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Decomposing graphs into internally-disjoint induced paths

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Abstract. Let G be a non-trivial, simple, finite, connected and undirected graph. An acyclic graphoidal decomposition (AGD) of G is a collection ψ of non-trivial paths in G that are internally vertex-disjoint such that every edge of G lies in exactly one path of ψ. By imposing the condition of induceness on the paths in ψ, the concept of induced acyclic graphoidal decomposition ($IAGD$) of a graph G has been defined in [**2**]. The minimum number of paths in such a decomposition of G is called the induced acyclic graphoidal decomposition number denoted by $\eta_{ia}(G)$. In this paper we obtain certain bounds of the parameter $\eta_{ia}(G)$ and initiate a study on graphs admitting an $IAGD$ satisfying the Helly property.

1. Introduction

The graphs considered here are non-trivial, simple, finite, connected and undirected. The order and size of a graph $G = (V, E)$ are denoted by n and m respectively. For terms not defined here we refer to [**6**].

A *decomposition* of a graph G is a collection ψ of its subgraphs such that every edge of G lies in exactly one member of ψ. Several variations of decomposition have been introduced and well studied by imposing conditions on the members of the decomposition. Path cover [**7**], unrestricted path cover [**8**], (a, b)- decomposition [**11**] are some such variations. In this sequence, Acharya and Sampathkumar [**1**] introduced the notion of *graphoidal decomposition(GD)* which is a decomposition ψ of a graph G into internally-disjoint paths and cycles (that is, every vertex of G is an internal vertex of at most one member of ψ). The *graphoidal decomposition number* $\eta(G)$ is the minimum cardinality of a GD of G. Arumugam and Suseela [**4**] called a GD none of whose members is a cycle as an *acyclic graphoidal decomposition (AGD)* and studied the corresponding parameter $\eta_a(G)$. An AGD wherein any two paths have at most one vertex in common is a *simple acyclic graphiodal decomposition (SAGD)* and the minimum cardinality of such a decomposition is $\eta_{as}(G)$; this was introduced in [**3**]. Motivated by the observation that every path in a $SAGD$ is an induced path

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but not conversely, Arumugam [**2**] defined the concept of induced acyclic graphoidal decomposition as an AGD all of whose members are induced paths and he used $\eta_{ia}(G)$ to denote the minimum cardinality of such a decomposition. The study of this parameter was initiated in [**9**] where they determined the value of η_{ia} for several families of graphs such as complete graphs, complete bipartite graphs,wheels, unicyclic graphs and bicyclic graphs. We now extend the study of this parameter $\eta_{ia}(G)$ further by obtaining some bounds along with characterization of graphs attaining these bounds and investigating graphs admitting an $IAGD$ satisfying some specified property.

We need the following theorem that gives an expression for $\eta_{ia}(G)$ in terms of the order of G and the interior vertices which is similar to the one presented in [**10**]. By *internal vertices* of a path P, we mean the vertices on P other than its end vertices. For an $IAGD$ ψ of G, a vertex v is said to be *interior* to ψ if v is an internal vertex of an element of ψ and is called *exterior to* ψ otherwise.

THEOREM 1.1. *For every induced acyclic graphoidal decomposition ψ of a graph G, let t_ψ denote the number of vertices interior to ψ and let $t_{ia}(G) = max\ t_\psi$, where the maximum is taken over all the induced acyclic graphoidal decompositions ψ of G. Then $\eta_{ia}(G) = m - t_{ia}(G)$.*

REMARK 1.2. If there exists an edge-disjoint collection S of internally disjoint induced paths of G such that every vertex of G is an internal vertex of an element in S, then $\eta_{ia}(G) = m - n$. This is because S together with the edges of G not belonging to members of S yield an $IAGD$ ψ of G with $t_{ia}(G) = n$.

2. Bounds of η_{ia}

In this section we obtain some sharp bounds of η_{ia} and investigate the extremal graphs attaining these bounds.

