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DIG-Semigroups

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Abstract. In this paper, we introduce a new class of algebras that related to distributive implication groupoids(DIG) and semigroups, call it a DIG-semigroup. We also define the concept of left(resp. right) deductive systems(LDS (resp. RDS) for short) of a DIG-semigroup and of unit divisors in DIG-semigroups. The notion of DIG-homomorphisms between DIG-semigroups is introduced and investigate some of their properties and the quotient of DIG-semigroup via deductive systems is constructed.

1. Introduction

The concept of Hilbert algebras was introduced in early 50-ties by L.Henkin and T.Skolem for some investigations of implication in intuitionistic and other classical logics. In 60-ties, these algebras were studied especially, by A. Diego [**5**] from algebraic point of view. Later, Hilbert algebras were treated by D. Busneag [**2**], Y. B. Jun [**6**], I. Chajda and R. Halas [**3**] etc. I. Chajda and R. Halas introduced the concept of implication groupoid as a generalization of a Hilbert algebra and studied some connections among ideals, deductive systems and congruence kernels whenever the implication groupoid is distributive [**4**]. Later, Bandaru [**1**], given a subset of a distributive implication groupoid, the smallest ideal containing it is constructed and a characterization of ideals in a distributive implication groupoid using upper sets is given.

K. H. Kim et al. introduced a new class of algebras related to Hilbert algebras and semigroups called a HS-algebra and studied some properties of HS-algebras [**7, 8**]. They characterized congruence relation in terms of both left and right compatible relation and constructed quotient HS-algebra whenever HS-algebra is commutative.

In this paper, by combining distributive implication groupoids and semigroups, we introduce the notion of DIG-semigroups as a generalization of HS-algebras. We describe left(resp. right) deductive systems(LDS (resp. RDS) for short) generated by a nonempty subset in a DIG-semigroup as a simple form and the element of $< D \cup E >_l$ (resp. $< D \cup E >_r$) where D and E are LDS(resp. RDS) of a DIG-semigroup X. Also, we construct the quotient of DIG-semigroup via deductive systems.

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2. Preliminaries

We recall some basic definitions and results that are necessary in the sequel.

Definition 2.1. [**2**] *A Hilbert algebra is an algebra* $\mathcal{H} = (H, *, 1)$ *of type* $(2, 0)$ *satisfying the axioms*
$(H1)$ $x * (y * x) = 1$
$(H2)$ $(x * (y * z)) * ((x * y) * (x * z)) = 1$
$(H3)$ $x * y = 1$ *and* $y * x = 1$ *imply* $x = y$.

Definition 2.2. [**4**] *An algebra* $(X, *, 1)$ *of type (2,0) is called a Distributive Implication Groupoid(DIG) if it satisfies the following identities:*
(1) $x * x = 1$
(2) $1 * x = x$
(3) $x * (y * z) = (x * y) * (x * z)$ *for all* $x, y, z \in X$.

One can observe that, every Hilbert algebra is a distributive implication groupoid but converse need not be true.

Example 2.3. [**4**] *Let* $X = \{1, a, b, c, d\}$. *The operation '$*$' is defined by*

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	1
b	1	a	1	1	d
c	1	a	1	1	d
d	1	1	c	c	1

Then $(X, *, 1)$ *is a distributive implication groupoid but not a Hilbert algebra.*

In every distributive implication groupoid, one can introduce the so called induced relation $\leqslant$ by the setting

$$x \leqslant y \text{ if and only if } x * y = 1$$

Lemma 2.4. [**4**] *Let* $(X, *, 1)$ *be a distributive implication groupoid. Then* X *satisfies the identities*

$$x * 1 = 1 \text{ and } x * (y * x) = 1$$

Moreover, the induced relation $\leqslant$ *is a quasi-order on* X *and the following relationships are satisfied*
(i) $x \leqslant 1$ (ii) $x \leqslant y * x$ (iii) $x * ((x * y) * y) = 1$ (iv) $1 \leqslant x$ *implies* $x = 1$
(v) $y * z \leqslant (x * y) * (x * z)$ (vi) $x \leqslant y$ *implies* $y * z \leqslant x * z$
(vii) $x * (y * z) \leqslant y * (x * z)$
$(viii)$ $x * y \leqslant (y * z) * (x * z)$

Definition 2.5. [**4**] *Let* $(X, *, 1)$ *be a distributive implication groupoid. A subset* $I \subseteq X$ *is called an ideal of* X *if, for all* $x, y \in X$,
$(I1)$ $1 \in X$
$(I2)$ $x \in I$ *and* $x * y \in I$ *imply* $y \in I$.

