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On Some Properties and Inequalities for the Nielsen's β-Function

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Abstract. In this study, we obtain some convexity, monotonicity and additivity properties as well as some inequalities involving the Nielsen's β-function which was introduced in 1906.

1. Introduction and Preliminaries

The Nielsen's β-function, $\beta(x)$ which was introduced in [**9**] is defined as

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t}\,dt, \quad x > 0 \tag{1.1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+x}, \quad x > 0 \tag{1.2}$$

and by change of variables, the representation (1.1) can be written as

$$\beta(x) = \int_0^\infty \frac{e^{-xt}}{1+e^{-t}}\,dt, \quad x > 0. \tag{1.3}$$

The function $\beta(x)$ is also defined as [**9**]

$$\beta(x) = \frac{1}{2}\left\{\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right\} \tag{1.4}$$

where $\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function and $\Gamma(x)$ is the Euler's Gamma function. See also [**1**], [**3**], [**5**] and [**7**].

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It is known that function $\beta(x)$ satisfies the following properties [**1**],[**9**].

$$\beta(x+1) = \frac{1}{x} - \beta(x), \tag{1.5}$$

$$\beta(x) + \beta(1-x) = \frac{\pi}{\sin \pi x}. \tag{1.6}$$

In particular, $\beta(1) = \ln 2$, $\beta\left(\frac{1}{2}\right) = \frac{\pi}{2}$, $\beta\left(\frac{3}{2}\right) = 2 - \frac{\pi}{2}$ and $\beta(2) = 1 - \ln 2$.

PROPOSITION 1.1. The function $\beta(x)$ is related to the classical Euler's beta function, $B(x,y)$ in the following ways.

$$\beta(x) = -\frac{d}{dx}\left\{\ln B\left(\frac{x}{2}, \frac{1}{2}\right)\right\}, \tag{1.7}$$

$$\beta(x) + \beta(1-x) = B(x, 1-x). \tag{1.8}$$

PROOF. By the Euler's beta function $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, we obtain

$$B\left(\frac{x}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{x+1}{2}\right)}. \tag{1.9}$$

Then by taking the logarithmic derivative of (1.9) and using (1.4), we obtain

$$\begin{aligned}\frac{d}{dx}\left\{\ln B\left(\frac{x}{2}, \frac{1}{2}\right)\right\} = \frac{1}{2}\frac{B'\left(\frac{x}{2}, \frac{1}{2}\right)}{B\left(\frac{x}{2}, \frac{1}{2}\right)} &= \frac{1}{2}\left\{\frac{\Gamma'\left(\frac{x}{2}\right)}{\Gamma\left(\frac{x}{2}\right)} - \frac{\Gamma'\left(\frac{x+1}{2}\right)}{\Gamma\left(\frac{x+1}{2}\right)}\right\} \\ &= \frac{1}{2}\left\{\psi\left(\frac{x}{2}\right) - \psi\left(\frac{x+1}{2}\right)\right\} \\ &= -\beta(x)\end{aligned}$$

yielding the result (1.7). The result (1.8) follows easily from the relation (1.6).

REMARK 1.2. The function $\beta(x)$ is referred to as the *incomplete beta function* in [**1**] and [**7**]. However, this should not be confused with the incomplete beta function which is usually defined as

$$B(a; x, y) = \int_0^a t^{x-1}(1-t)^{y-1}\, dt, \quad x > 0, y > 0$$

or the regularized incomplete beta function which is defined as

$$I_a(x,y) = \frac{B(a;x,y)}{B(x,y)}, \quad x > 0, y > 0.$$

Also, the function should not be confused with Dirichlet's beta function which is defined as [**4**]

$$\beta^*(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^x} = \frac{1}{\Gamma(x)}\int_0^\infty \frac{t^{x-1}}{e^t + e^{-t}}\, dt, \quad x > 0.$$

We shall use the notations $\mathbb{N} = \{1, 2, 3, \dots,\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ in the rest of the paper.