THEOREM 2.1. *For any graph G, $\eta_{ia}(G) \leqslant m$ with equality holding only for complete graphs.*

PROOF. The inequality $\eta_{ia}(G) \leqslant m$ is obvious as the edge set $E(G)$ itself is a trivial $IAGD$ for any graph G. Now assume that $\eta_{ia}(G) = m$ and G is not complete. Then G contains at least one pair of non-adjacent vertices u and v. Let P be a shortest $u - v$ path in G. Since P is an induced path, the collection $\psi = \{P\} \cup (E(G) \setminus E(P))$ is an induced acyclic graphoidal decomposition of G with $|\psi| < m$ and hence $\eta_{ia} \neq m$. Converse follows as any induced path in a complete graph is of length one. □

Now we proceed to obtain a bound for η_{ia} for a given graph G in terms of its diameter and characterize graphs for which the bound is attained. In this connection we describe a graph $K^{(u,v)}$ as follows.

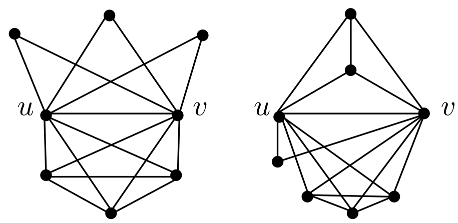

Figure 1

The graph $K^{(u,v)}$ is obtained by pasting at least two complete graphs of order more than two on the same edge uv as shown in Figure 1. Note that complete graphs cannot be considered as the graph $K^{(u,v)}$.

THEOREM 2.2. *If G is a graph with diameter d, then $\eta_{ia}(G) \leqslant m-d+1$. Further, equality holds if and only if G is isomorphic to a graph all of whose blocks are either complete or the graph $K^{(u,v)}$ with u and v as cut vertices of G such that the block-cutpoint graph of G is a caterpillar.*

PROOF. Let $P=(v_1,v_2,\ldots,v_{d+1})$ be a diametrical path in G. Obviously P is an induced path. Therefore, the inequality $\eta_{ia}(G) \leqslant m-d+1$ is immediate; because $\psi = \{P\} \cup (E(G) \setminus E(P))$ is an *IAGD* of G with cardinality $m-d+1$.

Now, let G be a graph with $\eta_{ia}(G) = m-d+1$. Obviously, G is complete in the case when $d=1$. Assume $d \geqslant 2$. Denote the vertices of G lying outside P by $u_1,u_2,\ldots,u_{n-d-1}$. Now, if there exists an induced path P' having a vertex u_k, where $1 \leqslant k \leqslant n-d-1$, as an internal vertex then the paths P and P' together with the edges not covered by these paths form an *IAGD* of G with cardinality less than $m-d+1$, which is a contradiction. We will get a similar contradiction if we assume that one of the end vertices v_1 and v_{d+1} of the path P is an internal vertex of an induced path in G. Thus none of the vertices $u_1,u_2,\ldots,u_{n-d-1},v_1,$ and v_{d+1} is an internal vertex of any induced path in G.
Further, for a vertex u_k lying outside P, if $d(u_k,P) \geqslant 2$, let v_k be a vertex on P nearest to u_k and let P' be a shortest path between u_k and v_k. Then P' will be an induced path of length more than one containing a vertex u_s not lying on P as an internal vertex. But this is impossible as seen in the above paragraph. Therefore every vertex of G not on P is adjacent to a vertex on P.
Thus what we have proved is that for any diametrical path P, the following conditions hold.

(a) No vertex in the set consisting the end vertices of P along with the vertices outside P can be internal vertex of any induced path.
(b) Every vertex outside P has a neighbour on P.

Now, one can observe that the following are some immediate consequences of the above two conditions.

(i) If v_i and v_j are two vertices on P adjacent to a vertex u_k not on P, then v_i and v_j are adjacent.
(ii) If u_k is a vertex not on P, then u_k is adjacent to at most two vertices of P.