Theorem 2.6. [**4**] *Let* $\mathcal{X} = (X, *, 1)$ *be a distributive implication groupoid. Then a subset* $I \subseteq X$ *is an ideal of* $\mathcal{X}$ *if and only if*
(1) $1 \in I$

(2) $x \in X, y \in I$ *imply* $x * y \in I$.
(3) $x \in X, y_1, y_2 \in I$ *imply* $(y_2 * (y_1 * x)) * x \in I$

THEOREM 2.7. [**4**] *Let I be an ideal of a distributive implication groupoid $\mathcal{X} = (X, *, 1)$. If $a \in I$ and $a \leqslant b$, then $b \in I$.*

THEOREM 2.8. [**1**] *Let I be a subset of a distributive implication groupoid X containing 1. Then $I \in \mathcal{I}(X)$, the set of all ideals of X if and only if for any $a, b \in I$ and $x \in X$, $a * (b * x) = 1$ implies $x \in I$.*

DEFINITION 2.9. [**8**] *An HS-algebra is a non-empty set X with two binary operations '$\odot$' and '$*$' and constant '1' satisfying the axioms:*
$(DIGS1) : (X, \odot)$ *is a semigroup.*
$(DIGS2) : (X, *, 1)$ *is a Hilbert algebra.*
$(DIGS3) : x \odot (y * z) = (x \odot y) * (x \odot z)$ *and*
$(x * y) \odot z = (x \odot z) * (y \odot z)$, *for all $x, y, z \in X$.*

3. DIG-Semigroups

In this section we introduce the notion of DIG-semigroup and study its properties.

DEFINITION 3.1. *A distributive implication groupoid-semigroup (simply DIG-semigroup) is a non-empty set X with two binary operations '$\odot$' and '$*$' and constant '1' satisfying the axioms:*
$(DIGS1) : (X, \odot)$ *is a semigroup.*
$(DIGS2) : (X, *, 1)$ *is a distributive implication groupoid.*
$(DIGS3) : x \odot (y * z) = (x \odot y) * (x \odot z)$ *and*
$(x * y) \odot z = (x \odot z) * (y \odot z)$, *for all $x, y, z \in X$.*

Clearly, every DIG-semigroup is a DIG but converse need not be true.

EXAMPLE 3.2. *Let $X = \{1, a, b, c\}$. Define the operation '$*$' by*

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	1	1	c
c	1	b	b	1

Define $\odot$ on X by $x \odot y = x * y$, for all $x, y \in X$. Then $a \odot (a \odot b) = a \odot 1 = a * 1 = 1 \neq b = 1 * b = (a * a) \odot b = (a \odot a) \odot b$. Thus $\odot$ is not associative. Hence X is a distributive implication groupoid, but is not a DIG-semigroup.

EXAMPLE 3.3. *Let $X = \{1, a, b, c\}$. The operations '$\odot$' and '$*$' are defined by*

$\odot$	1	a	b	c
1	1	1	1	1
a	1	1	a	1
b	1	a	b	1
c	1	1	1	1

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	1	1	c
c	1	a	b	1

Then $(X, \odot, *, 1)$ is a DIG-semigroup.

The proof of the following proposition is straightforward.

PROPOSITION 3.4. *Every HS-algebra is a DIG-semigroup*

By the following example we show that the converse of above proposition need not be true.

EXAMPLE 3.5. *Let $X = \{1, a, b, c\}$. The operations '$\odot$' and '$*$' are defined by*

$\odot$	1	a	b	c
1	1	1	1	1
a	1	1	1	1
b	1	1	1	1
c	1	1	1	c

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	1	1	c
c	1	b	b	1

Then $(X, \odot, *, 1)$ is a DIG-semigroup, but is not an HS-algebra

In every DIG-semigroup X, one can introduce the so called induced relation $\leqslant$ by the setting for all $x, y \in X$

$$x \leqslant y \text{ if and only if } x * y = 1$$

Clearly $\leqslant$ is reflexive.
From now on, $(X, \odot, *, 1)$ or simply X is a DIG-semigroup.

