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Some dual definite integrals for Bessel functions of the first kind

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Abstract. Based on known definite integrals of Bessel functions of the first kind, we obtain exact solutions to unknown definite integrals using the method of integral transforms from Hankel's transform.

1. Introduction

In Cohl (2012) [**1**], orthogonality and Hankel's transform are used to generate solutions to new definite integrals based on known integrals. In this paper, we use the method of integral transforms to create new integrals from a variety of known integrals containing Bessel functions of the first kind J_ν. In this method, we use the closure relation for Bessel functions of the first kind to generate a guess for a function to use in Hankel's transform. This guess may be incorrect if it does not satisfy the first condition for Hankel's transform, in which case a new definite integral is not generated. If the guess satisfies the condition, restrictions on ν must then be adjusted to satisfy the other condition of Hankel's transform. For the functions used in this paper, superscripts and subscripts refer to lists of parameters. Any exceptions to this will be clear from context.

As far as we are aware, the 40 definite integrals over Bessel functions of the first kind that we present in this manuscript, do not currently appear in the literature. An extension of the survey presented in this manuscript can be used to mechanically compute new definite integrals from pre-existing definite integrals over Bessel functions of the first kind.

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1.1. Application of Hankel's transform. We use the following result where for $x \in (0,\infty)$ we define

$$F(r \pm 0) := \lim_{x \to r\pm} F(x);$$

see Watson (1944) [**7**, p. 456].

Theorem 1.1. *Let $F : (0,\infty) \to \mathbf{C}$ be such that*

$$\int_0^\infty \sqrt{x}\,|F(x)|\,dx < \infty, \tag{1.1}$$

and let $\nu \geqslant -\frac{1}{2}$. Then

$$\frac{1}{2}(F(r+0)+F(r-0)) = \int_0^\infty uJ_\nu(ur)\int_0^\infty xF(x)J_\nu(ux)\,dx\,du \tag{1.2}$$

provided that the positive number r lies inside an interval in which $F(x)$ has finite variation.

The effort described in this paper was motivated by the large collection of Bessel function of the first kind integrals which exist in the book "Table of Integrals, Series, and Products" [**4**]. Not counting Theorem 2.1 (which stands alone), the method of integral transforms was applied to the Bessel function definite integrals appearing in Sections 6.51 and 6.52 of [**4**]. This method can be applied to many definite integrals appearing in Sections 6.5-6.7 of [**4**].

For the definite integrals presented in this manuscript, we have directly verified that (1.1) is satisfied. This is easily accomplished by analyzing the behavior of the integrands in a small neighborhood of the endpoints $\{0,\infty\}$. For this paper, this technique produced 30 theorems including 40 definite integrals which are given below. The method of integral transforms does not always succeed in producing new definite integrals because the conditions on the Hankel transform (1.1) is not satisfied. Some cases of this are shown in Section 7.

2. Polynomial, rational, algebraic, and power functions

Theorem 2.1. *Let $b,c > 0$, $\nu > -\frac{1}{2}$, $t \in \mathbf{C} \setminus (-\infty,0]$. Then*

$$\int_0^\infty (\Delta(a,b,c))^{2\nu-1}a^{1-\nu}J_\nu(at)da = 2^{1-\nu}\sqrt{\pi}\,\Gamma\left(\nu+\tfrac{1}{2}\right)\left(\tfrac{bc}{t}\right)^\nu J_\nu(bt)J_\nu(ct), \tag{2.1}$$

*where $\Delta : [0,\infty)^3 \to [0,\infty)$ (Heron's formula [**5**]), defined by*

$$\Delta(a,b,c) := \sqrt{s(s-a)(s-b)(s-c)},$$

$s = (a+b+c)/2$, is the area of a triangle with sides of length a, b, and c.

Proof. We apply Theorem 1.1 to the function $F_\nu^{b,c} : (0,\infty) \to \mathbf{C}$ defined by

$$F_\nu^{b,c}(t) := 2^{1-\nu}\sqrt{\pi}\,\Gamma\left(\nu+\tfrac{1}{2}\right)\left(\tfrac{bc}{t}\right)^\nu J_\nu(bt)J_\nu(ct),$$

where $\Gamma : \mathbf{C} \setminus -\mathbf{N}_0 \to \mathbf{C}$ is Euler's gamma function defined in [**3**, (5.2.1)], and $J_\nu : \mathbf{C} \setminus (-\infty, 0] \to \mathbf{C}$, (order) $\nu \in \mathbf{C}$, is the Bessel function of the first kind defined in [**3**, (10.2.2)]. The desired result is obtained from Sonine's formula [**6**]

$$\int_0^\infty J_\nu(at)J_\nu(bt)J_\nu(ct)t^{1-\nu}dt = \frac{2^{\nu-1}(\Delta(a,b,c))^{2\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)(abc)^\nu},$$

where $\operatorname{Re} a > 0$, $b,c > 0$, $\operatorname{Re}\nu > -\frac{1}{2}$. □

THEOREM 2.2. *Let* $\nu > \frac{1}{2}$, $\mu > 0$, $\alpha,\beta > 0$, $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\int_0^\alpha b^\nu J_{\nu-1}(bz)\, db = \alpha^\nu z^{-1} J_\nu(\alpha z), \tag{2.2}$$

$$\int_\beta^\infty a^{1-\mu} J_\mu(az)\, da = \beta^{1-\mu} z^{-1} J_{\mu-1}(\beta z). \tag{2.3}$$

Proof. By applying Theorem 1.1 to the functions $F_\nu^\alpha : (0,\infty) \to \mathbf{C}$ and $G_\mu^\beta : (0,\infty) \to \mathbf{C}$ defined by $F_\nu^\alpha(x) := \alpha^\nu x^{-1} J_\nu(\alpha x)$, $G_\mu^\beta(x) := \beta^{1-\mu} x^{-1} J_{\mu-1}(\beta x)$, we obtain the desired results from the known integral [**4**, (6.512.3)]

$$\int_0^\infty J_\nu(\alpha x)J_{\nu-1}(\beta x)dx = \begin{cases} \alpha^{-\nu}\beta^{\nu-1} & \text{if } \beta < \alpha, \\ (2\beta)^{-1} & \text{if } \beta = \alpha, \\ 0 & \text{if } \beta > \alpha, \end{cases}$$

where $\operatorname{Re}\nu > 0$. □

THEOREM 2.3. *Let* $\nu \geqslant -\frac{1}{2}$, $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\int_0^\infty \frac{c^{\nu+1}}{1+c^2} J_\nu(cz)dc = K_\nu(z). \tag{2.4}$$

Proof. We are given the integral [**4**, (6.521.2)]

$$\int_0^\infty xK_\nu(ax)J_\nu(bx)dx = \frac{b^\nu}{a^\nu(b^2+a^2)},$$

where $\operatorname{Re} a > 0$, $b > 0$, $\operatorname{Re}\nu > -1$. By applying Theorem 1.1 to the function $F_\nu^a : (0,\infty) \to \mathbf{C}$ defined by $F_\nu^a(x) := a^\nu K_\nu(ax)$, we obtain the following integral

$$\int_0^\infty \frac{b^{\nu+1}}{b^2+a^2} J_\nu(bx)db = a^\nu K_\nu(ax),$$

where $\operatorname{Re} a > 0$, $\nu \geqslant -\frac{1}{2}$, $x \in \mathbf{C} \setminus (-\infty, 0]$. With the substitutions $z = ax$ and $c = b/a$, we obtain the desired result. □

Note that when the method of integral transforms is applied to $F_a(x) := aK_1(ax)$ given the integral [**4**, (6.521.7)]

$$\int_0^\infty xK_1(ax)J_1(bx) = \frac{b}{a(a^2+b^2)},$$

where $a, b > 0$, we obtain the integral generated from [**4**, (6.521.2)] when $\nu = 1$.