(iii) If u_r and u_s are two adjacent vertices outside P, then $N(u_r) \cap V(P) = N(u_s) \cap V(P)$. That is, every pair of adjacent vertices u_r and u_s will form a triangle with each of the vertices of P to which u_r and/or u_s are adjacent.

It is not difficult to see that all the above observations lead us to the desired graph. Conversely, suppose G is the graph given in the statement of the theorem. As the block-cutpoint graph of G is a caterpillar, we see that for any diametrical path P of G, the condition (a) observed above holds and hence $t_{ia}(G) \leqslant d-1$ which implies that $\eta_{ia}(G) \geqslant m-d+1$. Since the other inequality always holds, we have the desired result. □

EXAMPLE 2.3. The graph given in Figure 2 illustrates a graph described in the statement of the above theorem. Here $d = 13$, $m = 45$ and $\eta_{ia} = 33$.

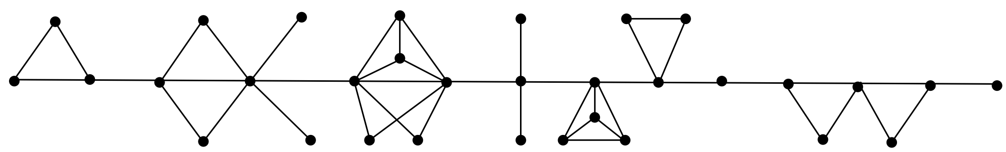

Figure 2

THEOREM 2.4. *For any graph G, we have $\eta_{ia}(G) \geqslant \Delta(G) - 1$ and the bound is sharp.*

PROOF. As a vertex in G is an internal vertex of at most one vertex of a path in any *IAGD*, the inequality follows. Further, the bound is attained for several classes of graphs. One familiar class is the graph obtained from the wheel $W_n = K_1 + C_{n-1}$ where $n \geqslant 5$ by subdividing the $(n-1)$ spokes. Some more families of graphs for which the bound is attained is given in Figure 3. Note that graphs homeomorphic to these families also attain the bound $\eta_{ia} = \Delta - 1$. □

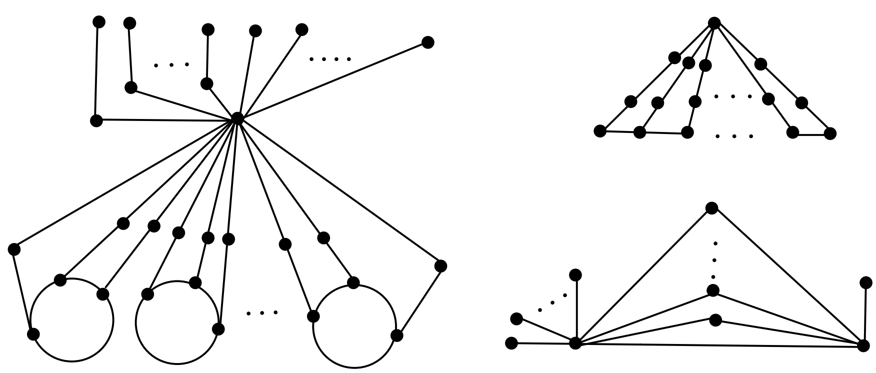

Figure 3

The problem of characterizing the graphs attaining the lower bound given in the above theorem seems to be difficult. However, here we settle the small case when $\Delta = 3$.