By differentiating m times of (1.1), (1.2) and (1.3), we obtain

$$\beta^{(m)}(x) = \int_0^1 \frac{(\ln t)^m t^{x-1}}{1+t}\, dt, \quad x > 0 \tag{1.10}$$

$$= (-1)^m m! \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+x)^{m+1}}, \quad x > 0 \tag{1.11}$$

$$= (-1)^m \int_0^{\infty} \frac{t^m e^{-xt}}{1+e^{-t}}\, dt, \quad x > 0 \tag{1.12}$$

for $m \in \mathbb{N}_0$. It is clear that $\beta^{(0)}(x) = \beta(x)$. In particular, we have

$$\beta^{(m)}(1) = (-1)^m m! \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^{m+1}} = (-1)^m m! \eta(m+1), \quad m \in \mathbb{N}_0 \tag{1.13}$$

$$= (-1)^m m! \left(1 - \frac{1}{2^m}\right) \zeta(m+1), \quad m \in \mathbb{N} \tag{1.14}$$

where $\eta(x)$ is the Dirichlet's eta function and $\zeta(x)$ is the Riemann zeta function defined as

$$\eta(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^x},\ x > 0 \quad \text{and} \quad \zeta(x) = \sum_{k=1}^{\infty} \frac{1}{k^x},\ x > 1.$$

Then by differentiating m times of (1.4) and (1.5), we obtain respectively

$$\beta^{(m)}(x+1) = \frac{(-1)^m m!}{x^{m+1}} - \beta^{(m)}(x) \tag{1.15}$$

and

$$\beta^{(m)}(x) = \frac{1}{2^{m+1}} \left\{ \psi^{(m)}\left(\frac{x+1}{2}\right) - \psi^{(m)}\left(\frac{x}{2}\right) \right\}. \tag{1.16}$$

For rational arguments $x = \frac{p}{q}$, the function $\psi^{(m)}(x)$ takes the form

$$\psi^{(m)}\left(\frac{p}{q}\right) = (-1)^{m+1} m! q^{m+1} \sum_{k=0}^{\infty} \frac{1}{(qk+p)^{m+1}}, \quad m \geqslant 1 \tag{1.17}$$

which implies

$$\psi^{(m)}\left(\frac{3}{4}\right) - \psi^{(m)}\left(\frac{1}{4}\right) = (-1)^{m+1} m! 4^{m+1} \left\{ \sum_{k=0}^{\infty} \frac{1}{(4k+3)^{m+1}} - \sum_{k=0}^{\infty} \frac{1}{(4k+1)^{m+1}} \right\}. \tag{1.18}$$

Let $x = \frac{1}{2}$ in (1.16). Then we obtain

$$\beta^{(m)}\left(\frac{1}{2}\right) = \frac{1}{2^{m+1}} \left\{ \psi^{(m)}\left(\frac{3}{4}\right) - \psi^{(m)}\left(\frac{1}{4}\right) \right\} \tag{1.19}$$

which by (1.18) can be written as

$$\beta^{(m)}\left(\frac{1}{2}\right) = (-1)^{m+1} m! 2^{m+1} \left\{ \sum_{k=0}^{\infty} \frac{1}{(4k+3)^{m+1}} - \sum_{k=0}^{\infty} \frac{1}{(4k+1)^{m+1}} \right\}. \tag{1.20}$$

Now let $m = 1$ in (1.20). Then we obtain

$$\beta'\left(\frac{1}{2}\right) = 4 \left\{ \sum_{k=0}^{\infty} \frac{1}{(4k+3)^2} - \sum_{k=0}^{\infty} \frac{1}{(4k+1)^2} \right\} = -4G \tag{1.21}$$

where $G = 0.915965594177...$ is the Catalan's constant.

Remark 1.3. The Catalan's constant has several interesting representations [**2**], and amongst them are:

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2},$$

$$G = -\frac{\pi^2}{8} + 2 \sum_{k=0}^{\infty} \frac{1}{(4k+1)^2}, \tag{1.22}$$

$$G = \frac{\pi^2}{8} - 2 \sum_{k=0}^{\infty} \frac{1}{(4k+3)^2}. \tag{1.23}$$

Thus, (1.21) is a consequence (1.22) and (1.23) .

