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Certain weighted integral inequalities involving the fractional hypergeometric operators

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Abstract. In this paper, the Gauss hypergeometric function fractional integral operator is used to generate some new weighted fractional integral inequalities.

1. Introduction

The fractional integral inequalities have many applications in fractional differential equations, numerical quadrature, transform theory, probability and statistical problems. For details, we refer to [**1, 2, 4, 6, 8, 10, 15, 16, 17, 18**] and the references therein. Recently, by applying the different fractional integral operators such as Riemann-Liouville fractional integral operators, Hadamard fractional operators, fractional $q-$integral operators, Saigo fractional integral operators and fractional hypergeometric operators, many researchers have obtained a lot of fractional integral inequalities and applications, we refer to [**3, 5, 13, 14, 15, 19, 20, 21, 22, 23, 24**]. In [**12**] Dahmani established some new classes of fractional integral inequalities using the Riemann-Liouville fractional integral operators. Dahmani *et al.* [**10, 11**] derived certain integral inequalities involving the fractional $q-$integral operators. Also, Chinchane *et al.* [**7**] and Yang [**27**] established some fractional integral inequalities using Hadamard fractional integral operators and Saigo fractional integral operators respectively. Recently, Baleanu *et al.* [**3, 4**], Choi [**8**] and Wang *et al.* [**26**] established some integral inequalities by using the Gauss hypergeometric function fractional operators, introduced by Curiel and Galue [**9**]. Motivated by the results presented in [**10, 11, 12**], the main aim of this paper is to establish some new weighted fractional integral inequalities involving the Gauss hypergeometric function fractional operators.

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2. Fractional Calculus

In this section, we give some necessary definitions and mathematical preliminaries of fractional calculus operators which are used further in this paper, we can see [**9, 17, 23, 25**].

Definition 1. *A real valued function $f(t)$, is said to be in the space $\mathbb{C}_\mu([0,\infty))$, $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$ such that $f(t) = t^p g(t)$, where $g(t) \in \mathbb{C}([0,\infty))$.*

Definition 2. *Let $\alpha > 0, \mu > -1, \beta, \eta \in \mathbb{R}$, then, a generalized fractional integral $I_t^{\alpha,\beta,\eta,\mu}$ of order α for a real-valued continuous function $f(t)$ is defined by*

$$\begin{aligned} I_t^{\alpha,\beta,\eta,\mu}[f(t)] &= \tfrac{x^{-\alpha-\beta-2\eta}}{\Gamma(\alpha)} \int_0^t x^\mu (t-x)^{\alpha-1} \\ &\times {}_2F_1\left(\alpha+\beta+\mu, -\eta; \alpha; 1-\tfrac{x}{t}\right) f(x)\,dx, \end{aligned} \tag{2.1}$$

where, the function ${}_2F_1(.)$ appearing as a kernel for the operator (2.1) is the Gaussian hypergeometric function defined by

$${}_2F_1(\epsilon, \varepsilon; \kappa; t) = \sum \tfrac{(\epsilon)_n (\varepsilon)_n}{(\kappa)_n} \tfrac{t^n}{n!}\ , \tag{2.2}$$

and $(\epsilon)_n$ is the Pochhammer symbol

$$(\epsilon)_n = \epsilon(\epsilon+1)...(\epsilon+n-1),\ (\epsilon)_0 = 1\ . \tag{2.3}$$

It may be noted that the Pochhammer symbol in terms of the gamma function is defined by

$$(\epsilon)_n = \tfrac{\Gamma(\epsilon+n)}{\Gamma(\epsilon)},\ n > 0\ , \tag{2.4}$$

where the gamma function is given by

$$\Gamma(\epsilon) = \int_0^\infty e^{-u} u^{\epsilon-1} du\ . \tag{2.5}$$

For $f(t) = t^\rho$ in (2.1), we get

$$I_t^{\alpha,\beta,\eta,\mu}\left[t^{\rho-1}\right] = \tfrac{\Gamma(\mu+\rho)\Gamma(\rho-\beta+\eta)}{\Gamma(\rho-\beta)\Gamma(\rho+\alpha+\eta+\mu)} (t-a)^{\varpi-\beta-\mu-1}\ , \tag{2.6}$$

where $\alpha, \beta, \eta, \rho \in \mathbb{R}, \mu > -1, \mu+\rho > 0$ and $\rho - \beta + \eta > 0$.