THEOREM 2.5. *Let G be a connected graph with $\Delta = 3$ which is not a tree. Then $\eta_{ia}(G) = \Delta - 1$ if and only if G is homeomorphic to one of the graphs G_1, G_2, G_3, G_4, G_5 and G_6 given in Figure 4.*

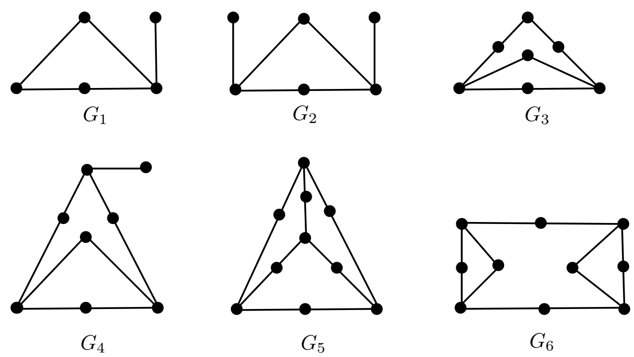

Figure 4

PROOF. Let us first prove that for any graph G with $\eta_{ia}(G) = \Delta - 1$, where $\Delta \geqslant 3$, the number n_Δ of vertices of degree Δ is given by

$$n_\Delta \leqslant \begin{cases} 4 & \text{if} \quad \Delta = 3 \\ 3 & \text{if} \quad \Delta = 4 \\ 2 & \text{if} \quad \Delta \geqslant 5 \end{cases} \tag{2.1}$$

Let ψ be a minimum *IAGD* of G. Then ψ contains $\Delta - 1$ paths and hence there exist at most $2(\Delta - 1)$ vertices of G that are end vertices of some paths in ψ. Note that each of the n_Δ vertices is an end vertex of at least $\Delta - 2$ paths in ψ and hence we have $n_\Delta(\Delta - 2) \leqslant 2(\Delta - 1)$ so that $n_\Delta \leqslant \frac{2(\Delta-1)}{\Delta-2}$ which implies the required inequality.

Now, for the given graph G, if $\Delta(G) = 3$ and $\eta_{ia}(G) = \Delta - 1$, then by the inequality (2.1), we have $n_\Delta \leqslant 4$. Let $\psi = \{P_1, P_2\}$ be a minimum *IAGD* of G. Then by examining all possible configurations of P_1 and P_2, it can be seen that when $n_\Delta = 1$, G is homeomorphic to G_1; when $n_\Delta = 2$, G is homeomorphic to G_2 or G_3; for $n_\Delta = 3$, G is homeomorphic to G_4 and finally when $n_\Delta = 4$, G is homeomorphic to either G_5 or G_6. $\square$

We close this section by presenting a bound for η_{ia} in terms of the clique number.

THEOREM 2.6. *If G is a graph with clique number ω, then $\eta_{ia}(G) \geqslant \binom{\omega}{2}$ and the bound is sharp.*

PROOF. If H is a maximum clique in G, then every edge of H is a member of any *IAGD* of G and therefore $\eta_{ia}(G) \geqslant |E(H)| = \binom{\omega}{2}$. Obviously, the bound is attained for complete graphs. Anther example is the graph G obtained from a complete graph H by attaching a path of any length at each vertex of H. In this case $\eta_{ia}(G) = |E(H)| = \binom{\omega(G)}{2}$. □

3. η_{ia} and other Path Decomposition Parameters

This section establishes relationship of η_{ia} with some graphoidal decomposition parameters such as η_a and η_{as}.

We have observed that induced acyclic graphoidal decomposition and simple acyclic graphoidal decomposition are special cases of acyclic graphoidal decomposition. Further, for a given graph G any simple acyclic graphoidal decomposition of G will be an induced acyclic graphoidal decomposition and any induced acyclic graphoidal decomposition of G will be an acyclic graphoidal decomposition of G. Hence we have the following result.

Theorem 3.1. *For any graph G, we have $\eta_a(G) \leqslant \eta_{ia}(G) \leqslant \eta_{as}(G)$.*

Remark 3.2. The inequalities given in the above theorem are strict. That is, all can be equal or they are all distinct as evident from the graphs given in Figure 5.