THEOREM 2.4. *Let* $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\int_0^\infty \frac{c}{(1+c^2)^2} J_0(cz)\, dc = \frac{z}{2} K_1(z). \tag{2.5}$$

Proof. We are given the integral [**4**, (6.521.12)]

$$\int_0^\infty x^2 K_1(ax) J_0(bx) = \frac{2a}{(a^2+b^2)^2},$$

where $a > b > 0$. By applying Theorem 1.1 to the function $F_a : (0,\infty) \to \mathbf{C}$ defined by $F_a(x) := \frac{x}{2a} K_1(ax)$, we obtain the following integral

$$\int_0^\infty \frac{bJ_0(bx)}{(a^2+b^2)^2} db = \frac{x}{2a} K_1(ax),$$

where $a > 0$, $x \in \mathbf{C} \setminus (-\infty, 0]$. With the substitutions $z = ax$ and $c = b/a$, we obtain the desired result. ☐

THEOREM 2.5. *Let* $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\int_0^\infty \frac{c^2}{(1+c^2)^2} J_1(cz) dc = \frac{z}{2} K_0(z). \tag{2.6}$$

Proof. We are given the integral [**4**, (6.521.12)]

$$\int_0^\infty x^2 K_0(ax) J_1(bx) dx = \frac{2b}{(a^2+b^2)^2},$$

where a,$b > 0$. By applying Theorem 1.1 to the function $F_a : (0,\infty) \to \mathbf{C}$ defined by $F_a(x) := \frac{x}{2} K_0(ax)$, we obtain the following integral

$$\int_0^\infty \frac{b^2 J_1(bx)}{(a^2+b^2)^2} db = \frac{x}{2} K_0(ax),$$

where $a > 0$, $x \in \mathbf{C} \setminus (-\infty, 0]$. With the substitutions $z = ax$ and $c = b/a$, we obtain the desired result. ☐

THEOREM 2.6. *Let* $\gamma > 0$, $\nu \geqslant -\frac{1}{2}$, $\operatorname{Re}\alpha > |\operatorname{Im}\beta|$, $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\int_0^\infty bJ_\nu(bz) \frac{l_1^\nu}{l_2^\nu (l_2^2 - l_1^2)} db = K_0(\alpha z) J_\nu(\gamma z), \tag{2.7}$$

$$\int_0^\infty cJ_\nu(cz) \frac{l_1^\nu}{l_2^\nu (l_2^2 - l_1^2)} dc = K_0(\alpha z) J_\nu(\beta z), \tag{2.8}$$

where l_1 *and* l_2 *are defined as*

$$l_1 := \frac{1}{2}\left[\sqrt{(b+c)^2 + a^2} - \sqrt{(b-c)^2 + a^2}\right], \tag{2.9}$$

$$l_2 := \frac{1}{2}\left[\sqrt{(b+c)^2 + a^2} + \sqrt{(b-c)^2 + a^2}\right]. \tag{2.10}$$

Proof. By applying Theorem 1.1 to the function $F_\nu^{a,c} : (0,\infty) \to \mathbf{C}$ and $G_\nu^{a,b} : (0,\infty) \to \mathbf{C}$ defined by $F_\nu^{a,c}(x) := K_0(ax)J_\nu(cx)$, $G_\nu^{a,b}(x) := K_0(ax)J_\nu(bx)$, we obtain the desired result from the known integral [**4**, (6.522.12)]

$$\int_0^\infty xK_0(ax)J_\nu(bx)J_\nu(cx)dx = \frac{l_1^\nu}{l_2^\nu(l_2^2 - l_1^2)},$$

where $c > 0$, $\operatorname{Re}\nu > -1$, $\operatorname{Re} a > |\operatorname{Im} b|$. □

THEOREM 2.7. *Let* $\operatorname{Re} b > \operatorname{Re} a$, $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\int_0^\infty \frac{cJ_0(cz)}{(a^4 + b^4 + c^4 - 2a^2b^2 + 2a^2c^2 + 2b^2c^2)^{1/2}} dc = I_0(az)K_0(bz). \tag{2.11}$$

Let $\operatorname{Re} b > \operatorname{Re} c$, $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\int_0^\infty \frac{aJ_0(az)}{l_2^2 - l_1^2} da = I_0(cz)K_0(bz). \tag{2.12}$$

where l_1 *and* l_2 *are defined in* (2.9) *and* (2.10).

Proof. By applying Theorem 1.1 to the function $F_a^b : (0,\infty) \to \mathbf{C}$ and $G_c^b : (0,\infty) \to \mathbf{C}$ defined by $F_a^b(x) := I_0(ax)K_0(bx)$, $G_c^b(x) := I_0(cx)K_0(bx)$, we obtain the desired results from the known integrals (see [**4**, (6.522.4)])

$$\int_0^\infty xI_0(ax)K_0(bx)J_0(cx)dx = (a^4 + b^4 + c^4 - 2a^2b^2 + 2a^2c^2 + 2b^2c^2)^{-1/2},$$

where $\operatorname{Re} b > \operatorname{Re} a$, $c > 0$, and

$$\int_0^\infty xI_0(cx)K_0(bx)J_0(ax)dx = \frac{1}{l_2^2 - l_1^2},$$

where $\operatorname{Re} b > \operatorname{Re} c$, $a > 0$, respectively. □

THEOREM 2.8. *Let* $\gamma > 0$, $\nu \geqslant -\frac{1}{2}$, $\operatorname{Re}\alpha > |\operatorname{Im}\beta|$, $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\int_0^\infty \frac{b^{\nu+1}}{(l_2^2 - l_1^2)^{2\nu+1}} J_\nu(bz)db = \frac{z^\nu(\alpha\gamma)^{-\nu}\sqrt{\pi}}{2^{3\nu}\,\Gamma(\nu + \frac{1}{2})} K_\nu(\alpha z)J_\nu(\gamma z), \tag{2.13}$$

$$\int_0^\infty \frac{c^{\nu+1}}{(l_2^2 - l_1^2)^{2\nu+1}} J_\nu(cz)dc = \frac{z^\nu(\alpha\beta)^{-\nu}\sqrt{\pi}}{2^{3\nu}\,\Gamma(\nu + \frac{1}{2})} K_\nu(\alpha z)J_\nu(\beta z), \tag{2.14}$$

where l_1 *and* l_2 *are defined in* (2.9) *and* (2.10).