3. Hypergeometric Fractional Integral Inequalilties

In this section, we firstly prove some weighted fractional integral inequalities concerning the Gauss hypergeometric function fractional integral operators.

Theorem 3. *Let f be positive and continuous function on $[0,\infty)$, and let $w : [0,\infty) \to \mathbb{R}^+$ be positive continuous function. Then we have*

$$\begin{aligned} &I_t^{\alpha,\beta,\eta,\mu}\left[w(t) f^{\sigma+\theta}(t)\right] I_t^{\alpha,\beta,\eta,\mu}\left[w(t) t^\sigma f^\delta(t)\right] \\ &\leq I_t^{\alpha,\beta,\eta,\mu}\left[w(t) f^{\sigma+\delta}(t)\right] I_t^{\alpha,\beta,\eta,\mu}\left[w(t) t^\sigma f^\theta(t)\right], \end{aligned} \tag{3.1}$$

for all $t > 0, \alpha > \max(0, -\beta-\mu), \beta < 1, \mu > -1$ and $\beta - 1 < \eta < 0$.

PROOF. Since the function f is positive and continuous on $[0,\infty)$, then for all $\sigma>0, \delta\geq\theta>0$, $x,y\in(0,t), t>0$, we can write

$$(y^{\sigma}f^{\sigma}(x)-x^{\sigma}f^{\sigma}(y))\left(f^{\delta-\theta}(x)-f^{\delta-\theta}(y)\right)\geq 0 \ , \tag{3.2}$$

which implies that

$$y^{\sigma}f^{\delta-\theta}(y)f^{\sigma}(x)+x^{\sigma}f^{\delta-\theta}(x)f^{\sigma}(y)\leq y^{\sigma}f^{\sigma+\delta-\theta}(x)+x^{\sigma}f^{\sigma+\delta-\theta}(y) \ . \tag{3.3}$$

Consider

$$\begin{aligned}\psi(t,x)&=\frac{t^{-\alpha-\beta-2\mu}x^{\mu}(t-x)^{\alpha-1}}{\Gamma(\alpha)}\ {}_2F_1\left(\alpha+\beta+\mu,-\eta;\alpha;1-\tfrac{x}{t}\right)\\ &=\frac{x^{\mu}(t-x)^{\alpha-1}}{\Gamma(\alpha)}t^{-\alpha-\beta-2\mu}+\frac{(\alpha+\beta+\mu)(-\eta)x^{\mu}(t-x)^{\alpha}}{\Gamma(\alpha+1)}t^{-\alpha-\beta-2\mu-1}\\ &\quad+\frac{(\alpha+\beta+\mu)(-\eta)(\alpha+\beta+\mu+1)(-\eta+1)x^{\mu}(t-x)^{\alpha+1}}{2\Gamma(\alpha+2)}t^{-\alpha-\beta-2\mu-2}+\ldots.\end{aligned} \tag{3.4}$$

where $x\in(0,t)$ and $t>0$. We observe that each term of the above series is positive in view of the conditions stated with Theorem 3 and hence, the function $\psi(t,x)$ remains positive, for all $x\in(0,t)$, $t>0$.