LEMMA 3.6. *The induced relation $\leqslant$ on X is a quasi-order(i.e., reflexive and transitive relation) on X.*

PROOF. Let $x, y, z \in X$ and $x \leqslant y, y \leqslant z$. Then $x * y = 1 = y * z$ and

$$x * z = 1 * (x * z) = (x * y) * (x * z) = x * (y * z) = x * 1 = 1.$$

Therefore $x \leqslant z$. Hence $\leqslant$ is a quasi-order on X. □

THEOREM 3.7. *The induced quasi-order $\leqslant$ on X is an order if and only if $(X, \odot, *, 1)$ is an HS-algebra.*

PROOF. Suppose $\leqslant$ is an order on X. Then, by antisymmetry of $\leqslant$, $(X, *, 1)$ is a Hilbert algebra. Hence $(X, \odot, *, 1)$ is an HS-algebra. Converse is clear. □

PROPOSITION 3.8. *In X, the following holds:*
(i) $1 \odot x = x \odot 1 = 1$.
(ii) $x \leqslant y \Rightarrow z \odot x \leqslant z \odot y, x \odot z \leqslant y \odot z$.

PROOF. (i) $1 \odot x = (x * x) \odot x = (x \odot x) * (x \odot x) = 1$ and
$x \odot 1 = x \odot (x * x) = (x \odot x) * (x \odot x) = 1$.
(ii) Let $x \leqslant y$ and $z \in X$. Then $x * y = 1$ and $(z \odot x) * (z \odot y) = z \odot (x * y) = z \odot 1 = 1$.
Also $(x \odot z) * (y \odot z) = (x * y) \odot z = 1 \odot z = 1$. Therefore *(ii)* holds. □

DEFINITION 3.9. *A non-empty subset D of X is called a left(resp. right) deductive system(LDS resp. RDS) if it satisfies*
($DS1$) $x \odot a \in D$(resp. $a \odot x \in D$) for all $x \in X$ and $a \in D$
($DS2$) For any $x, y \in X, x * y \in D, x \in D \Rightarrow y \in D$.
If D is both left and right deductive system of X, then D is called a deductive system(DS) of X.

EXAMPLE 3.10. *Let $X = \{1, a, b, c\}$. The operations '$\odot$' and '$*$' are defined by*

$\odot$	1	a	b	c
1	1	1	1	1
a	1	1	a	1
b	1	a	b	1
c	1	1	1	c

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	1	1	c
c	1	a	b	1

Then $(X, \odot, *, 1)$ is a DIG-semigroup. Clearly $D = \{1, a\}$ is a DS of X. But $E = \{1, b\}$ is not a DS of X, since $a \odot b = a \notin E$.

DEFINITION 3.11. *A non-empty subset S of X is called a subalgebra of X if $x*y \in S$ and $x \odot y \in S$, for all $x, y \in S$.*

THEOREM 3.12. *Every deductive system of X is a subalgebra of X.*

PROOF. Let D be a deductive system of X and $a, b \in D$. Then $a \odot b \in D$. Since $b \leqslant a * b$, we have, by $(DS2)$, $a * b \in D$. □

The converse of the above theorem need not be true, in Example 3.10, the set $E = \{1, b\}$ is a subalgebra of X, but not deductive system of X.

THEOREM 3.13. *Let X with $x * y = (x \odot y) * y$ for all $x, y \in X$. Then the following holds:*

(1) $x \leqslant x \odot y$.
(2) $x \leqslant y$ *if and only if* $x \odot y \leqslant y$.
(3) *If* $x \leqslant y$, *then* $x \odot y \leqslant y \odot x$.
(4) *If* $x \odot y = 1$, *then* $x * y = y$.