Equivalently, by letting $m = 1$ in (1.19) we obtain

$$\beta'\left(\frac{1}{2}\right) = \frac{1}{4} \left\{ \psi'\left(\frac{3}{4}\right) - \psi'\left(\frac{1}{4}\right) \right\} = -4G$$

since $\psi'\left(\frac{1}{4}\right) = \pi^2 + 8G$ and $\psi'\left(\frac{3}{4}\right) = \pi^2 - 8G$. See [**1**] and [**6**].
By using (1.13), (1.14), (1.15) and (1.21), we derive the following special values.

$$\begin{aligned}
\beta'(1) &= -\frac{1}{2}\zeta(2) = -\frac{\pi^2}{12}, \\
\beta'(2) &= -1 + \frac{\pi^2}{12}, \\
\beta'(3) &= \frac{3}{4} - \frac{\pi^2}{12}, \\
\beta'\left(\frac{3}{2}\right) &= 4(G-1), \\
\beta'\left(\frac{5}{2}\right) &= \frac{40}{9} - 4G.
\end{aligned}$$

More special values may be derived by using similar procedures. As shown in [**1**] and [**5**], the Nielsen's β-function is very useful in evaluating certain integrals.

2. Main Results

To start with, we recall the following well-known definitions.

Definition 2.1. A function $f : (0, \infty) \to \mathbb{R}$ is said to be completely monotonic if f has derivatives of all order and

$$(-1)^k f^{(k)}(x) \geqslant 0 \quad \text{for} \quad x \in (0, \infty), \quad k \in \mathbb{N}_0.$$

Definition 2.2. A function $f : I \to \mathbb{R}^+$ is said to be logarithmically convex if

$$\log f(ux + vy) \leqslant u \log f(x) + v \log f(y)$$

or equivalently

$$f(ux + vy) \leqslant (f(x))^u (f(y))^v$$

for each $x, y \in I$ and $u, v > 0$ such that $u + v = 1$.

Lemma 2.3. For $x > 0$, the following statements hold .

(1) $\beta(x)$ is decreasing.
(2) $\beta^{(m)}(x)$ is positive and decreasing if m is even.
(3) $\beta^{(m)}(x)$ is negative and increasing if m is odd.
(4) $\left|\beta^{(m)}(x)\right|$ is decreasing for all $m \in \mathbb{N}$.

Proof. These follow easily from (1.3) and (1.12).

Remark 2.4. Furthermore, it follows from (1.12) that

$$(-1)^k \beta^{(k)}(x) = (-1)^{2k} \int_0^\infty \frac{t^k e^{-xt}}{1 + e^{-t}} \, dt \geqslant 0,$$

for $x > 0$ and $k \in \mathbb{N}_0$. Thus, the function $\beta(x)$ is completely monotonic. More generally, $\beta^{(m)}(x)$ is completely monotonic if m is even and $-\beta^{(m)}(x)$ is completely monotonic if m is odd. To see this, observe that for $x > 0$ and $k, m \in \mathbb{N}_0$, we obtain

$$(-1)^k \beta^{(m+k)}(x) = (-1)^{m+2k} \int_0^\infty \frac{t^{m+k} e^{-xt}}{1 + e^{-t}} \, dt \geqslant (\leqslant) 0$$

respectively for even(odd) m.

Remark 2.5. Since $f(x) = -\beta'(x)$ is convex, then by the classical Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a} \int_a^b f(x) \, dx \leqslant \frac{f(a) + f(b)}{2},$$

for a convex function $f : [a, b] \to \mathbb{R}$, we obtain the inequality

$$\frac{\beta'(a) + \beta'(b)}{2} \leqslant \frac{\beta(b) - \beta(a)}{b - a} \leqslant \beta'\left(\frac{a+b}{2}\right), \tag{2.1}$$

where $a, b > 0$. Alternatively, since $\beta'(x)$ is continuous and concave (i.e. $\beta'''(x) < 0$) on $(0, \infty)$, then by Theorem 1 of [**8**], we obtain the result (2.1).