Proof. By applying Theorem 1.1 to the functions $F_\nu^{a,c} : (0,\infty) \to \mathbf{C}$, $G_\nu^{a,b} : (0,\infty) \to \mathbf{C}$, defined by

$$F_\nu^{a,c}(x) := \frac{x^\nu(ac)^{-\nu}\sqrt{\pi}}{2^{3\nu}\,\Gamma(\nu + \frac{1}{2})} K_\nu(ax)J_\nu(cx),$$

$$G_\nu^{a,b}(x) := \frac{x^\nu(ab)^{-\nu}\sqrt{\pi}}{2^{3\nu}\,\Gamma(\nu + \frac{1}{2})} K_\nu(ax)J_\nu(bx),$$

we obtain the desired results from the known integral [**4**, (6.522.15)]

$$\int_0^\infty x^{\nu+1} J_\nu(bx) K_\nu(ax) J_\nu(cx)dx = \frac{2^{3\nu}(abc)^\nu\,\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\,(l_2^2-l_1^2)^{2\nu+1}},$$

where $\operatorname{Re} a > |\operatorname{Im} b|$, $c > 0$. ☐

THEOREM 2.9. *Let* $\gamma > 0$, $\operatorname{Re}\beta \geqslant |\operatorname{Im}\alpha|$, $\operatorname{Re}\alpha > 0$, $\operatorname{Re} p > |\operatorname{Re} q|$, $\operatorname{Re} q > 0$, $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\int_0^\infty \frac{a^2 J_1(az)(a^2+\beta^2-\gamma^2)}{[(a^2+\beta^2+\gamma^2)^2-4a^2\gamma^2]^{3/2}} da = \frac{z}{2} K_0(\beta z) J_0(\gamma z), \tag{2.15}$$

$$\int_0^\infty \frac{cJ_0(cz)(\alpha^2+\beta^2-c^2)}{[(\alpha^2+\beta^2+c^2)^2-4\alpha^2c^2]^{3/2}} dc = \frac{z}{2\alpha} J_1(\alpha z) K_0(\beta z). \tag{2.16}$$

$$\int_0^\infty J_1(bz) \frac{2b^2(p^2+b^2-\gamma^2)}{(l_2^2-l_1^2)^3} db = zK_0(pz)J_0(\gamma z), \tag{2.17}$$

$$\int_0^\infty J_0(cz) \frac{c(p^2+q^2-c^2)}{(l_2^2-l_1^2)^3} dc = \frac{z}{2q} J_1(qz) K_0(pz), \tag{2.18}$$

where l_1 *and* l_2 *are defined in* (2.9) *and* (2.10).

Proof. By applying Theorem 1.1 to the functions $F_b^c : (0,\infty) \to \mathbf{C}$, $G_a^b : (0,\infty) \to \mathbf{C}$, $H_a^c : (0,\infty) \to \mathbf{C}$, $I_b^a : (0,\infty) \to \mathbf{C}$ defined by $F_b^c(x) := xK_0(bx)J_0(cx)$, $G_a^b(x) := \frac{x}{2a} J_1(ax)K_0(bx)$, $H_a^c(x) := xK_0(ax)J_0(cx)$, $I_b^a(x) := \frac{x}{2b} J_1(bx)K_0(ax)$, we obtain the desired results from the known integrals (see [**4**, (6.525.1)])

$$\int_0^\infty x^2 J_1(ax) K_0(bx) J_0(cx)dx = \frac{2a(a^2+b^2-c^2)}{[(a^2+b^2+c^2)^2-4a^2c^2]^{3/2}},$$

where $c > 0$, $\operatorname{Re} b \geqslant |\operatorname{Re} a|$, $\operatorname{Re} a > 0$,

$$\int_0^\infty x^2 J_1(bx) K_0(ax) J_0(cx)dx = \frac{2b(a^2+b^2-c^2)}{(l_2^2-l_1^2)^3},$$

where $c > 0$, $\operatorname{Re} a > |\operatorname{Im} b|$, $\operatorname{Re} b > 0$. ☐

THEOREM 2.10. *Let* $\operatorname{Re} a > 0$, $\nu \geqslant -\frac{1}{2}$, $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\int_0^\infty \frac{J_\nu(bz)}{\sqrt{b^2+4a^2}} db = I_{\nu/2}(az) K_{\nu/2}(az). \tag{2.19}$$

Proof. By applying Theorem 1.1 to the function $F_\nu^a : (0,\infty) \to \mathbf{C}$ defined by

$$F_\nu^a(x) := I_{\nu/2}(ax) K_{\nu/2}(ax),$$

we obtain the desired result from the known integral [**4**, (6.522.9)]

$$\int_0^\infty x I_{\nu/2}(ax) K_{\nu/2}(ax) J_\nu(bx)dx = b^{-1}(b^2+4a^2)^{-1/2},$$

where $b > 0$, $\operatorname{Re} a > 0$, $\operatorname{Re}\nu > -1$. ☐

THEOREM 2.11. *Let* $a > 0$, $\nu \geqslant -\frac{1}{2}$, $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\int_{2a}^{\infty} \frac{J_\nu(bz)}{\sqrt{b^2 - 4a^2}} db = -\frac{\pi}{2} J_{\nu/2}(az) Y_{\nu/2}(az). \tag{2.20}$$

Proof. By applying Theorem 1.1 to the function $F_\nu^a : (0, \infty) \to \mathbf{C}$ defined by

$$F_\nu^a(x) := -\frac{\pi}{2} J_{\nu/2}(ax) Y_{\nu/2}(ax),$$

we obtain the desired result from the known integral [**4**, (6.522.10)]

$$\int_0^\infty x J_{\nu/2}(ax) Y_{\nu/2}(ax) J_\nu(bx) dx = \begin{cases} 0 & \text{if } 0 < b < 2a, \\ -2\pi^{-1} b^{-1} (b^2 - 4a^2)^{-1/2} & \text{if } 2a < b, \end{cases}$$

where $\operatorname{Re} \nu > -1$. □

THEOREM 2.12. *Let* $\operatorname{Re} a > 0$, $\nu \geqslant -\frac{1}{2}$, $\operatorname{Re} \mu \geqslant \frac{3}{2}$, $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\int_0^\infty \frac{J_\nu(bz)}{\sqrt{b^2 + 4a^2}} \left[b + (b^2 + 4a^2)^{1/2} \right]^\mu db = 2^\mu a^\mu I_{(\nu-\mu)/2}(az) K_{(\nu+\mu)/2}(az). \tag{2.21}$$