PROOF. Suppose X satisfies $x * y = (x \odot y) * y$, for all $x, y \in X$. Then

(1) $x*(x\odot y) = (x\odot(x\odot y))*(x\odot y) = ((x\odot x)\odot y)*(x\odot y) = ((x\odot x)*x)\odot y = (x * x) \odot y = 1 \odot y = 1$.
(2) It is clear.
(3) Let $x \leqslant y$. Then $x * y = (x \odot y) * y = 1$. Then $(x \odot y) * (y \odot x) = ((x\odot y)\odot(y\odot x))*(y\odot x) = (((x\odot y)\odot y)*y)\odot x = ((x\odot y)*y)\odot x = 1\odot x = 1$.
(4) Let $x \odot y = 1$. Then $x * y = (x \odot y) * y = 1 * y = y$.

□

EXAMPLE 3.14. *Let $X = \{1, a, b, c\}$. The operations '$\odot$' and '$*$' are defined by*

$\odot$	1	a	b	c
1	1	1	1	1
a	1	1	1	1
b	1	1	1	1
c	1	1	1	c

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	1	1	c
c	1	b	b	1

Then $(X, \odot, *, 1)$ is a DIG-semigroup. But $a * a = 1 \neq a = 1 * a = (a \odot a) * a$. Also $a \odot b = 1$, but $a * b = 1 \neq b$.

Hence the condition $x * y = (x \odot y) * y$, for all $x, y \in X$ is necessary to prove Theorem 3.13.

Example 3.15. *Let $X = \{1, a, b, c\}$. Define the operations '$\odot$' and '$*$' by*

$\odot$	1	a	b	c
1	1	1	1	1
a	1	a	1	a
b	1	1	b	b
c	1	a	b	c

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	a
c	1	1	1	1

Then $(X, \odot, *, 1)$ is a DIG-semigroup with $x * y = (x \odot y) * y$ for all $x, y \in X$.

Definition 3.16. *An element $a(\neq 1) \in X$ is said to be a left unit divisor if*

$$\text{there exists } b(\neq 1) \in X \text{ such that } (a \odot b) = 1$$

An element $a(\neq 1) \in X$ is said to be a right unit divisor if

$$\text{there exists } b(\neq 1) \in X \text{ such that } b \odot a = 1$$

An element of X which is both left and right unit divisor is called a unit divisor of X.

In Example 3.3, a, b, c are unit divisors.

Theorem 3.17. *If there are no left(resp. right) unit divisors in X, then X satisfies the left(resp. right) cancellation law for the operation $\odot$.*

Proof. Let $x, y, z \in X$ be such that $x \odot y = x \odot z$ and $x \neq 1$. Then

$$x \odot (y * z) = (x \odot y) * (x \odot z) = 1$$

and

$$x \odot (z * y) = (x \odot z) * (x \odot y) = 1$$

Since X has no left unit divisor, it follows that $y*z = 1 = z*y$ so that $y = z$. Similarly we can show the right cancellation law for the operation $\odot$. □

Theorem 3.18. *If X satisfies the left(resp. right) cancellation law for the operation $\odot$ i.e.,*

$$x \odot y = x \odot z (\text{ resp. } y \odot x = z \odot x) \Rightarrow y = z \text{ for all } x, y, z \in X$$

then X contains no left(resp. right) unit divisors in X.

Proof. Let X satisfying left cancellation law for the operation $\odot$ and assume that $x \odot y = 1$ where $x \neq 1$. Then $x \odot y = 1 = x \odot 1$ and hence $y = 1$. Similarly it holds for the right case. Hence there is no left(resp. right) unit divisors in X. □

Let $(X, *, 1)$ be a distributive implication groupoid and $a, b \in X$. Then the set

$$A(a, b) = \{x \in X \mid a * (b * x) = 1\}$$

is non-empty since $1, a, b \in A(a, b)$.

Proposition 3.19. *If D is a left deductive system(LDS) of X, then $A(a, b) \subseteq D$, for all $a, b \in D$.*

Proof. Let $x \in A(a, b)$ where $a, b \in D$. Then $a*(b*x) = 1 \in D$ and so $x \in D$(by DS2). Therefore $A(a, b) \subseteq D$. □

The following theorem can be proved easily.