Multiplying both sides of (3.2) by $\psi(t,x)w(x)f^{\theta}(x)$ and integrating with respect to x over $(0,t)$, we get

$$\begin{aligned}&y^{\sigma}f^{\delta-\theta}(y)I_t^{\alpha,\beta,\eta,\mu}\left[w(t)f^{\theta+\sigma}(t)\right]+f^{\sigma}(y)I_t^{\alpha,\beta,\eta,\mu}\left[w(t)t^{\sigma}f^{\delta}(t)\right]\\ &\leq y^{\sigma}I_t^{\alpha,\beta,\eta,\mu}\left[w(t)f^{\sigma+\delta}(t)\right]+f^{\sigma+\delta-\theta}(y)I_t^{\alpha,\beta,\eta,\mu}\left[w(t)t^{\sigma}f^{\theta}(t)\right].\end{aligned} \tag{3.5}$$

Now, multiplying both sides of (3.5) by $\psi(t,y)w(y)f^{\theta}(y)$ and integrating with respect to y over $(0,t)$, we can write

$$\begin{aligned}&I_t^{\alpha,\beta,\eta,\mu}\left[w(t)f^{\sigma+\theta}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)t^{\sigma}f^{\delta}(t)\right]\\ &+I_t^{\alpha,\beta,\eta,\mu}\left[w(t)t^{\sigma}f^{\delta}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)f^{\sigma+\theta}(t)\right]\\ &\leq I_t^{\alpha,\beta,\eta,\mu}\left[w(t)f^{\sigma+\delta}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)t^{\sigma}f^{\theta}(t)\right]\\ &+I_t^{\alpha,\beta,\eta,\mu}\left[w(t)t^{\sigma}f^{\theta}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)f^{\sigma+\delta}(t)\right],\end{aligned} \tag{3.6}$$

which implies (3.1). □

Theorem 4. *Let f be positive and continuous function on $[0,\infty)$, and let $w:[0,\infty)\to\mathbb{R}^{+}$ be positive continuous function. Then, for all $t>0$ and for all $\delta\geq\theta>0,\sigma>0$, we have*

$$
\begin{aligned}
&I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,t^{\sigma}f^{\delta}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f^{\sigma+\theta}(t)\right]\\
&+I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,t^{\sigma}f^{\delta}(t)\right]I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,f^{\sigma+\theta}(t)\right]\\
&\leq I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,t^{\sigma}f^{\theta}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f^{\sigma+\delta}(t)\right]\\
&+I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,t^{\sigma}f^{\theta}(t)\right]I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,f^{\sigma+\delta}(t)\right],
\end{aligned}
\tag{3.7}
$$

where $\alpha>\max(0,-\beta-\mu),\beta<1,\mu>-1,\ \beta-1<\eta<0,\ \omega>\max(0,-\lambda-\varpi),\lambda<1,\varpi>-1,\ \lambda-1<\gamma<0.$

Proof. Now multiplying both sides of (3.2) by the quantity $\varphi(t,y)\,w(y)\,f^{\theta}(y)$, where

$$
\varphi(t,y)=\frac{t^{-\omega-\lambda-2\varpi}y^{\varpi}(t-y)^{\alpha-1}}{\Gamma(\alpha)}\ {}_2F_1\left(\omega+\lambda+\varpi,-\varpi;\alpha;1-\frac{y}{t}\right),y\in(0,t),t>0\ ,
\tag{3.8}
$$

in view of the arguments mentioned above in the proof of Theorem 3. We can see that the function $\varphi(t,y)$ remains positive under the conditions stated with Theorem 4. Integrating the resulting inequality obtained with respect to y from 0 to t, we obtain

$$
\begin{aligned}
&f^{\sigma}(x)\,I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,t^{\sigma}f^{\delta}(t)\right]+x^{\sigma}f^{\delta-\theta}(x)\,I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,f^{\sigma+\theta}(t)\right]\\
&\leq f^{\sigma+\delta-\theta}(x)\,I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,t^{\sigma}f^{\theta}(t)\right]+x^{\sigma}I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,f^{\sigma+\delta}(t)\right].
\end{aligned}
\tag{3.9}
$$