Proof. By applying Theorem 1.1 to the function $F_\nu^{\mu,a} : (0, \infty) \to \mathbf{C}$ defined by

$$F_\nu^{\mu,a}(x) := 2^\mu a^\mu I_{(\nu-\mu)/2}(ax) K_{(\nu+\mu)/2}(ax),$$

we obtain the desired result from the known integral [**4**, (6.522.12)]

$$\int_0^\infty x I_{(\nu-\mu)/2}(ax) K_{(\nu+\mu)/2}(ax) J_\nu(bx) dx = 2^{-\mu} a^{-\mu} b^{-1} (b^2 + 4a^2)^{-1/2} \left[b + (b^2 + 4a^2)^{1/2} \right]^\mu,$$

where $\operatorname{Re} a > 0$, $b > 0$, $\operatorname{Re} \nu > -1$, $\operatorname{Re}(\nu - \mu) > -2$. □

THEOREM 2.13. *Let* $|\operatorname{Re} a| < \operatorname{Re} b$, $x \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\int_0^\infty \frac{c J_0(cx)(b^2 + c^2 - a^2)}{[(a^2 + b^2 + c^2)^2 - 4a^2 b^2]^{3/2}} dc = \frac{x}{2b} I_0(ax) K_1(bx). \tag{2.22}$$

Proof. By applying Theorem 1.1 to the function $F_a^b : (0, \infty) \to \mathbf{C}$ defined by $F_a^b(x) := \frac{x}{2b} I_0(ax) K_1(bx)$, we obtain the desired result from the known integral [**4**, (6.525.2)]

$$\int_0^\infty x^2 I_0(ax) K_1(bx) J_0(cx) dx = 2b(b^2 + c^2 - a^2) \left[(a^2 + b^2 + c^2)^2 - 4a^2 b^2 \right]^{-3/2},$$

where $\operatorname{Re} b > |\operatorname{Re} a|$, $c > 0$. □

3. Bessel and Struve functions

THEOREM 3.1. *Let $\nu > 0$, $z \in \mathbf{C} \setminus (-\infty, 0]$. Then*

$$\int_0^\infty J_\nu(cz) J_{2\nu}\left(2\sqrt{c}\right) dc = \frac{1}{z} J_\nu\left(\frac{1}{z}\right). \tag{3.1}$$

Proof. We are given the integral [**4**, (6.514.1)]

$$\int_0^\infty J_\nu\left(\frac{a}{x}\right) J_\nu(bx) dx = b^{-1} J_{2\nu}\left(2\sqrt{ab}\right),$$

where $\operatorname{Re}\nu > 0$, $a,b > 0$. By applying Theorem 1.1 to the function $F_\nu^a : (0,\infty) \to \mathbf{C}$ defined by $F_\nu^a(x) := x^{-1} J_\nu\left(ax^{-1}\right)$, we obtain the following integral

$$\int_0^\infty J_\nu(bx) J_{2\nu}\left(2\sqrt{ab}\right) db = x^{-1} J_\nu\left(\frac{a}{x}\right),$$

where $\nu > 0$, $a > 0$, $x \in \mathbf{C} \setminus (-\infty, 0]$. By making the substitutions $x = az$, $c = ba$, we obtain the desired result. □

THEOREM 3.2. *Let $-\frac{1}{2} \leqslant \nu < \frac{5}{2}$, $z \in \mathbf{C} \setminus (-\infty, 0]$. Then*

$$\int_0^\infty c J_\nu(cz) \left[e^{i(\nu+1)\pi/2} K_{2\nu}\left(2e^{i\pi/4}\sqrt{c}\right) + e^{-i(\nu+1)\pi/2} K_{2\nu}\left(2e^{-i\pi/4}\sqrt{c}\right)\right] dc = \frac{1}{z^3} K_\nu\left(\frac{1}{z}\right). \tag{3.2}$$

Proof. We are given the integral [**4**, (6.514.3)]

$$\int_0^\infty J_\nu\left(\frac{a}{x}\right) K_\nu(bx) dx = b^{-1} e^{i(\nu+1)\pi/2} K_{2\nu}\left(2e^{i\pi/4}\sqrt{ab}\right) + b^{-1} e^{-i(\nu+1)\pi/2} K_{2\nu}\left(2e^{-i\pi/4}\sqrt{ab}\right),$$

where $a > 0$, $\operatorname{Re} b > 0$, $|\operatorname{Re}\nu| < \frac{5}{2}$. By applying Theorem 1.1 to the function $F_\nu^b : (0,\infty) \to \mathbf{C}$ defined by $F_\nu^b(x) := bx^{-3} K_\nu(bx^{-1})$, we obtain the following integral

$$\int_0^\infty a J_\nu(ax) \left[e^{i(\nu+1)\pi/2} K_{2\nu}\left(2e^{i\pi/4}\sqrt{ab}\right) + e^{-i(\nu+1)\pi/2} K_{2\nu}\left(2e^{-i\pi/4}\sqrt{ab}\right)\right] da = \frac{b}{x^3} K_\nu\left(\frac{b}{x}\right),$$

where $\operatorname{Re} b > 0$, $-\frac{1}{2} \leqslant \nu < \frac{5}{2}$, $x \in \mathbf{C} \setminus (-\infty, 0]$. With the substitutions $x = bz$, $c = ba$, we obtain the desired result. □

THEOREM 3.3. *Let $|\nu| < \frac{1}{2}$, $z \in \mathbf{C} \setminus (-\infty, 0]$. Then*

$$\int_0^\infty J_\nu(cz) \left[K_{2\nu}\left(2\sqrt{c}\right) - \frac{\pi}{2} Y_{2\nu}\left(2\sqrt{c}\right)\right] dc = -\frac{\pi}{2z} Y_\nu\left(\frac{1}{z}\right). \tag{3.3}$$

Proof. We are given the integral [**4**, (6.514.4)]

$$\int_0^\infty Y_\nu\left(\frac{a}{x}\right) J_\nu(bx) dx = -\frac{2b^{-1}}{\pi} \left[K_{2\nu}\left(2\sqrt{ab}\right) - \frac{\pi}{2} Y_{2\nu}\left(2\sqrt{ab}\right)\right],$$

where $a,b > 0$, $|\mathrm{Re}\,\nu| < \frac{1}{2}$. By applying Theorem 1.1 to the function $F_\nu^a : (0,\infty) \to \mathbf{C}$ defined by

$$F_\nu^a(x) := -\frac{\pi}{2x} Y_\nu\left(\frac{a}{x}\right),$$

we obtain the following integral

$$\int_0^\infty J_\nu(bx)\left[K_{2\nu}\left(2\sqrt{ab}\right) - \frac{\pi}{2} Y_{2\nu}\left(2\sqrt{ab}\right)\right] db = -\frac{\pi}{2x} Y_\nu\left(\frac{a}{x}\right),$$

where $a > 0$, $|\nu| < \frac{1}{2}$, $x \in \mathbf{C} \setminus (-\infty, 0]$. With the substitutions $x = az$, $c = ab$, we obtain the desired result. □