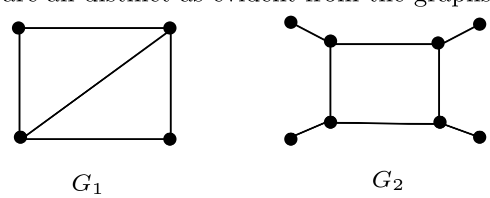

Figure 5

For the graph G_1, we see that $\eta_a(G_1) = 2$, $\eta_{ia}(G_1) = 3$ and $\eta_{as}(G_1) = 4$. On the other hand all the three parameters are equal to 4 if we consider the graph G_2.

From the following theorems we infer that the differences of η_{ia} with each of the graphoidal decomposition parameters η_a and η_{as} can be arbitrarily large.

Theorem 3.3. *Given any positive integer k, there exists a graph G such that $\eta_{ia}(G) - \eta_a(G) = k$.*

Proof. Consider the graph G given in Figure 6 obtained by attaching k triangles at a single vertex, say v. Then G is of order $2k + 1$ and size $3k$.

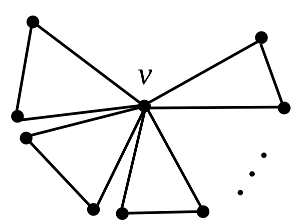

Figure 6

Now, if $k = 1$, then $G = K_3$ so that $\eta_a(G) = 2$ and $\eta_{ia}(G) = 3$. Further, when $n \geqslant 2$ for any *IAGD* of G the only vertex that can be made interior is the vertex v and therefore from Theorem 1.1 it follows that $\eta_{ia} = 3k - 1$. A similar argument shows that $\eta_a(G) = 3k - k - 1$ and hence $\eta_{ia}(G) - \eta_a(G) = k$. □

Theorem 3.4. *Given any positive integer k, there exists a graph G such that $\eta_{as}(G) - \eta_{ia}(G) = k$.*

Proof. Consider the graph G given in Figure 7 obtained by attaching k cycles of length four at one single vertex v.

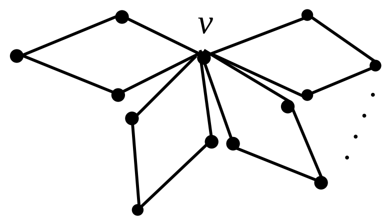

Figure 7

By the similar argument given in the proof of the theorem above, we can prove that $\eta_{as}(G) - \eta_{ia}(G) = k$. □

4. *IAGD* and Helly Property

Recall that a collection $\mathcal{F}$ of subsets of a non-empty set has the *Helly Property* if the members of each pairwise intersecting subfamily $\mathcal{F}'$ of $\mathcal{F}$ has an element in common. Suppose $\mathcal{F}$ is a collection of paths in a graph G. Then $\mathcal{F}$ has the Helly property means that whenever any two paths in a subfamily $\mathcal{F}'$ of $\mathcal{F}$ have a vertex in common, there must be a vertex common to all the paths in $\mathcal{F}'$.

For example, if G is the graph given in Figure 8, the collection $\psi_1 = \{(v_1, v_2, v_4, v_3), (v_4, v_5, v_8), (v_2, v_6, v_7)\}$ has the Helly property, whereas the collection $\psi_2 = \{(v_1, v_2, v_4), (v_4, v_5, v_8), (v_2, v_6, v_5)\}$ does not have the Helly property.

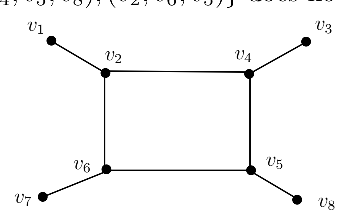

Figure 8

In this section we investigate the graphs admitting an induced acyclic graphoidal decomposition that satisfies the Helly property.

Theorem 4.1. *Every triangle-free graph admits an IAGD satisfying the Helly property.*

Proof. Let G be a graph with no triangles. Consider the *IAGD* $\psi = |E(G)|$, the edge set of G. Then the members of any pairwise intersecting subfamily of ψ form a star in G and so the center vertex of this star is a vertex common to all the members of the subfamily. Hence ψ satisfies the Helly property.