THEOREM 2.6. Let $m, n \in \mathbb{N}_0$, $a > 1$, $\frac{1}{a} + \frac{1}{b} = 1$ such that $\frac{m}{a} + \frac{n}{b} \in \mathbb{N}_0$. Then, the inequality

$$\left|\beta^{\left(\frac{m}{a}+\frac{n}{b}\right)}\left(\frac{x}{a}+\frac{y}{b}\right)\right| \leqslant \left|\beta^{(m)}(x)\right|^{\frac{1}{a}} \left|\beta^{(n)}(y)\right|^{\frac{1}{b}} \tag{2.2}$$

holds for $x, y > 0$.

PROOF. By the relation (1.12) and the Hölder's inequality, we obtain

$$\begin{aligned}\left|\beta^{\left(\frac{m}{a}+\frac{n}{b}\right)}\left(\frac{x}{a}+\frac{y}{b}\right)\right| &= \int_0^\infty \frac{t^{\left(\frac{m}{a}+\frac{n}{b}\right)} e^{-\left(\frac{x}{a}+\frac{y}{b}\right)t}}{1+e^{-t}}\,dt \\ &= \int_0^\infty \frac{t^{\frac{m}{a}} e^{-\frac{xt}{a}}}{(1+e^{-t})^{\frac{1}{a}}} \cdot \frac{t^{\frac{n}{b}} e^{-\frac{yt}{b}}}{(1+e^{-t})^{\frac{1}{b}}}\,dt \\ &\leqslant \left(\int_0^\infty \frac{t^m e^{-xt}}{1+e^{-t}}\,dt\right)^{\frac{1}{a}} \left(\int_0^\infty \frac{t^n e^{-yt}}{1+e^{-t}}\,dt\right)^{\frac{1}{b}} \\ &= \left|\beta^{(m)}(x)\right|^{\frac{1}{a}} \left|\beta^{(n)}(y)\right|^{\frac{1}{b}}\end{aligned}$$

which completes the proof.

REMARK 2.7. Note that the absolute signs in (2.2) are not required if m and n are even.

REMARK 2.8. If $m = n$ is even in Theorem 2.6, then the inequality (2.2) becomes

$$\beta^{(m)}\left(\frac{x}{a}+\frac{y}{b}\right) \leqslant \left(\beta^{(m)}(x)\right)^{\frac{1}{a}} \left(\beta^{(m)}(y)\right)^{\frac{1}{b}} \tag{2.3}$$

which implies that the function $\beta^{(m)}(x)$ is logarithmically convex for even m. Moreover, if $m = 0$ in (2.3), then we obtain

$$\beta\left(\frac{x}{a}+\frac{y}{b}\right) \leqslant (\beta(x))^{\frac{1}{a}} (\beta(y))^{\frac{1}{b}} \tag{2.4}$$

implies that $\beta(x)$ is logarithmically convex.

REMARK 2.9. Let $a = b = 2$, $x = y$ and $m = n + 2$ in Theorem 2.6. Then we obain the Turan-type inequality

$$\left|\beta^{(n+1)}(x)\right|^2 \leqslant \left|\beta^{(n+2)}(x)\right| \left|\beta^{(n)}(x)\right|. \tag{2.5}$$

Furthermore, if $n = 0$ in (2.5) then we get

$$(\beta'(x))^2 \leqslant \beta''(x)\beta(x). \tag{2.6}$$

THEOREM 2.10. Let $m \in \mathbb{N}_0$ be even. Then the function

$$Q(x) = e^{ax}\beta^{(m)}(x) \tag{2.7}$$

is convex for $x > 0$ and any real number a.