THEOREM 3.4. *Let* $\nu \geqslant -\frac{1}{4}$, $\mu > -\frac{1}{2}$, $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\int_0^\infty J_{2\nu}(cz) J_\nu\left(\frac{c^2}{4}\right) c\,dc = 2J_\nu\left(z^2\right), \tag{3.4}$$

$$\int_0^\infty J_\mu(cz) J_\mu\left(\frac{1}{4c}\right) dc = z^{-1} J_{2\mu}\left(\sqrt{z}\right). \tag{3.5}$$

Proof. We are given the integral [**4**, (6.516.1)]

$$\int_0^\infty J_{2\nu}\left(a\sqrt{x}\right) J_\nu(bx) dx = b^{-1} J_\nu\left(\frac{a^2}{4b}\right),$$

where $\mathrm{Re}\,\nu > -\frac{1}{2}$, $a,b > 0$. By applying Theorem 1.1 to the functions $F_\nu^b : (0,\infty) \to \mathbf{C}$ and $G_\mu^a : (0,\infty) \to \mathbf{C}$ defined by $F_\nu^b(x) := 2bJ_\nu\left(bx^2\right)$, $G_\mu^a(x) := x^{-1} J_{2\mu}\left(a\sqrt{x}\right)$, we obtain the following integrals

$$\int_0^\infty a J_{2\nu}(ax) J_\nu\left(\frac{a^2}{4\beta}\right) da = 2\beta J_\nu\left(\beta x^2\right),$$

$$\int_0^\infty J_\mu(bx) J_\mu\left(\frac{\alpha^2}{4b}\right) db = x^{-1} J_{2\mu}(\alpha\sqrt{x}),$$

where $\alpha,\beta > 0$, $\nu \geqslant -\frac{1}{4}$, $\mu > -\frac{1}{2}$, $z \in \mathbf{C} \setminus (-\infty, 0]$. With the substitutions $z^2 = bx^2$, $c = a/\sqrt{b}$, and $\sqrt{z} = a\sqrt{x}$, $c = b/a^2$ respectively, we obtain the desired results. □

THEOREM 3.5. *Let* $\nu > -1$, $\mu \geqslant -\frac{1}{2}$, $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\int_0^\infty J_{\nu/2}(cz) J_{\nu/2}\left(\frac{1}{4c}\right) dc = z^{-1} J_\nu\left(\sqrt{z}\right), \tag{3.6}$$

$$\int_0^\infty c J_\mu(cz) J_{\mu/2}\left(\frac{c^2}{4}\right) dc = 2J_{\mu/2}(z^2). \tag{3.7}$$

Proof. We are given the integral [**4**, (6.526.1)]

$$\int_0^\infty x J_{\nu/2}(ax^2) J_\nu(bx) dx = \frac{1}{2a} J_{\nu/2}\left(\frac{b^2}{4a}\right),$$

where $a,b>0$, $\mathrm{Re}\,\nu>-1$. By applying Theorem 1.1 to the function $F^b_\nu:(0,\infty)\to\mathbf{C}$ defined by $F^b_\nu(x):=x^{-1}J_\nu\left(b\sqrt{x}\right)$, we obtain the following integrals

$$\int_0^\infty J_{\nu/2}(ax)J_{\nu/2}\left(\frac{\beta^2}{4a}\right)da=x^{-1}J_\nu\left(\beta\sqrt{x}\right),$$

$$\int_0^\infty bJ_\mu(bx)J_{\mu/2}\left(\frac{b^2}{4\alpha}\right)db=2\alpha J_{\mu/2}(\alpha x^2),$$

where $\alpha,\beta>0$, $\nu>-1$, $\mu\geqslant-\frac{1}{2}$, $x\in\mathbf{C}\setminus(-\infty,0]$. With the substitutions $\sqrt{z}=\beta\sqrt{x}$, $c=a/\beta^2$, and $z^2=\alpha x^2$, $x=z\sqrt{\alpha}$, we obtain the desired results. □

THEOREM 3.6. *Let* $\nu\geqslant-\frac{1}{4}$, $x\in\mathbf{C}\setminus(-\infty,0]$. *Then*

$$\int_0^\infty a^2J_{2\nu}(ax)J_{\nu+1/2}(a^2)da=\frac{x}{4}J_{\nu-1/2}\left(\frac{x^2}{4}\right). \tag{3.8}$$

Proof. By applying Theorem 1.1 to the function $F_\nu:(0,\infty)\to\mathbf{C}$ defined by $F_\nu(x):=\frac{x}{4}J_{\nu-1/2}\left(\frac{x^2}{4}\right)$, we obtain the desired result from the known integral [**4**, (6.527.1)]

$$\int_0^\infty J_{2\nu}(2ax)J_{\nu-1/2}(x^2)dx=\frac{1}{2}a\,J_{\nu+1/2}(a^2),$$

where $a>0$, $\mathrm{Re}\,\nu>-\frac{1}{2}$. □

THEOREM 3.7. *Let* $\nu\geqslant-\frac{1}{4}$, $x\in\mathbf{C}\setminus(-\infty,0]$. *Then*

$$\int_0^\infty a^2J_{2\nu}(ax)J_{\nu-1/2}(a^2)da=\frac{x}{4}J_{\nu+1/2}\left(\frac{x^2}{4}\right). \tag{3.9}$$

Proof. By applying Theorem 1.1 to the function $F_\nu:(0,\infty)\to\mathbf{C}$ defined by $F_\nu(x):=\frac{x}{4}J_{\nu+1/2}\left(\frac{x^2}{4}\right)$, we obtain the desired result from the known integral [**4**, (6.527.1)]

$$\int_0^\infty J_{2\nu}(2ax)J_{\nu+1/2}(x^2)dx=\frac{1}{2}a\,J_{\nu-1/2}(a^2),$$

where $a>0$, $\mathrm{Re}\,\nu>-2$. □

THEOREM 3.8. *Let* $\nu\geqslant-\frac{1}{2}$, $z\in\mathbf{C}\setminus(-\infty,0]$. *Then*

$$\int_0^\infty cJ_\nu(cz)\mathbf{H}_{\nu/2}\left(\frac{c^2}{4}\right)dc=-2Y_{\nu/2}(z^2). \tag{3.10}$$