Next, multiplying both sides of (3.9) by $\psi(t,x)\,f^{\theta}(x)$ and integrating with respect to x from 0 to t, we obtain

$$
\begin{aligned}
&I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,t^{\sigma}f^{\delta}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f^{\sigma+\theta}(t)\right]\\
&+I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,f^{\sigma+\theta}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,t^{\sigma}f^{\delta}(t)\right]\\
&\leq I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,t^{\sigma}f^{\theta}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f^{\sigma+\delta}(t)\right]\\
&+I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,f^{\sigma+\delta}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,t^{\sigma}f^{\theta}(t)\right].
\end{aligned}
\tag{3.10}
$$

which implies (3.7). □

Remark 5. *Applying Theorem 4 for $\alpha=\omega,\ \beta=\lambda,\eta=\gamma$ and $\mu=\varpi$,we obtain Theorem 3.*

THEOREM 6. *Let f and h be two positive and continuous functions on $[0,\infty)$ and $w:[0,\infty)\to\mathbb{R}^{+}$ positive continuous functions. Then for all $t>0$ and $\delta\geq\theta>0,\sigma>0$, we have*

$$\begin{aligned}&I_t^{\alpha,\beta,\eta,\mu}\left[w(t)f^{\sigma}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)h^{\sigma}(t)f^{\delta}(t)\right]\\&\leq I_t^{\alpha,\beta,\eta,\mu}\left[w(t)f^{\delta+\sigma}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[h^{\sigma}(t)f^{\theta}(t)\right],\end{aligned}\tag{3.11}$$

where $\alpha>\max(0,-\beta-\mu),\beta<1,\mu>-1$ and $\beta-1<\eta<0$.

PROOF. Let $x,y\in(0,y),t>0$, for any $\delta\geq\theta>0,\sigma>0$. Then we have

$$\left(h^{\sigma}(y)f^{\sigma}(x)-h^{\sigma}(x)f^{\sigma}(y)\right)\left(f^{\delta-\theta}(x)-f^{\delta-\theta}(y)\right)\geq 0\ ,\tag{3.12}$$

which implies that

$$h^{\sigma}(y)f^{\delta-\theta}(y)f^{\sigma}(x)+f^{\sigma}(y)h^{\sigma}(x)f^{\delta-\theta}(x)\leq h^{\sigma}(y)f^{\delta+\sigma-\theta}(x)+f^{\sigma+\delta-\theta}(y)h^{\sigma}(x)\ .\tag{3.13}$$

Multiplying both sides of (3.13) by $\psi(t,x)w(x)f^{\theta}(x)$, then integrating the resulting inequality with respect to x over $(0,t)$, we obtain

$$\begin{aligned}&h^{\sigma}(y)f^{\delta-\theta}(y)I_t^{\alpha,\beta,\eta,\mu}\left[w(t)f^{\sigma}(t)\right]+f^{\sigma}(y)I_t^{\alpha,\beta,\eta,\mu}\left[w(t)h^{\sigma}(t)f^{\delta}(t)\right]\\&\leq h^{\sigma}(y)I_t^{\alpha,\beta,\eta,\mu}\left[w(t)f^{\delta+\sigma}(t)\right]+f^{\sigma+\delta-\theta}(y)I_t^{\alpha,\beta,\eta,\mu}\left[w(t)h^{\sigma}(t)f^{\theta}(t)\right].\end{aligned}\tag{3.14}$$

Multiplying now both sides of (3.14) by $\psi(t,y)w(y)f^{\theta}(y)$, then integrating the resulting inequality with respect to y over $(0,t)$, we obtain

$$\begin{aligned}&I_t^{\alpha,\beta,\eta,\mu}\left[w(t)f^{\sigma}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)h^{\sigma}(t)f^{\delta}(t)\right]\\&+I_t^{\alpha,\beta,\eta,\mu}\left[w(t)h^{\sigma}(t)f^{\delta}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)f^{\sigma}(t)\right]\\&\leq I_t^{\alpha,\beta,\eta,\mu}\left[w(t)f^{\delta+\sigma}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[h^{\sigma}(t)f^{\theta}(t)\right]\\&+I_t^{\alpha,\beta,\eta,\mu}\left[h^{\sigma}(t)f^{\theta}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)f^{\delta+\sigma}(t)\right].\end{aligned}\tag{3.15}$$