THEOREM 3.20. *Let $\{D_i\}_{i\in I}$ be an arbitrary collection of LDSs of X. Then $\bigcap_{i\in I} D_i$ is also a LDS of X.*

For any subset D of X, the intersection of all LDS(resp. RDS) of X containing D is called the LDS(resp. RDS) generated by D and is denoted by $< D >_l$(resp. $< D >_r$). It is clear that if D and E are subsets of X satisfying $D \subseteq E$, then $< D >_l \subseteq < E >_l$(resp. $< D >_r \subseteq < E >_r$) and if D is a LDS(resp. RDS) of X, then $< D >_l = D$(resp. $< D >_r = D$).

THEOREM 3.21. *Let D be a non-empty subset of X such that $X \odot D \subseteq D$(resp. $D \odot X \subseteq D$). Then*

$$< D >_l = \{a \in X \mid y_n * (\cdots * (y_1 * a) \dots) = 1 \textit{ for some } y_1, y_2, \dots, y_n \in D\}$$

$$< D >_r = \{a \in X \mid y_n * (\cdots * (y_1 * a) \dots) = 1 \textit{ for some } y_1, y_2, \dots, y_n \in D\}$$

PROOF. Let $x \in X, b \in B$. Where

$$B = \{a \in X \mid y_n * (\cdots * (y_1 * a) \dots) = 1 \text{ for some } y_1, y_2, \dots, y_n \in D\}$$

Then there exist $y_1, y_2, \dots, y_n \in D$ such that $y_n * (\cdots * (y_1 * b) \dots) = 1$. Hence $1 = x \odot 1 = x \odot (y_n * (\cdots * (y_1 * b) \dots)) = (x \odot y_n) * (\cdots * ((x \odot y_1) * (x \odot b)) \dots)$(resp. $1 = 1 \odot x = (y_n * (\cdots * (y_1 * b) \dots)) \odot x = (y_n \odot x) * (\cdots * ((y_1 \odot x) * (b \odot x)) \dots)$). Since $x \odot y_i \in D$(resp.$y_i \odot x \in D$) for $i = 1, 2, \dots, n$, we have $x \odot b \in B$(resp. $b \odot x \in B$). Let $x, a \in X$ be such that $a * x \in B$ and $a \in B$. Then there exist $y_1, y_2, \dots, y_n, z_1, \dots, z_m \in D$ such that $y_n * (\cdots * (y_1 * (a * x)) \dots) = 1$ and $z_m * (\cdots * (z_1 * a) \dots) = 1$. Hence $a * (y_n * (\cdots * (y_1 * x) \dots) = 1)$ i.e., $a \leqslant y_n * (\cdots * (y_1 * x) \dots)$. Also, $1 = z_m * (\cdots * (z_1 * a) \dots) \leqslant z_m * (\cdots * (z_1 * (y_n * (\dots (y_1 * x) \dots))) \dots)$. Thus

$$z_m * (\cdots * (z_1 * (y_n * (\dots (y_1 * x) \dots))) \dots) = 1$$

which implies that $x \in B$. Therefore B is a LDS(resp. RDS) of X. Obviously, $D \subseteq B$. Let G be a LDS(resp. RDS) containing D. To show $B \subseteq G$, let a be an element of B. Then there exist $y_1, y_2, \dots, y_n \in D$ such that $y_n * (\cdots * (y_1 * a) \dots) = 1$. Then $a \in G$. Therefore $B \subseteq G$. Hence $B = < D >_l$ (resp. $< D >_r$). □

In the following example we show that the union of LDS(resp. RDS's) D and E may not be LDS(resp. RDS) of X.

EXAMPLE 3.22. *Let $X = \{1, a, b, c, d\}$. The operations '$\odot$' and '$*$' are defined by*

$\odot$	1	a	b	c	d
1	1	1	1	1	1
a	1	1	1	1	1
b	1	1	1	1	1
c	1	1	1	1	1
d	1	1	1	1	d

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	d
b	1	a	1	a	d
c	1	1	1	1	d
d	1	1	b	b	1

Then $(X, \odot, *, 1)$ is a DIG-semigroup. We know that $D = \{1, a\}$ and $E = \{1, b\}$ are LDS of X but $D \cup E = \{1, a, b\}$ is not a LDS of X since $b * c = a \in D \cup E, c \notin D \cup E$. We can observe that if $D = \{1, a, c\} \subseteq X$ such that $X \odot D \subseteq D$(resp. $D \odot X \subseteq D$) then $< D >_l$ (resp. $< D >_r$) $= \{1, a, b, c\}$.