Proof. We are given the integral [**4**, (6.526.4)]

$$\int_0^\infty xY_{\nu/2}(ax^2)J_\nu(bx)dx=-\frac{1}{2a}\mathbf{H}_{\nu/2}\left(\frac{b^2}{4a}\right),$$

where $a > 0$, $\operatorname{Re} b > 0$, $\operatorname{Re} \nu > -1$ and $\mathbf{H}_\nu : \mathbf{C} \to \mathbf{C}$, for $\nu \in \mathbf{N}_0$, is the Struve function defined in [**3**, (11.2.1)]. By applying Theorem 1.1 to the function $F_\nu^a : (0,\infty) \to \mathbf{C}$ defined by $F_\nu^a(x) := -2aY_{\nu/2}(ax^2)$, we obtain the following integral

$$\int_0^\infty bJ_\nu(bx)\mathbf{H}_{\nu/2}\left(\frac{b^2}{4a}\right)db = -2aY_{\nu/2}(ax^2),$$

where $a > 0$, $\nu \geqslant -\frac{1}{2}$, $x \in \mathbf{C} \setminus (-\infty, 0]$. With the substitutions $z^2 = ax^2$, $c = b/\sqrt{a}$, we obtain the desired result. □

4. Exponential, logarithmic and inverse trigonometric functions

THEOREM 4.1. *Let* $\nu \geqslant -\frac{1}{2}$, $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\int_0^\infty J_\nu(cz)e^{-2/c}c^{-1}dc = 2J_\nu(2\sqrt{z})K_\nu(2\sqrt{z}). \tag{4.1}$$

Proof. We are given the integral [**4**, (6.526.4)]

$$\int_0^\infty xJ_\nu(2\sqrt{ax})K_\nu(2\sqrt{ax})J_\nu(bx)dx = \frac{1}{2}b^{-2}e^{-2a/b},$$

where $\operatorname{Re} a > 0$, $b > 0$, $\operatorname{Re} \nu > -1$. By applying Theorem 1.1 to the function $F_\nu^a : (0,\infty) \to \mathbf{C}$ defined by $F_\nu^a(x) := 2J_\nu(2\sqrt{ax})K_\nu(2\sqrt{ax})$, we obtain the following integral

$$\int_0^\infty b^{-1}J_\nu(bx)e^{-2a/b}db = 2J_\nu(2\sqrt{ax})K_\nu(2\sqrt{ax}),$$

where $\operatorname{Re} a > 0$, $\nu \geqslant -\frac{1}{2}$, $x \in \mathbf{C} \setminus (-\infty, 0]$. With the substitutions $z = ax$, $c = b/a$, we obtain the desired result. □

THEOREM 4.2. *Let* $a > 0$, $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\int_0^a J_1(bz)\ln\left(1 - \frac{b^2}{a^2}\right)db = -\pi z^{-1}Y_0(az). \tag{4.2}$$

Proof. We apply Theorem 1.1 to the function $F_a : (0,\infty) \to \mathbf{C}$ defined by $F_a(x) := -\pi x^{-1}Y_0(ax)$, where $Y_\nu : \mathbf{C} \setminus (-\infty, 0] \to \mathbf{C}$, (order) $\nu \in \mathbf{C}$, is the Bessel function of the second kind defined in [**3**, (10.2.3)]. We obtain the desired result from the known integral [**4**, (6.512.6)]

$$\int_0^\infty J_1(bx)Y_0(ax)dx = -\frac{b^{-1}}{\pi}\ln\left(1 - \frac{b^2}{a^2}\right),$$

where $0 < b < a$. □

THEOREM 4.3. *Let* $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\int_0^\infty J_1(cz)\ln(1 + c^2)dc = 2z^{-1}K_0(z). \tag{4.3}$$

Proof. We are given the integral [**4**, (6.512.9)]

$$\int_0^\infty K_0(ax)J_1(bx)dx = \frac{1}{2b}\ln\left(1+\frac{b^2}{a^2}\right),$$

where a,$b > 0$ and $K_\nu : \mathbf{C} \setminus (-\infty, 0] \to \mathbf{C}$, (order) $\nu \in \mathbf{C}$, is the modified Bessel function of the second kind defined in [**3**, (10.25.3)]. We apply Theorem 1.1 to the function $F_a : (0,\infty) \to \mathbf{C}$ defined by $F_a(x) := 2x^{-1}K_0(ax)$, and obtain the following integral

$$\int_0^\infty J_1(bx)\ln\left(1+\frac{b^2}{a^2}\right) db = 2x^{-1}K_0(ax),$$

where $a > 0$, $x \in \mathbf{C} \setminus (-\infty, 0]$. With the substitutions $z = ax$ and $c = b/a$, we obtain the desired result. □

THEOREM 4.4. *Let $a > 0$, $z \in \mathbf{C} \setminus (-\infty, 0]$. Then*

$$\int_0^{2a} \sin^{-1}\left(\frac{b}{2a}\right) J_1(bz)db = \frac{\pi}{2z}[J_0^2(az) - J_0(2az)]. \tag{4.4}$$

Proof. By applying Theorem 1.1 to the function $F_a : (0,\infty) \to \mathbf{C}$ defined by

$$F_a(x) := \frac{\pi}{2x}J_0^2(ax),$$

we obtain the desired result from the known integral [**4**, (6.513.9)]

$$\int_0^\infty J_0^2(ax)J_1(bx)dx = \begin{cases} b^{-1} & \text{if } 0 < 2a < b, \\ \dfrac{2}{\pi b}\sin^{-1}\left(\dfrac{b}{2a}\right) & \text{if } 0 < b < 2a. \end{cases}$$

□

5. Hypergeometric and Legendre functions

THEOREM 5.1. *Let $n \in \mathbf{N}_0$, $\mu > 0$, $\nu > \frac{1}{2}$, $t \in \mathbf{C} \setminus (-\infty, 0]$. Then*

$$\int_0^\alpha J_{\nu-n-1}(bt)\, {}_2F_1\left(\begin{matrix}\nu, -n\\ \nu-n\end{matrix}; \frac{b^2}{\alpha^2}\right) b^{\nu-n}\, db = \frac{n!\alpha^{\nu-n}\Gamma(\nu-n)J_{\nu+n}(\alpha t)}{t\Gamma(\nu)}, \tag{5.1}$$

$$\int_\beta^\infty J_{\mu+n}(at)\, {}_2F_1\left(\begin{matrix}\mu, -n\\ \mu-n\end{matrix}; \frac{\beta^2}{a^2}\right) \frac{da}{a^{\mu-n-1}} = \frac{n!\beta^{-\mu+n+1}\Gamma(\mu-n)J_{\mu-n-1}(\beta t)}{t\Gamma(\mu)}. \tag{5.2}$$