Theorem 6 is thus proved. □

THEOREM 7. *Let f and h be two positive and continuous functions on $[0,\infty)$ and let $w:[0,\infty)\to\mathbb{R}^{+}$ be positive continuous function. Then for any $t>0,\delta\geq\theta>0,\sigma>0$, we have*

$$\begin{aligned}&h^{\sigma}(y)f^{\delta-\theta}(y)I_t^{\alpha,\beta,\eta,\mu}\left[w(t)f^{\sigma}(t)\right]+f^{\sigma}(y)I_t^{\alpha,\beta,\eta,\mu}\left[w(t)h^{\sigma}(t)f^{\delta}(t)\right]\\&\leq h^{\sigma}(y)I_t^{\alpha,\beta,\eta,\mu}\left[w(t)f^{\delta+\sigma}(t)\right]+f^{\sigma+\delta-\theta}(y)I_t^{\alpha,\beta,\eta,\mu}\left[w(t)h^{\sigma}(t)f^{\theta}(t)\right],\end{aligned}\tag{3.16}$$

where $\alpha>\max(0,-\beta-\mu),\beta<1,\mu>-1,\beta-1<\eta<0,\omega>\max(0,-\lambda-\varpi),\lambda<1,\varpi>-1$, $\lambda-1<\gamma<0$.

Proof. Multiplying the inequality (3.14) by $\psi(t,y)\,w(y)\,f^{\theta}(y)$ and integrating with respect to y over $(0,t)$, we get

$$\begin{aligned}
&I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f^{\sigma}(t)\right] I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,h^{\sigma}(t)\,f^{\delta}(t)\right]\\
&+I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,h^{\sigma}(t)\,f^{\delta}(t)\right] I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,f^{\sigma}(t)\right]\\
&\leq I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f^{\delta+\sigma}(t)\right] I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,h^{\sigma}(t)\,f^{\theta}(t)\right]\\
&+I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,h^{\sigma}(t)\,f^{\theta}(t)\right] I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,f^{\sigma+\delta-\theta}(y)\right].
\end{aligned} \tag{3.17}$$

The result is proved. □

Remark 8. *For $\alpha=\omega$, $\beta=\lambda, \eta=\gamma$ and $\mu=\varpi$, Theorem 7 immediately is reduced to Theorem 6.*

Next, we shall propose a new generalization of weighted fractional integral inequalities using a family of n positive functions defined on $[0,\infty)$.

Theorem 9. *Let f_i, $i=1,...,n$ be n positive and continuous fonctions on $[0,\infty)$ and let $w:[0,\infty)\to\mathbb{R}^{+}$. Then, for all $t>0, \sigma>0, \delta\geq\theta_q>0$, $q\in\{1,...,n\}$, the following fractional inequality*

$$\begin{aligned}
&I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right] I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,t^{\sigma}f_q^{\delta}(t)\prod_{i\neq q}^{n}f_i^{\theta_i}(t)\right]\\
&\leq I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,t^{\sigma}\prod_{i=1}^{n}f_i^{\theta_i}(t)\right] I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\delta+\sigma}(t)\prod_{i\neq q}^{n}f_i^{\theta_i}(t)\right],
\end{aligned} \tag{3.18}$$

is valid for any $\alpha>\max(0,-\beta-\mu)$, $\beta<1, \mu>-1, \beta-1<\eta<0$.

Proof. Suppose f_i, $i=1,...,n$ be n positive continuous fonctions on $[0,\infty)$, then we can write

$$\left(y^{\sigma}f_q^{\sigma}(x)-x^{\sigma}f_q^{\sigma}(y)\right)\left(f_q^{\delta-\theta_q}(x)-f_q^{\delta-\theta_q}(y)\right)\geq 0\ , \tag{3.19}$$

for any fixed $q\in\{1,...,n\}$ and for any $\sigma>0$, $\delta\geq\theta_q>0$, $x,y\in(0,t)$, $t>0$.