PROOF. Let m be even and a be any real number. Then for $x > 0$,

$$\begin{aligned} Q'(x) &= ae^{ax}\beta^{(m)}(x) + e^{ax}\beta^{(m+1)}(x), \\ Q''(x) &= a^2e^{ax}\beta^{(m)}(x) + 2ae^{ax}\beta^{(m+1)}(x) + e^{ax}\beta^{(m+2)}(x) \\ &= e^{ax}\left[a^2\beta^{(m)}(x) + 2a\beta^{(m+1)}(x) + \beta^{(m+2)}(x)\right]. \end{aligned}$$

The quadratic function $f(a) = a^2\beta^{(m)}(x) + 2a\beta^{(m+1)}(x) + \beta^{(m+2)}(x)$ has a discriminant $\Delta = 4\left[\left(\beta^{(m+1)}(x)\right)^2 - \beta^{(m)}(x)\beta^{(m+2)}(x)\right] \leqslant 0$ as a result of (2.5). Then, since $\beta^{(m)}(x) > 0$, it follows that $f(a) \geqslant 0$. Thus, $Q''(x) \geqslant 0$ and this completes the proof.

THEOREM 2.11. Let $m \in \mathbb{N}_0$ be even. Then the function

$$P(x) = \left[\beta^{(m)}(x)\right]^{\alpha} \tag{2.8}$$

is convex for $x > 0$ and $\alpha > 0$.

PROOF. Let m be even, $x > 0$ and $\alpha > 0$. Then

$$\ln P(x) = \alpha \ln \beta^{(m)}(x) \quad \text{implies} \quad \frac{P'(x)}{P(x)} = \alpha\frac{\beta^{(m+1)}(x)}{\beta^{(m)}(x)}.$$

That is,

$$P'(x) = \alpha P(x)\frac{\beta^{(m+1)}(x)}{\beta^{(m)}(x)}$$

and then

$$\begin{aligned} P''(x) &= P(x)\left\{\left(\frac{P'(x)}{P(x)}\right)^2 + \alpha\left[\frac{\beta^{(m+2)}(x)\beta^{(m)}(x) - (\beta^{(m+1)}(x))^2}{[\beta^{(m)}(x)]^2}\right]\right\} \\ &\geqslant 0 \end{aligned}$$

as a result of (2.5).

THEOREM 2.12. Let $m \in \mathbb{N}_0$ be even. Then the function

$$U(x) = \frac{\beta^{(m)}(kx)}{\left[\beta^{(m)}(x)\right]^k} \tag{2.9}$$

is increasing if $k > 1$ and decreasing if $0 < k \leqslant 1$.

PROOF. For $x > 0$ and m even, define a function S by

$$S(x) = \frac{\beta^{(m+1)}(x)}{\beta^{(m)}(x)}.$$

Then direct differentiation yields

$$S'(x) = \frac{\beta^{(m+2)}(x)\beta^{(m)}(x) - \left(\beta^{(m+1)}(x)\right)^2}{\left[\beta^{(m)}(x)\right]^2}$$

and by (2.5), we conclude that $S'(x) \geqslant 0$. Hence $S(x)$ is increasing. Next, let $u(x) = \ln U(x)$. Then we obtain

$$u'(x) = k\left[\frac{\beta^{(m+1)}(kx)}{\beta^{(m)}(kx)} - \frac{\beta^{(m+1)}(x)}{\beta^{(m)}(x)}\right].$$

Since $S(x)$ is increasing, it follows that $u'(x) > 0$ if $k > 1$ and $u'(x) \leqslant 0$ if $0 < k \leqslant 1$. This completes the proof.

Corollary 2.13. Let $m \in \mathbb{N}_0$ be even and $0 < x \leqslant y$. Then the inequality

$$\left(\frac{\beta^{(m)}(y)}{\beta^{(m)}(x)}\right)^k \leqslant \frac{\beta^{(m)}(ky)}{\beta^{(m)}(kx)} \tag{2.10}$$

is satisfied if $k > 1$. It reverses if $0 < k \leqslant 1$.

Proof. This follows from the monotonicity property of $U(x)$ as defined in (2.9).

Theorem 2.14. Let $m \in \mathbb{N}_0$ be even and $a > 0$. Then for $x > 0$, the function

$$\Omega(x) = \frac{\beta^{(m)}(a)}{\beta^{(m)}(x+a)}$$

is increasing and logarithmically concave, and the inequality

$$1 < \frac{\beta^{(m)}(a)}{\beta^{(m)}(x+a)} \tag{2.11}$$

is satisfied.