Theorem 3.23. *Let D and E be LDS of X. Then*

$$< D \cup E >_l \ (\text{resp. } < D \cup E >_r) = \{a \in X \mid x * (y * a) = 1 \text{ for some } x \in D, y \in E\}$$

Proof. Let $H = \{a \in X \mid x * (y * a) = 1$ for some $x \in D, y \in E\}$. Clearly, $H \subseteq < D \cup E >_l$ (resp. $< D \cup E >_r$). Let $b \in < D \cup E >_l$ (resp. $< D \cup E >_r$). Then, by Theorem 3.21, there exist $y_1, y_2, \dots, y_n \in D \cup E$ such that $y_n * (\cdots * (y_1 * b) \dots) = 1$. If $y_i \in D$ for all $i = 1, 2, \dots, n$, then $b \in D$. If $y_i \in E$, for all $i = 1, 2, \dots, n$, then $b \in E$. Hence $b \in H$. If some of $y_1, y_2, \dots, y_n \in D$ and others belong to E, then we can assume that $y_1, y_2, \dots, y_k \in D$ and $y_{k+1}, \dots, y_n \in E$ for $1 \leqslant k < n$, without loss of generality. Let $p = y_k * (\cdots * (y_1 * b) \dots)$. Then $y_n * (\cdots * (y_{k+1} * p) \dots) = 1$ and hence $p \in E$. Let $q = p * b = y_k * (\cdots * (y_1 * b) \dots) * b$. Then

$$\begin{aligned}
1 &= [y_k * (\cdots * (y_1 * b) \dots)] * [y_k * (\cdots * (y_1 * b) \dots)] \\
&= y_k * [y_k * (\cdots * (y_1 * b) \dots) * (\cdots * (y_1 * b) \dots)] \\
&= y_k * [\cdots * (y_1 * (y_k * (\dots (y_1 * b) \dots)) * b) \dots] \\
&= y_k * [\cdots * (y_1 * q) \dots]
\end{aligned}$$

and so $q \in D$. Since $p * (q * b) = 1$, we have $b \in H$. So that $< D \cup E >_l$ (resp. $< D \cup E >_r$) $\subseteq H$. □

We denote the set of all deductive systems of X by $\mathcal{D}(X)$. Let $D_1, D_2 \in \mathcal{D}(X)$. We define the meet of D_1 and D_2 by $D_1 \wedge D_2 = D_1 \cap D_2$ and the join of D_1 and D_2 by $D_1 \vee D_2 =< D_1 \cup D_2 >$. We note that $(\mathcal{D}(X), \vee, \wedge)$ is a lattice. Also, $\{1\} \in \mathcal{D}(X)$ and $X \in \mathcal{D}(X)$ and it is almost evident that the set theoretical intersection of an arbitrary set of deductive systems of X is deductive system of X again. Hence, the set $\mathcal{D}(X)$ forms a complete lattice with respect to set inclusion where the operation meet coincides with set intersection, the least(or greatest) element of $\mathcal{D}(X)$ is $\{1\}$(or X respectively).

4. DIG-homomorphism of DIG-semigroups

In this section, we introduce DIG-homomorphisms of DIG-semigroups and study their properties.

Definition 4.1. *Let X and Y be two DIG-semigroups. A mapping $\phi : X \to Y$ is called a DIG-homomorphism if for all $a, b \in X$,*

$$\phi(a * b) = \phi(a) * \phi(b) \text{ and } \phi(a \odot b) = \phi(a) \odot \phi(b).$$

A DIG-homomorphism ϕ is called a DIG-monomorphism(resp. DIG-epimorphism) if it is injective(resp. surjective). A bijective DIG-homomorphism is called a DIG-isomorphism. For any DIG-homomorphism $\phi : X \to Y$ the set $\{x \in X \mid \phi(x) = 1\}$ is called the kernel of ϕ, denoted by $\ker \phi$ and the set $\{\phi(x) \mid x \in X\}$ is called the image of ϕ, denoted by $Im(\phi)$. We denote by $Hom(X, Y)$ the set of all DIG-homomorphisms of DIG-semigroups from X to Y.