From (3.19), we obtain

$$y^{\sigma}f_q^{\delta-\theta_q}(y)\,f_q^{\sigma}(x)+f_q^{\sigma}(y)\,x^{\sigma}f_q^{\delta-\theta_q}(x)\leq y^{\sigma}f_q^{\delta+\sigma-\theta_q}(x)+f_q^{\delta+\sigma-\theta_q}(y)\,x^{\sigma}\ , \tag{3.20}$$

Now, multiplying both sides of (3.20) by $\psi(t,x)\,w(x)\prod_{i=1}^{n}f_i^{\theta_i}(x)$ and integrating with respect to x from 0 to t, we obtain

$$\begin{aligned}
&y^{\sigma}f_q^{\delta-\theta_q}(y)\,I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]\\
&+f_q^{\sigma}(y)\,I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,t^{\sigma}f_q^{\delta}(t)\prod_{i\neq q}^{n}f_i^{\theta_i}(t)\right]\\
&\leq y^{\sigma}I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\delta+\sigma}(t)\prod_{i\neq q}^{n}f_i^{\theta_i}(t)\right]\\
&+f_q^{\delta+\sigma-\theta_q}(y)\,I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,t^{\sigma}\prod_{i=1}^{n}f_i^{\theta_i}(t)\right].
\end{aligned} \tag{3.21}$$

Next, multiplying the inequality (3.21) by $\psi(t,y)\,w(y)\prod_{i=1}^{n} f_i^{\theta_i}(y)$ and integrating with respect to y from 0 to t, we can write

$$\begin{aligned}
&I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,t^{\sigma} f_q^{\delta}(t)\prod_{i\neq q}^{n} f_i^{\theta_i}(t)\right]\\
&+I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,t^{\sigma} f_q^{\delta}(t)\prod_{i\neq q}^{n} f_i^{\theta_i}(t)\right] I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right]\\
&\leq I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\delta+\sigma}(t)\prod_{i\neq q}^{n} f_i^{\theta_i}(t)\right] I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,t^{\sigma}\prod_{i=1}^{n} f_i^{\theta_i}(t)\right]\\
&+I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,t^{\sigma}\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\delta+\sigma}(t)\prod_{i\neq q}^{n} f_i^{\theta_i}(t)\right].
\end{aligned} \tag{3.22}$$

This ends the proof of Theorem 9. □

THEOREM 10. *Let $f_i,\ i=1,...,n$ be n positive continuous fonctions on $[0,\infty)$, $w:[0,\infty)\to\mathbb{R}^{+}$. Then for any $t>0$ and for all $\sigma>0$, $\delta\geq\theta_q>0, q\in\{1,...,n\}$ we have*

$$\begin{aligned}
&I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,t^{\sigma} f_q^{\delta}(t)\prod_{i\neq q}^{n} f_i^{\theta_i}(t)\right] I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right]\\
&+I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,t^{\sigma} f_q^{\delta}(t)\prod_{i\neq q}^{n} f_i^{\theta_i}(t)\right] I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,f_q^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right]\\
&\leq I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\delta+\sigma}(t)\prod_{i\neq q}^{n} f_i^{\theta_i}(t)\right] I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,t^{\sigma}\prod_{i=1}^{n} f_i^{\theta_i}(t)\right]\\
&+I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,f_q^{\delta+\sigma}(t)\prod_{i\neq q}^{n} f_i^{\theta_i}(t)\right] I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,t^{\sigma}\prod_{i=1}^{n} f_i^{\theta_i}(t)\right].
\end{aligned} \tag{3.23}$$

where $\alpha>\max(0,-\beta-\mu), \beta<1, \mu>-1, \beta-1<\eta<0, \omega>\max(0,-\lambda-\varpi), \lambda<1, \varpi>-1, \lambda-1<\gamma<0$.