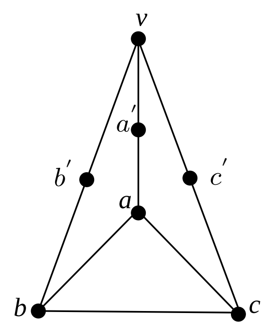

Figure 9

□

Corollary 4.2. *Every bipartite graph admits an IAGD satisfying the Helly property.*

Note that the converse of Theorem 4.1 is not true. For example, the graph given in Figure 9 contains a triangle. But the collection $\psi = \{(a, b, b', v), (b, c, c', v), (c, a, a', v)\}$ is an *IAGD* satisfying the Helly property.

The following observation was made in the book by Balakrishnan and Ranganathan [**5**].

Theorem 4.3 ([**5**]). *Every family of subtrees of a tree satisfies the Helly property.*

It is obvious from the above theorem that every *IAGD* of a tree satisfies the Helly property. In this connection, it is quiet natural to ask the question: " Are there any other classes of graphs wherein every *IAGD* satisfies the Helly property?". The following theorem answers this question.

Theorem 4.4. *Every induced acyclic graphoidal decomposition of a graph G satisfies the Helly property if and only if G is a tree.*

Proof. Suppose G is a graph in which every *IAGD* satisfies the Helly property. Let G contains a cycle $C = (v_1, v_2, \dots, v_k, v_1)$ where $k \geqslant 3$. Let $P_1 = (v_1, v_2)$, $P_2 = (v_2, v_3)$ and $P_3 = (v_3, v_4, \dots, v_k, v_1)$. Then $\psi = \{P_1, P_2, P_3\} \cup \{E(G) \setminus E(C)\}$ is an *IAGD* of G. Clearly, $\{P_1, P_2, P_3\}$ is a pairwise intersecting subfamily of paths in ψ, whereas there exists no vertex in G common to the paths P_1, P_2 and P_3. Thus ψ does not satisfy the Helly property, which is a contradiction. Therefore, G is acyclic and hence is a tree. The converse follows from Theorem 4.3. □

As seen in the proof of Theorem 4.1, in the absence of triangles, the edge set serves as a required *IAGD* with the Helly property. Certainly, the edge set will not be a minimum *IAGD* for a graph, of course other than complete graphs. So, looking for graphs admitting a minimum *IAGD* with the Helly property would be an interesting problem. Certainly, trees are such graphs as seen in Theorem 4.4. Here, we provide more classes of such graphs.

Example 4.5. If G is the graph obtained from a cycle C of length greater than three by attaching a path of any length to every vertex of C, then G has a minimum *IAGD* satisfying the Helly property.

Proof. Let $C = (v_1, v_2, \dots, v_k, v_1)$, where $k \geqslant 4$. Let $Q_1, Q_2, \dots, Q_k$ be the paths attached to the vertices $v_1, v_2, \dots, v_k$ respectively with v_i as the terminal vertex. Now, for each $i = 1, 2, \dots, k-1$, let Q'_i be the path consisting of Q_i followed by the edge $v_i v_{i+1}$ and let Q'_k be the path Q_k followed by the edge $v_k v_1$. Then $\psi = \{Q'_1, Q'_2, \dots, Q'_k\}$ is a minimum *IAGD* satisfying the Helly property as any pairwise intersecting subfamily of paths in ψ contains at most two paths. □

Example 4.6. Let G be a graph consisting of exactly two vertices of degree $\Delta \geqslant 3$, say u and v, and all other vertices are of degree two as in Figure 10. Denote the edge-disjoint $u - v$ paths with u as origin by $P_1, P_2, \dots, P_\Delta$. Also, assume that at least one of these paths, say the path P_1 has length more than three and all others have length at least two. Let x be an internal vertex of P_1.