Proof. By applying Theorem 1.1 to the functions $G_n^{\nu,\alpha} : (0,\infty) \to \mathbf{C}$ and $H_n^{\mu,\beta} : (0,\infty) \to \mathbf{C}$ defined by

$$G_n^{\nu,\alpha}(t) := \frac{n!\alpha^{\nu-n}\Gamma(\nu-n)J_{\nu+n}(\alpha t)}{t\Gamma(\nu)},$$

$$H_n^{\mu,\beta}(t) := \frac{n!\beta^{-\mu+n+1}\Gamma(\mu-n)J_{\mu-n-1}(\beta t)}{t\Gamma(\mu)},$$

we obtain the desired results from the known integral [**4**, (6.512.2)]

$$\int_0^\infty J_{\nu+n}(\alpha t)J_{\nu-n-1}(\beta t)dt = \begin{cases} \frac{\beta^{\nu-n-1}\Gamma(\nu)}{\alpha^{\nu-n}n!\,\Gamma(\nu-n)}\, {}_2F_1\left(\begin{matrix}\nu,-n\\ \nu-n\end{matrix};\frac{\beta^2}{\alpha^2}\right) & \text{if } 0<\beta<\alpha,\\ (-1)^n(2\alpha)^{-1} & \text{if } \beta=\alpha,\\ 0 & \text{if } \beta>\alpha,\end{cases}$$

where $\operatorname{Re}\nu > 0$ and ${}_2F_1 : \mathbf{C}^2 \times (\mathbf{C} \setminus -\mathbf{N}_0) \times (\mathbf{C} \setminus [1,\infty)) \to \mathbf{C}$ is the hypergeometric function defined in [**3**, (15.2.1)]. ☐

THEOREM 5.2. *Let* $\nu \geqslant -\frac{1}{2}$, $\nu > -2\operatorname{Re}\mu - 1$, $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\text{(5.3)}\quad \int_0^\infty P^{-\mu}_{-1/2+\nu/2}\left(\sqrt{1+\frac{4}{c^2}}\right) Q^{-\mu}_{-1/2+\nu/2}\left(\sqrt{1+\frac{4}{c^2}}\right) J_\nu(cz)dc = \frac{e^{-\mu\pi i}\,\Gamma\left(\frac{\nu-2\mu+1}{2}\right)}{z\Gamma\left(\frac{\nu+2\mu+1}{2}\right)} I_\mu(z)K_\mu(z).$$

Proof. We are given the integral [**4**, (6.513.3)]

$$\int_0^\infty I_\mu(ax)K_\mu(ax)J_\nu(bx)dx = \frac{e^{\mu\pi i}\Gamma\left(\frac{\nu+2\mu+1}{2}\right)}{b\Gamma\left(\frac{\nu-2\mu+1}{2}\right)} P^{-\mu}_{-1/2+\nu/2}\left(\sqrt{1+\frac{4a^2}{b^2}}\right) Q^{-\mu}_{-1/2+\nu/2}\left(\sqrt{1+\frac{4a^2}{b^2}}\right),$$

where $\operatorname{Re} a > 0$, $b > 0$, $\operatorname{Re}\nu > -1$, $\operatorname{Re}\nu + 2\mu > -1$ and $P^\mu_\nu : \mathbf{C} \setminus (-\infty, 1] \to \mathbf{C}$, for $\nu+\mu \notin -\mathbf{N}$ with degree ν and order μ, and $Q^\mu_\nu : \mathbf{C} \setminus (-\infty, 1] \to \mathbf{C}$, for $\nu+\mu \notin -\mathbf{N}$ with degree ν and order μ, are the associated Legendre functions of the first [**3**, (14.3.6)] and second [**3**, (14.3.7)] kind respectively. By applying Theorem 1.1 to the function $F^{\mu,a}_\nu : (0,\infty) \to \mathbf{C}$ defined by

$$F^{\mu,a}_\nu(x) := \frac{e^{-\mu\pi i}\,\Gamma\left(\frac{\nu-2\mu+1}{2}\right)}{x\Gamma\left(\frac{\nu+2\mu+1}{2}\right)} I_\mu(ax)K_\mu(ax),$$

we obtain the following integral

$$\int_0^\infty P^{-\mu}_{-1/2+\nu/2}\left(\sqrt{1+\frac{4a^2}{b^2}}\right) Q^{-\mu}_{-1/2+\nu/2}\left(\sqrt{1+\frac{4a^2}{b^2}}\right) J_\nu(bx)db = \frac{e^{-\mu\pi i}\,\Gamma\left(\frac{\nu-2\mu+1}{2}\right)}{x\Gamma\left(\frac{\nu+2\mu+1}{2}\right)} I_\mu(ax)K_\mu(ax),$$

where $\operatorname{Re} a > 0$, $\nu \geqslant -\frac{1}{2}$, $\nu > -\operatorname{Re} 2\mu - 1$, $x \in \mathbf{C} \setminus (-\infty, 0]$. By making the substitutions $z = ax$, $c = b/a$, we obtain the desired result. ☐

THEOREM 5.3. *Let* $\nu > \mp\operatorname{Re}\mu - 1$, $\nu \geqslant -\frac{1}{2}$, $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\text{(5.4)}\quad \int_0^\infty J_\nu(cz)\left[Q^{-\mu}_{-1/2+\nu/2}\left(\sqrt{1+\frac{4}{c^2}}\right)\right]^2 dc = \frac{e^{-2\mu\pi i}\Gamma\left(\frac{1+\nu-2\mu}{2}\right)}{z\Gamma\left(\frac{1+\nu+2\mu}{2}\right)}[K_\mu(z)]^2.$$

Proof. We are given the integral [**4**, (6.513.5)]

$$\int_0^\infty [K_\mu(ax)]^2 J_\nu(bx)dx = \frac{e^{2\mu\pi i}\Gamma\left(\frac{1+\nu+2\mu}{2}\right)}{b\Gamma\left(\frac{1+\nu-2\mu}{2}\right)}\left[Q^{-\mu}_{-1/2+\nu/2}\left(\sqrt{1+\frac{4a^2}{b^2}}\right)\right]^2,$$

where $\mathrm{Re}\, a > 0$, $b > 0$, $\mathrm{Re}\left(\frac{\nu}{2} \pm \mu\right) > -\frac{1}{2}$. By applying Theorem 1.1 to the function $F_\nu^{\mu,a} : (0,\infty) \to \mathbf{C}$ defined by

$$F_\nu^{\mu,a}(x) := \frac{e^{-2\mu\pi i}\Gamma\left(\frac{1+\nu-2\mu}{2}\right)}{x\Gamma\left(\frac{1+\nu+2\mu}{2}\right)}[K_\mu(ax)]^2,$$

we obtain the following integral

$$\int_0^\infty J_\nu(bx)\left[Q^{-\mu}_{-1/2+\nu/2}\left(\sqrt{1+\frac{4a^2}{b^2}}\right)\right]^2 db = \frac{e^{-2\mu\pi i}\Gamma\left(\frac{1+\nu-2\mu}{2}\right)}{x\Gamma\left(\frac{1+\nu+2\mu}{2}\right)}[K_\mu(ax)]^2,$$

where $\mathrm{Re}\, a > 0$, $\nu > \mp\mathrm{Re}\,\mu - 1$, $\nu \geqslant -\frac{1}{2}$, $x \in \mathbf{C} \setminus (-\infty, 0]$. With the substitutions $z = ax$, $c = b/a$, we obtain the desired result. □