Example 4.2. *Let $X = \{1, a, b, c\}$ and $Y = \{1, x, y, z\}$. The operations '$\odot$' and '$*$' are defined by*

⊙	1	a	b	c
1	1	1	1	1
a	1	1	1	1
b	1	1	1	1
c	1	1	1	c

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	1	1	c
c	1	b	b	1

⊙	1	x	y	z
1	1	1	1	1
x	1	1	1	1
y	1	1	y	1
z	1	1	1	z

$*$	1	x	y	z
1	1	x	y	z
x	1	1	y	z
y	1	x	1	z
z	1	1	y	1

Then $(X, \odot, *, 1)$ *and* $(Y, \odot, *, 1)$ *are DIG-semigroups. Define a map* $\phi : X \to Y$ *by*

$$\phi(r) = \begin{cases} 1, & \text{if } r = 1,a,b \\ z, & \text{if } r = c \end{cases}$$

Then ϕ *is a DIG-homomorphism from* X *into* Y*.*

Proposition 4.3. *Suppose that* $\phi : X \to Y$ *is a DIG-homomorphism of DIG-semigroups. Then, for* $x, y \in X$*,* (i) $\phi(1) = 1$ (ii) *If* $x * y = 1$*, then* $\phi(x) * \phi(y) = 1$

Proof. Since $\phi(1) = \phi(1 * 1) = \phi(1) * \phi(1) = 1, (i)$ holds. Let $x, y \in X$ and $x * y = 1$. Then $\phi(x) * \phi(y) = \phi(x * y) = \phi(1) = 1$. □

Note 4.4. *Suppose that* $\phi : X \to Y$ *is a DIG-homomorphism of DIG-semigroups. Then* ϕ *is a monomorphism if and only if* $\ker \phi = \{1\}$*.*

Proposition 4.5. *Let* X, Y *be DIG-semigroups and* $\phi \in Hom(X, Y)$*. Then*
(i) $\phi(x \odot 1) = \phi(1 \odot x) = 1$
(ii) $\phi(1 * x) = \phi(x)$
(iii) $\phi(x * 1) = \phi(1)$ *for all* $x \in X$*.*

Proposition 4.6. *Let* $\phi : X \to Y$ *be a homomorphism of DIG-semigroups. If* $x \in X$ *is a left(resp. right) unit divisor of* X*, then* $\phi(x)$ *is left(resp. right) unit divisor of* Y*.*

Proof. Let $x \in X$ be a left unit divisor of X. Then there exists $y(\neq 1) \in X$ such that $x \odot y = 1$. Now $y \in X$ implies that $\phi(y) \in Y$ and $\phi(x) \odot \phi(y) = \phi(x \odot y) = \phi(1) = 1$. □

Note 4.7. *Let* X, Y *and* Z *be DIG-semigroups. If* $\phi \in Hom(X, Y)$ *and* $\psi \in Hom(Y, Z)$*, then* $\psi \circ \phi \in Hom(X, Z)$*.*

Proposition 4.8. *Let* X *and* Y *be DIG-semigroups and* B *a left(resp.right) deductive system of* Y*. Then for any* $\phi \in Hom(X, Y)$*,*$\phi^{-1}(B)$ *is a left(resp. right) deductive system of* X *containing* $\ker \phi$*.*

Proof. Let $x \in X$ and $y \in \phi^{-1}(B)$. Then $\phi(y) \in B$ and $\phi(x \odot y) = \phi(x) \odot \phi(y)$. Since B is a left deductive system of Y, we have $\phi(x \odot y) \in B$ i.e., $x \odot y \in \phi^{-1}(B)$. Hence $X \odot \phi^{-1}(B) \subseteq \phi^{-1}(B)$. Now, let $x, y \in X$ be such that $y \in \phi^{-1}(B)$ and $y * x \in \phi^{-1}(B)$. Then $\phi(y) \in B$ and $\phi(y * x) = \phi(y) * \phi(x) \in B$. Since B is a left deductive system, we have $\phi(x) \in B$ i.e., $x \in \phi^{-1}(B)$. Hence $\phi^{-1}(B)$ is a left deductive system of X. Since $\{1\} \subseteq B, \ker\phi = \phi^{-1}(\{1\}) \subseteq \phi^{-1}(B)$.