Proof. Define μ for $m \in \mathbb{N}_0$ even, $a > 0$ and $x > 0$ by

$$\mu(x) = \ln \Omega(x) = \ln \beta^{(m)}(a) - \ln \beta^{(m)}(x+a).$$

Then

$$\mu'(x) = -\frac{\beta^{(m+1)}(x+a)}{\beta^{(m)}(x+a)} > 0$$

which implies that $\mu(x)$ in increasing. Consequently, $\Omega(x) = e^{\mu(x)}$ is increasing. Next, we have

$$(\ln \Omega(x))'' = -\left[\frac{\beta^{(m+2)}(x+a)\beta^{(m)}(x+a) - (\beta^{(m+1)}(x+a))^2}{[\beta^{(m)}(x+a)]^2}\right] \leqslant 0$$

which implies that $\Omega(x)$ is logarithmically concave. Furthermore,

$$\lim_{x\to 0^+} \Omega(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} \Omega(x) = \infty.$$

Then since $\Omega(x)$ is increasing, we obtain the result (2.11).

Theorem 2.15. Let $m \in \mathbb{N}_0$. Then the following inequalities hold for $x, y > 0$.

$$\beta^{(m)}(x+y) \leqslant \beta^{(m)}(x) + \beta^{(m)}(y) \tag{2.12}$$

if m is even, and

$$\beta^{(m)}(x+y) \geqslant \beta^{(m)}(x) + \beta^{(m)}(y) \tag{2.13}$$

if m is odd.

PROOF. Let m be even and $H(x) = \beta^{(m)}(x+y) - \beta^{(m)}(x) - \beta^{(m)}(y)$. Then for a fixed y, we obtain

$$\begin{aligned} H'(x) &= \beta^{(m+1)}(x+y) - \beta^{(m+1)}(x) \\ &= (-1)^{(m+1)} \int_0^\infty \frac{t^m \left(e^{-(x+y)t} - e^{-xt}\right)}{1+e^{-t}}\, dt \\ &= -\int_0^\infty \frac{t^m e^{-xt}}{1+e^{-t}} \left(e^{-yt} - 1\right) dt \\ &\geqslant 0. \end{aligned}$$

Hence, $H(x)$ is increasing. Moreover,

$$\begin{aligned} \lim_{x\to\infty} H(x) &= \lim_{x\to\infty} \left[\beta^{(m)}(x+y) - \beta^{(m)}(x) - \beta^{(m)}(y)\right] \\ &= (-1)^m \lim_{x\to\infty} \left[\int_0^\infty \frac{t^m}{1+e^{-t}} \left(e^{-(x+y)t} - e^{-xt} - e^{-yt}\right) dt\right] \\ &= -\int_0^\infty \frac{t^m e^{-yt}}{1+e^{-t}}\, dt \\ &\leqslant 0. \end{aligned}$$

Therefore, $H(x) \leqslant 0$ which gives the result (2.12). Similarly, for m odd, we obtain $H'(x) \leqslant 0$ and $\lim_{x\to\infty} H(x) \geqslant 0$ which implies that $H(x) \geqslant 0$ and this gives the result (2.13).

REMARK 2.16. Theorem 2.15 is another way of saying that the function $\beta^{(m)}(x)$ is subadditive if m is even, and superadditive if m is odd.

THEOREM 2.17. Let $m \in \mathbb{N}_0$. Then for m odd, the function $\beta^{(m)}(x)$ is star-shaped on $(0,\infty)$. That is,

$$\beta^{(m)}(\alpha x) \leqslant \alpha \beta^{(m)}(x) \tag{2.14}$$

for all $x \in (0,\infty)$ and $\alpha \in (0,1]$.