PROOF. We multiply the inequality (3.21) by $\varphi(t,y)\,w(y)\prod_{i=1}^{n} f_i^{\theta_i}(y)$ then we integrate the result with respect to y on $(0,t)$, we can write

$$\begin{aligned}
&f_q^{\sigma}(x)\,I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,t^{\sigma} f_q^{\delta}(t)\prod_{i\neq q}^{n} f_i^{\theta_i}(t)\right]\\
&+x^{\sigma} f_q^{\delta-\theta_q}(x)\,I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,f_q^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right]\\
&\leq f_q^{\delta+\sigma-\theta_q}(x)\,I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,t^{\sigma}\prod_{i=1}^{n} f_i^{\theta_i}(t)\right]\\
&+x^{\sigma} I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,f_q^{\delta+\sigma}(t)\prod_{i\neq q}^{n} f_i^{\theta_i}(t)\right].
\end{aligned} \tag{3.24}$$

Therefore,

$$
\begin{aligned}
& I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,t^{\sigma} f_q^{\delta}(t)\prod_{i\neq q}^{n} f_i^{\theta_i}(t)\right] I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] \\
& + I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,t^{\sigma} f_q^{\delta}(t)\prod_{i\neq q}^{n} f_i^{\theta_i}(t)\right] I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,f_q^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] \\
& \leq I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\delta+\sigma}(t)\prod_{i\neq q}^{n} f_i^{\theta_i}(t)\right] I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,t^{\sigma}\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] \\
& + I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,f_q^{\delta+\sigma}(t)\prod_{i\neq q}^{n} f_i^{\theta_i}(t)\right] I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,t^{\sigma}\prod_{i=1}^{n} f_i^{\theta_i}(t)\right].
\end{aligned} \tag{3.25}
$$

This completes the proof. □

REMARK 11. *If we take $\alpha=\omega$, $\beta=\lambda, \eta=\gamma$ and $\mu=\varpi$, in Theorem 10, we obtain Theorem 9.*

THEOREM 12. *Let f_i, $i=1,...,n$ and h be positive continuous functions on $[0,\infty), w:[0,\infty)\rightarrow\mathbb{R}^{+}$. Then, for all $\sigma>0, \delta\geq\theta_q>0, q\in\{1,...,n\}$, the following fractional inequality*

$$
\begin{aligned}
& I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,g^{\sigma}(t)\,f_q^{\delta}(t)\prod_{i\neq q}^{n} f^{\theta_i}(t)\right] \\
& \leq I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\delta+\sigma}(t)\prod_{i\neq q}^{n} f^{\theta_i}(t)\right] I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,g^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right].
\end{aligned} \tag{3.26}
$$

holds for any $t>0, \alpha>\max(0,-\beta-\mu), \beta<1, \mu>-1, \beta-1<\eta<0$.

PROOF. Let $x,y\in(0,t), t>0$, for any $\sigma>0, \delta\geq\theta_q>0, q\in\{1,...,n\}$. Then we have

$$
\left(h^{\sigma}(y)\,f_q^{\sigma}(x)-h^{\sigma}(x)\,f_q^{\sigma}(y)\right)\left(f_q^{\delta-\theta_q}(x)-f_q^{\delta-\theta_q}(y)\right)\geq 0\ . \tag{3.27}
$$

Consider

$$
h^{\sigma}(y)\,f_q^{\delta-\theta_q}(y)\,f_q^{\sigma}(x)+f_q^{\sigma}(y)\,h^{\sigma}(x)\,f_q^{\delta-\theta_q}(x)\leq h^{\sigma}(y)\,f_q^{\delta+\sigma-\theta_q}(x)+f_q^{\delta+\sigma-\theta_q}(y)\,h^{\sigma}(x)\ . \tag{3.28}
$$

Multiplying both sides of the above inequality by $\psi(t,x)\,w(x)\prod_{i=1}^{n} f_i^{\theta_i}(x)$ and integrating with respect to x over $(0,t)$, we obtain

$$
\begin{aligned}
& h^{\sigma}(y)\,f_q^{\delta-\theta_q}(y)\,I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] \\
& + f_q^{\sigma}(y)\,I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,h^{\sigma}(t)\,f_q^{\delta}(t)\prod_{i\neq q}^{n} f^{\theta_i}(t)\right] \\
& \leq h^{\sigma}(y)\,I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\delta+\sigma}(t)\prod_{i\neq q}^{n} f^{\theta_i}(t)\right] \\
& + I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n} f_i^{\theta_i}(t)\right] f_q^{\delta+\sigma-\theta_q}(y).
\end{aligned} \tag{3.29}
$$

