-1}Lu(x,t) = u(x,t) - u(x,0), \tag{2.4}$$

where $u(x,0) = f(x)$. Upon substitution, we have

$$u(x,t) = f(x) + kL^{-1}Ru(x). \tag{2.5}$$

In the classic Adomian decomposition method, we decompose the solution into the solution components u_n to be determined by recursion as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{2.6}$$

Upon substitution of Eq. (2.6) into Eq. (2.5), we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + k\int_0^t \sum_{n=0}^{\infty} \frac{\partial^2 u_n}{\partial x^2}(x,t)dt. \tag{2.7}$$

Then we establish an appropriate Adomian recursion scheme as

$$\left\{ \begin{array}{rcl} u_0(x,t) & = & f(x), \\ u_{n+1}(x,t) & = & k\int_0^t \frac{\partial^2 u_n}{\partial x^2}(x,t)dt, \ n \geqslant 0. \end{array} \right. \tag{2.8}$$

In view of (2.8), the components $u_0(x,t), \ u_1(x,t), \ u_2(x,t), ...$ are immediately determined as

$$\left\{ \begin{array}{rcl} u_0(x,t) & = & f(x), \\ u_1(x,t) & = & kf^{(2)}(x)t, \\ u_2(x,t) & = & k^2 f^{(4)}(x)\frac{t^2}{2!}, \\ u_3(x,t) & = & k^3 f^{(6)}(x)\frac{t^3}{3!}, \\ & \cdots & . \end{array} \right. \tag{2.9}$$

Consequently, the solution is given as

$$u(x,t) = \sum_{n=0}^{\infty} k^n f^{(2n)}(x)\frac{t^n}{n!}. \tag{2.10}$$

Since the Global Maximum Principle can be used to give a proof of uniqueness for this problem, the resulting formulas must agree with this. Thus

THEOREM 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $2n-$times differentiable in $\mathbb{R}$. Then

$$\int_{-\infty}^{\infty} f(\xi) e^{-\frac{(\xi-x)^2}{4kt}} d\xi = \sqrt{4k\pi t} \sum_{n=0}^{\infty} f^{(2n)}(x) \frac{(kt)^n}{n!}, \tag{2.11}$$

where $-\infty < x < \infty,\ t > 0,\ k > 0.$

PROOF. This follows simply from (1.2) and (2.10). □

COROLLARY 2.

$$\int_{-\infty}^{\infty} f(x) e^{-bx^2} dx = \sqrt{\frac{\pi}{b}} \sum_{n=0}^{\infty} f^{(2n)}(0) \frac{(\frac{1}{4b})^n}{n!},\ b > 0. \tag{2.12}$$

This is a new helpful tool in calculating the Gaussian integrals as a convenient convergent series.

3. Examples

In order to verify the accuracy of our present method, we present some elementary examples. The reader will find a more advanced approach to the evaluation of many related integrals in [**8**].

Example 3.1.

$$\int_{0}^{\infty} e^{-bx^2} dx = \sqrt{\frac{\pi}{4b}},\ b > 0. \tag{3.1}$$

This comes from letting $f(x) = 1$ in (2.12).

Example 3.2.

$$\int_{-\infty}^{\infty} e^{-bx^2} \cos(rx) dx = \sqrt{\frac{\pi}{b}} e^{-\frac{r^2}{4b}}, \quad \text{for all real numbers } r. \tag{3.2}$$

This follows simply by letting $f(x) = \cos(rx)$ and $f^{(2n)}(0) = (-1)^n r^{2n}, n \geqslant 0$ in (2.12).

Example 3.3.

$$\int_{-\infty}^{\infty} x^4 e^{-bx^2} dx = \frac{3}{4b^2}. \tag{3.3}$$

This follows simply by letting $f(x) = x^4$ and

$$f^{(2n)}(0) = \begin{cases} 4!,\ n = 2 \\ 0, otherwise \end{cases} \tag{3.4}$$

in (2.12).

Example 3.4.

$$\int_{-\infty}^{\infty} \left(e^x + e^{-x}\right) e^{-bx^2} dx = 2\sqrt{\frac{\pi}{b}} e^{\frac{1}{4b}}. \tag{3.5}$$

This follows simply by letting $f(x) = e^x + e^{-x}$ and $f^{(2n)}(0) = 2, n \geqslant 0$ in (2.12).

Example 3.5.

$$\int_{-\infty}^{\infty} e^{-p^2\xi^2-q\xi}d\xi = \sqrt{\frac{\pi}{p}}e^{\frac{q^2}{4p^2}}. \tag{3.6}$$

The evaluation of this integral follows directly from completing the square $-p^2\xi^2-q\xi = -p^2(\xi-\frac{q}{2p^2})^2+\frac{q^2}{4p^2}$ *and letting* $x=\frac{q}{2p^2}$, $p^2=\frac{1}{4kt}$ *and* $f(\xi)=1$ *in (2.11) .*

Example 3.6. *The integral*

$$\int_{-\infty}^{\infty} \xi e^{-p\xi^2-2q\xi}d\xi = \frac{q}{p}\sqrt{\frac{\pi}{p}}e^{\frac{q^2}{p}} \tag{3.7}$$

is evaluated from completing the square in the exponent and letting $f(\xi)=\xi$, $x=\frac{q}{p}$ *and* $p=\frac{1}{4kt}$ *in (2.11).*

Example 3.7. *The Mordell integral (1920) is defined as* [**9, 10**]

$$M(a,b,c,d) = \int_{-\infty}^{\infty} \frac{e^{a\xi^2+b\xi}}{e^{c\xi}+d}d\xi, \ Re(a)<0. \tag{3.8}$$

This type of integrals for particular values of a,b,c,d *was evaluated by Ramanujan* [**11**]. *Mordell (1933) classified these according to the values of the parameters and evaluated them in terms of Jacobi's theta and other related functions, using the method of complex contour integration. It can easily be evaluated by our Theorem 1. Indeed, completing the square* $a\xi^2+b\xi = a(\xi+\frac{b}{2a})^2-\frac{b^2}{4a}$, *and taking* $f(x)=\frac{1}{e^{cx}+d}$.
If we choose $b=0$, *that is* $x=0$ *and* $d=1$, *then* $f(0)=\frac{1}{2}$ *and* $f^{(2n)}(0)=0,\ n\geqslant 1$. *Letting these and* $a=-\frac{1}{4kt}$ *in (2.12), we readily obtain*

$$M(a,0,c,1) = \int_{-\infty}^{\infty} \frac{e^{a\xi^2}}{e^{c\xi}+1}d\xi = \frac{1}{2}\sqrt{\frac{-\pi}{a}}. \tag{3.9}$$

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