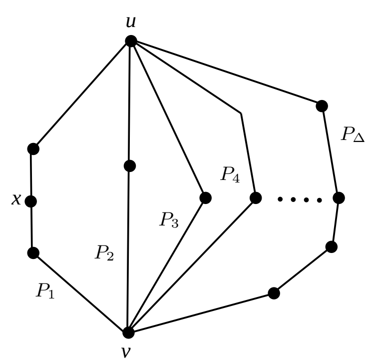

Figure 10

Let Q_1 be the path consisting of the $(x, u)-$section of P_1 followed by the path P_2 and let Q_2 be the path consisting of the $(x, v)-$section of P_1 followed by the path ${P_3}^{-1}$. Then Q_1 and Q_2 are induced paths and consequently $\psi = \{Q_1, Q_2, P_4, P_5, \dots, P_\Delta\}$ is a minimum *IAGD* satisfying the Helly property.

5. Open Problems

Here we list some interesting problems for further investigation.

(1) Find a characterization of graphs G for which
   (i) $\eta_{ia}(G) = m - n$.
   (ii) $\eta_{ia}(G) = \Delta(G) - 1$.
   (iii) $\eta_{ia}(G) = \binom{\omega}{2}$.

(2) In the case of trees, the parameters η_a, η_{ia} and η_{as} are equal and consequently the class of graphs for which these parameters are equal is non-empty. In this direction, determination of classes of graphs with $\eta_a = \eta_{ia} = eta_{as}$ would be interesting although little challenging.

(3) We have observed that triangle-free graphs admit an $IAGD$ satisfying the Helly Property and however the following problems are left open.
   (i) Obtain a necessary and sufficient condition for a graph to admit an $IAGD$ satisfy the Helly property.
   (ii) What are the graphs admitting a minimum $IAGD$ satisfy the Helly property?
   (iii) What are the graphs where every minimum $IAGD$ satisfy the Helly property?

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References

[1] B. D. Acharya and E. Sampathkumar, Graphoidal covers and graphoidal covering number of a graph, *Indian J.pure appl.Math.*, **18(10)**, (1987), 882 – 890.
[2] S. Arumugam, Path covers in graphs, Lecture Notes of the National Workshop on Decompositions of Graphs and Product Graphs held at Annamalai University, Tamil Nadu, during January 3-7, (2006).
[3] S. Arumugam and I. Sahul Hamid, Simple acyclic graphoidal covers in a graph, Australasian Journal of Combinatorics, **37**(2007), 243-255.
[4] S. Arumugam and J. Suresh Suseela, Acyclic graphoidal covers and path partitions in a graph, *Discrete Mathematics*, **190** (1998), 67 – 77.
[5] R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*, Springer-Verlag, New York, (2000).
[6] G. Chartrand and L. Lesniak, Graphs and Digraphs, Fourth Edition, CRC Press, Boca Raton (2004).
[7] F. Harary, Covering and packing in graphs I, *Ann. N. Y. Acad. Sci.*, **175** (1970), 198 – 205.

[8] F. Harary and A.J. Schwenk, Evolution of the path number of a graph, covering and packing in graphs II, Graph Theory and Computing, Ed. R. C. Rad, Academic Press, New York, (1972), 39 – 45.

[9] K. Ratan Singh and P. K. Das, Induced acyclic graphoidal covers in a graph, *World Academy of Science, Engineering and Technology*, **(68)** (2010), 38 – 44.

[10] I. Sahul Hamid and M. Joseph, Further Results on Induced Graphoidal Decomposition, *Discrete Mathematics, Algorithms and Applications*, **5**(1), (2013), 1350006, doi: 10.1142/S1793830913500067.

[11] N. Teypaz and C. Rapine, Graph Decomposition into Paths under Length Constraints, hal-00278189, version 1-9, May 2008, (Available at http://hal.archives-ouvertes.fr/docs/00/27/81/89/PDF/article.pdf).

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