Integrating both sides of (3.29) with respect to y over $(0,t)$, we obtain

$$\begin{aligned}&2I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,g^{\sigma}(t)\,f_q^{\delta}(t)\prod_{i\neq q}^{n}f^{\theta_i}(t)\right]\\ &\leq 2I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\delta+\sigma}(t)\prod_{i\neq q}^{n}f^{\theta_i}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,g^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right].\end{aligned} \tag{3.30}$$

This ends the proof. □

THEOREM 13. *Let f_i, $i=1,...,n$ and h be positive continuous functions on $[0,\infty)$, $w:[0,\infty)\to\mathbb{R}^+$. Then, for all $t>0$, $\sigma>0, \delta\geq\theta_q>0, q\in\{1,...,n\}$, we have*

$$\begin{aligned}&I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,h^{\sigma}(t)\,f_q^{\delta}(t)\prod_{i\neq q}^{n}f^{\theta_i}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]\\ &+I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,h^{\sigma}(t)\,f_q^{\delta}(t)\prod_{i\neq q}^{n}f^{\theta_i}(t)\right]I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,f_q^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]\\ &\leq I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\delta+\sigma}(t)\prod_{i\neq q}^{n}f^{\theta_i}(t)\right]I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]\\ &+I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,f_q^{\delta+\sigma}(t)\prod_{i\neq q}^{n}f^{\theta_i}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,g^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right].\end{aligned} \tag{3.31}$$

where $\alpha>\max(0,-\beta-\mu)$, $\beta<1$, $\mu>-1$, $\beta-1<\eta<0$, $\omega>\max(0,-\lambda-\varpi)$, $\lambda<1$, $\varpi>-1$, $\lambda-1<\gamma<0$.

PROOF. Multiplying (3.28) by $\varphi(t,y)\,w(y)\prod_{i=1}^{n}f_i^{\theta_i}(y)$ then we integrate the resulting inequality with respect to y on $(0,t)$, we obtain

$$\begin{aligned}&f_q^{\sigma}(x)\,I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,h^{\sigma}(t)\,f_q^{\delta}(t)\prod_{i\neq q}^{n}f^{\theta_i}(t)\right]\\ &+h^{\sigma}(x)\,f_q^{\delta-\theta_q}(x)\,I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,f_q^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]\\ &\leq f_q^{\delta+\sigma-\theta_q}(x)\,I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]\\ &+h^{\sigma}(x)\,I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,f_q^{\delta+\sigma}(t)\prod_{i\neq q}^{n}f^{\theta_i}(t)\right].\end{aligned} \tag{3.32}$$

The integration of (3.32) gives

$$\begin{aligned}&I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,h^{\sigma}(t)\,f_q^{\delta}(t)\prod_{i\neq q}^{n}f^{\theta_i}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]\\ &+I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,h^{\sigma}(t)\,f_q^{\delta}(t)\prod_{i\neq q}^{n}f^{\theta_i}(t)\right]I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,f_q^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]\\ &\leq I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,f_q^{\delta+\sigma}(t)\prod_{i\neq q}^{n}f^{\theta_i}(t)\right]\\ &+I_t^{\omega,\lambda,\gamma,\varpi}\left[w(t)\,f_q^{\delta+\sigma}(t)\prod_{i\neq q}^{n}f^{\theta_i}(t)\right]I_t^{\alpha,\beta,\eta,\mu}\left[w(t)\,h^{\sigma}(t)\prod_{i=1}^{n}f_i^{\theta_i}(t)\right].\end{aligned} \tag{3.33}$$

The proof is completed. □

REMARK 14. *Applying Theorem 13 for $\alpha = \omega$, $\beta = \lambda, \eta = \gamma$ and $\mu = \varpi$, we obtain Theorem 12.*

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