PROOF. Let m be odd and $T(x) = \beta^{(m)}(\alpha x) - \alpha\beta^{(m)}(x)$. Then for $x \in (0,\infty)$ and $\alpha \in (0,1]$, we have

$$\begin{aligned} T'(x) &= \alpha \left[\beta^{(m+1)}(\alpha x) - \beta^{(m+1)}(x)\right] \\ &\geqslant 0. \end{aligned}$$

Thus, $T(x)$ is increasing. Recall that $\beta^{(n)}(x)$ is decreasing for even n. Then since $0 < \alpha x \leqslant x$, we have $\beta^{(m+1)}(\alpha x) \geqslant \beta^{(m+1)}(x)$. Furthermore,

$$\begin{aligned} \lim_{x\to\infty} T(x) &= \lim_{x\to\infty} \left[\beta^{(m)}(\alpha x) - \alpha\beta^{(m)}(x)\right] \\ &= \lim_{x\to\infty} \left[\int_0^\infty \frac{t^m e^{-\alpha x t}}{1+e^{-t}}\, dt - \alpha \int_0^\infty \frac{t^m e^{-xt}}{1+e^{-t}}\, dt\right] \\ &= 0. \end{aligned}$$

Therefore, $T(x) \leqslant 0$ which completes the proof.

THEOREM 2.18. Let $m \in \mathbb{N}_0$. Then the inequality

$$\left[\beta^{(m)}(xy)\right]^2 \leqslant \beta^{(m)}(x)\beta^{(m)}(y) \tag{2.15}$$

holds for $x \geqslant 1$ and $y \geqslant 1$.

PROOF. We have $xy \geqslant x$ and $xy \geqslant y$ since $x \geqslant 1$ and $y \geqslant 1$. If m is even, then we obtain

$$0 < \beta^{(m)}(xy) \leqslant \beta^{(m)}(x)$$

and

$$0 < \beta^{(m)}(xy) \leqslant \beta^{(m)}(y)$$

since $\beta^{(m)}(x)$ is decreasing for even m (see Lemma 2.3). That implies

$$\left[\beta^{(m)}(xy)\right]^2 \leqslant \beta^{(m)}(x)\beta^{(m)}(y).$$

Also, if m is odd, then we have

$$0 > \beta^{(m)}(xy) \geqslant \beta^{(m)}(x)$$

and

$$0 > \beta^{(m)}(xy) \geqslant \beta^{(m)}(y)$$

since $\beta^{(m)}(x)$ is increasing for odd m, and that also implies

$$\left[\beta^{(m)}(xy)\right]^2 \leqslant \beta^{(m)}(x)\beta^{(m)}(y)$$

which completes the proof.

A generalization of Theorem 2.18 is given as follows.

THEOREM 2.19. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$ such that m is even. Then the inequality

$$\beta^{(m)}\left(\prod_{i=1}^{n} x_i\right) \leqslant \left(\prod_{i=1}^{n} \beta^{(m)}(x_i)\right)^{\frac{1}{n}} \tag{2.16}$$

holds for $x_i \geqslant 1$, $i = 1, 2, 3 \ldots, n$.

PROOF. Since $x_i \geqslant 1$ for $i = 1, 2, 3 \ldots, n$, we have $\prod_{i=1}^{n} x_i \geqslant x_j$ for $j = 1, 2, 3 \ldots, n$. Then for m even, we obtain

$$\begin{aligned} 0 < \beta^{(m)}\left(\prod_{i=1}^{n} x_i\right) &\leqslant \beta^{(m)}(x_1), \\ 0 < \beta^{(m)}\left(\prod_{i=1}^{n} x_i\right) &\leqslant \beta^{(m)}(x_2), \\ \vdots \qquad & \qquad \vdots \\ 0 < \beta^{(m)}\left(\prod_{i=1}^{n} x_i\right) &\leqslant \beta^{(m)}(x_n). \end{aligned}$$

Upon taking products of these inequalities, we obtain

$$\left[\beta^{(m)}\left(\prod_{i=1}^{n} x_i\right)\right]^n \leqslant \prod_{i=1}^{n} \beta^{(m)}(x_i)$$

which completes the proof.