6. Jacobi polynomials and Chebyshev polynomials of the first kind

THEOREM 6.1. *Let* $n \in \mathbf{N}_0$, $\nu > -n-1$, $\alpha,\beta > 0$, $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\int_\beta^\infty P_n^{(\nu,0)}\left(1 - \frac{2\beta^2}{a^2}\right) J_{\nu+2n+1}(az)a^{-\nu}\, da = z^{-1}\beta^{-\nu}J_\nu(\beta z), \tag{6.1}$$

$$\int_0^\alpha P_n^{(\nu,0)}\left(1 - \frac{2b^2}{\alpha^2}\right) J_\nu(bz)b^{\nu+1}\, db = z^{-1}\alpha^{\nu+1}J_{\nu+2n+1}(\alpha z). \tag{6.2}$$

Proof. By applying Theorem 1.1 to the functions $F_\nu^b : (0,\infty) \to \mathbf{C}$ and $G_\nu^{a,n} : (0,\infty) \to \mathbf{C}$ defined by $F_\nu^b(x) := x^{-1}b^{-\nu}J_\nu(bx)$, $G_\nu^{a,n}(x) := x^{-1}a^{\nu+1}J_{\nu+2n+1}(ax)$, we obtain the desired results from the known integral [**4**, (6.512.4)]

$$\int_0^\infty J_{\nu+2n+1}(ax)J_\nu(bx)dx = \begin{cases} b^\nu a^{-\nu-1}P_n^{(\nu,0)}\left(1-2a^{-2}b^2\right) & \text{if } 0 < b < a, \\ 0 & \text{if } 0 < a < b, \end{cases}$$

where $\mathrm{Re}\,\nu > -n-1$ and $P_n^{(\alpha,\beta)} : \mathbf{C} \to \mathbf{C}$ is the Jacobi polynomial defined in [**3**, (18.3.1)]. □

THEOREM 6.2. *Let* $a > 0$, $\nu \geqslant -\frac{1}{2}$, $z \in \mathbf{C} \setminus (-\infty, 0]$. *Then*

$$\int_0^{2a} \frac{J_\nu(bz)}{\sqrt{4a^2-b^2}} T_n\left(\frac{b}{2a}\right) db = \frac{\pi}{2} J_{(\nu+n)/2}(az)J_{(\nu-n)/2}(az). \tag{6.3}$$

Proof. By applying Theorem 1.1 to the function $F_\nu^{n,a} : (0,\infty) \to \mathbf{C}$ defined by

$$F_\nu^{n,a}(x) = \frac{\pi}{2} J_{(\nu+n)/2}(ax)J_{(\nu-n)/2}(ax),$$

we obtain the desired result from the known integral [**4**, (6.522.11)]

$$\int_0^\infty xJ_{(\nu+n)/2}(ax)J_{(\nu-n)/2}(ax)J_\nu(bx)dx = \begin{cases} 2\pi^{-1}b^{-1}(4a^2-b^2)^{-1/2}\,T_n(\frac{b}{2a}) & \text{if } 0<b<2a, \\ 0 & \text{if } 2a<b, \end{cases}$$

where $\operatorname{Re}\nu > -1$ and $T_n : \mathbf{C} \to \mathbf{C}$, for $n \in \mathbf{N}_0$, is the Chebyshev polynomial of the first kind found in [**3**, (18.3.1)]. □

7. Examples where the Hankel transforms fails

In the following examples, the potential use of the method given by Theorem 1.1 fails because the condition (1.1) can not be satisfied. This has been verified by analyzing the well-understood behavior of the integrands in a small neighborhood of the endpoints $\{0,\infty\}$.

(1) The definite integral [**4**, (6.512.1)] with $G_\nu^{\mu,b}(x) := \alpha(\nu,\mu)\Gamma(\nu+1)b^{-\nu}x^{-1}J_\nu(bx)$, $H_\nu^{\mu,a}(x) := \alpha(\nu,\mu)\Gamma(\nu+1)a^{\nu+1}x^{-1}J_\mu(ax)$,
where $\alpha(\nu,\mu) := \Gamma\left(\frac{\mu-\nu+1}{2}\right)/\Gamma\left(\frac{\mu+\nu+1}{2}\right)$.

(2) The definite integral [**4**, (6.514.1)] with $G_\nu^b(x) := bx^{-3}J_\nu(bx^{-1})$.

(3) The definite integral [**4**, (6.514.2)] with $F_\nu^b(x) := bx^{-3}Y_\nu(bx^{-1})$.

(4) The definite integrals [**4**, (6.516.2)], [**4**, (6.516.3)], [**4**, (6.516.4)], and [**4**, (6.516.7)] with respectively $F_\nu^b(x) := -2bY_\nu(bx^2)$, $F_\nu^b(x) := 4b\pi^{-1}K_\nu(bx^2)$, $F_\nu^a(x) := x^{-1}Y_{2\nu}(a\sqrt{x})$, and $F_\nu^a(x) := \frac{4\pi^{-1}x^{-1}}{\sec(\nu\pi)}K_{2\nu}(a\sqrt{x})$.

(5) The definite integral [**4**, (6.522.2)] with $F_\nu^{\mu,a}(x) := \frac{1}{2}e^{-2\mu\pi i}\beta(\nu,\mu)[K_\mu(ax)]^2$, where $\beta(\nu,\mu) := \Gamma(\frac{\nu}{2}-\mu)/\Gamma(1+\frac{\nu}{2}+\mu)$.

(6) The definite integrals [**4**, (6.522.6)] and [**4**, (6.522.8)] with $F_a(x) := -\frac{\pi}{2}J_0(ax)Y_0(ax)$, and $G_\nu^{\mu,a}(x) := \frac{1}{2}e^{-2\mu\pi i}\beta(\nu,\mu)K_{\mu-1/2}(ax)K_{\mu+1/2}(ax)$, respectively.

(7) The definite integral [**4**, (6.522.16)] with $F_\nu^{b,c}(x) := \sqrt{\pi}x^\nu\gamma(\nu)I_\nu(cx)K_\nu(bx)$, where $\gamma(\nu) := (8bc)^{-\nu}/\Gamma\left(\nu+\frac{1}{2}\right)$.

(8) The definite integrals [**4**, (6.526.2)] and [**4**, (6.526.3)] with $F_\nu^b(x) := 2x^{-1}Y_\nu(b\sqrt{x})$, $G_\nu^b(x) := \cos(\frac{\nu\pi}{2})K_\nu(b\sqrt{x})/(2\pi x)$, respectively.

(9) The definite integral [**4**, (6.526.6)] with $F_\nu^a(x) := 4a\pi^{-1}K_{\nu/2}(ax^2)$.

(10) The definite integral [**4**, (6.527.3)] with $F_\nu(x) := -xY_{\nu+1/2}\left(x^2/4\right)/4$.

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