THEOREM 2.20. Let $m, n \in \mathbb{N}_0$ and $s \geqslant 1$. Then, the inequality

$$\left(\left|\beta^{(m)}(x)\right| + \left|\beta^{(n)}(y)\right|\right)^{\frac{1}{s}} \leqslant \left|\beta^{(m)}(x)\right|^{\frac{1}{s}} + \left|\beta^{(n)}(y)\right|^{\frac{1}{s}} \tag{2.17}$$

holds for $x, y > 0$.

PROOF. Note that $u^s + v^s \leqslant (u+v)^s$, for $u, v \geqslant 0$ and $s \geqslant 1$. Then by the Minkowski's inequality, we obtain

$$\begin{aligned}
\left(\left|\beta^{(m)}(x)\right| + \left|\beta^{(n)}(y)\right|\right)^{\frac{1}{s}} &= \left(\int_0^\infty \frac{t^m e^{-xt}}{1+e^{-t}}\,dt + \int_0^\infty \frac{t^n e^{-yt}}{1+e^{-t}}\,dt\right)^{\frac{1}{s}} \\
&= \left(\int_0^\infty \left[\left(\frac{t^{\frac{m}{s}} e^{\frac{-xt}{s}}}{(1+e^{-t})^{\frac{1}{s}}}\right)^s + \left(\frac{t^{\frac{n}{s}} e^{\frac{-yt}{s}}}{(1+e^{-t})^{\frac{1}{s}}}\right)^s\right] dt\right)^{\frac{1}{s}} \\
&\leqslant \left(\int_0^\infty \left[\left(\frac{t^{\frac{m}{s}} e^{\frac{-xt}{s}}}{(1+e^{-t})^{\frac{1}{s}}}\right) + \left(\frac{t^{\frac{n}{s}} e^{\frac{-yt}{s}}}{(1+e^{-t})^{\frac{1}{s}}}\right)\right]^s dt\right)^{\frac{1}{s}} \\
&\leqslant \left(\int_0^\infty \frac{t^m e^{-xt}}{1+e^{-t}}\,dt\right)^{\frac{1}{s}} + \left(\int_0^\infty \frac{t^n e^{-yt}}{1+e^{-t}}\,dt\right)^{\frac{1}{s}} \\
&= \left|\beta^{(m)}(x)\right|^{\frac{1}{s}} + \left|\beta^{(n)}(y)\right|^{\frac{1}{s}}
\end{aligned}$$

which yields the desired result.

REMARK 2.21. Notice that $\left|\beta^{(m)}(x)\right| = (-1)^m \beta^{(m)}(x)$ for $m \in \mathbb{N}_0$ and $x > 0$. Then by the recurrence relation (1.15), we obtain

$$\left|\beta^{(m)}(x+1)\right| = \frac{m!}{x^{m+1}} - \left|\beta^{(m)}(x)\right| \tag{2.18}$$

which implies

$$\left|\beta^{(m)}(x)\right| \leqslant \frac{m!}{x^{m+1}}. \tag{2.19}$$

THEOREM 2.22. Let $m \in \mathbb{N}_0$ and $0 < a < b$. Then, there exists a $\lambda \in (a, b)$ such that

$$\left|\beta^{(m)}(b) - \beta^{(m)}(a)\right| \leqslant (b-a)\frac{(m+1)!}{\lambda^{m+2}} \tag{2.20}$$

PROOF. By the classical mean value theorem, there exist a $\lambda \in (a, b)$ such that $\frac{\beta^{(m)}(b)-\beta^{(m)}(a)}{b-a} = \beta^{(m+1)}(\lambda)$. Thus, $\frac{\left|\beta^{(m)}(b)-\beta^{(m)}(a)\right|}{(b-a)} = \left|\beta^{(m+1)}(\lambda)\right|$ and by (2.19), we obtain the result (2.20).

3. Conclusion

In this study, we obtained some convexity, monotonicity and additivity properties as well as some inequalities involving the Nielsen's β-function. The established results may be useful in evaluating or estimating certain integrals. Furthermore, the findings could provide useful information for further study of the function.

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