□

THEOREM 4.9. *Let X and Y be DIG-semigroups and $\phi : X \to Y$ be a DIG-epimorphism of DIG-semigroups. If D is a left(resp. right) deductive system of X, then $\phi(D)$ is a left(resp. right) deductive system of Y.*

PROOF. Let $x \in \phi(D)$ and $y \in Y$. Since ϕ is onto, there exist $a \in X$ and $b \in D$ such that $\phi(a) = y$ and $\phi(b) = x$. Then $a \odot b \in D$ implies that $y \odot x \in \phi(D)$. Hence $Y \odot \phi(D) \subseteq \phi(D)$. Now, suppose $a \in \psi(D), y \in Y$ and $a * y \in \phi(D)$. Since ϕ is onto, there exist $b \in D$ and $x \in X$ such that $\phi(b) = a$ and $\phi(x) = y$. Thus $\phi(b * x) = \phi(b) * \phi(x) = a * y$. So $b * x \in D$. It follows from (DS2) that $x \in D$. Hence $y = \phi(x) \in \phi(D)$. Therefore $\phi(D)$ is a left deductive system of Y. □

THEOREM 4.10. *Let $\phi : X \to Y$ be a DIG-homomorphism of DIG-semigroups. Then $\ker\phi$ is a deductive system of X.*

PROOF. Let $x \in X$ and $y \in \ker\phi$. Then $\phi(y) = 1$. Now, $\phi(x \odot y) = \phi(x) \odot \phi(y) = \phi(x) \odot 1 = 1$. Therefore $x \odot y \in \ker\phi$. Now, let $a * x \in \ker\phi$ and $a \in \ker\phi$. Then $\phi(a * x) = 1$ and hence $\phi(a) * \phi(x) = 1$. Therefore $\phi(x) = 1$. Hence $x \in \ker\phi$. $\ker\phi$ is a left deductive system of X. □

DEFINITION 4.11. *X is said to be commutative if $(x * y) * y = (y * x) * x$, for all $x, y \in X$.*

EXAMPLE 4.12. *Let $X = \{1, a, b, c\}$. Define the operations '$\odot$' and '$*$' by*

$\odot$	1	a	b	c
1	1	1	1	1
a	1	1	1	1
b	1	1	1	1
c	1	1	1	c

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	1	1	c
c	1	b	b	1

Then $(X, \odot, *, 1)$ is a commutative DIG-semigroup.

NOTE 4.13. *Every commutative DIG-semigroup is an HS-algebra.*

THEOREM 4.14. *Let X, Y and Z be commutative DIG-semigroups. Suppose that $\phi : X \to Y$ is a DIG-epimorphism and $\psi : X \to Z$ be a DIG-homomorphism. If $\ker\phi \subseteq \ker\psi$, then there exists a unique DIG-homomorphism $\gamma : Y \to Z$ such that $\gamma \circ \phi = \psi$.*

PROOF. Let $y \in Y$. Since ϕ is onto, there exists $x \in X$ such that $\phi(x) = y$. Define a mapping $\gamma : Y \to Z$ by $\gamma(y) = \psi(x)$. If $y = \phi(x_1) = \phi(x_2), x_1, x_2 \in X$, then $1 = \phi(x_1) * \phi(x_2) = \phi(x_1 * x_2)$. Hence $x_1 * x_2 \in \ker\phi$. Since $\ker\phi \subseteq \ker\psi$, we have $1 = \psi(x_1) * \psi(x_2) = \psi(x_1 * x_2)$. Similarly, we get that $\psi(x_2) * \psi(x_1) = 1$. Thus $\psi(x_1) = \psi(x_2)$. This means that γ is well-defined. Next we show that γ is a DIG-homomorphism. Let $a, b \in Y$. Then there exist $x_1, x_2 \in X$ such that $a = \phi(x_1)$ and $b = \phi(x_2)$. Now, we have

$$\gamma(a \odot b) = \gamma(\phi(x_1) \odot \phi(x_2)) = \gamma(\phi(x_1 \odot x_2)) = \psi(x_1) \odot \psi(x_2) = \gamma(a) \odot \gamma(b)$$

$$\gamma(a * b) = \gamma(\phi(x_1) * \phi(x_2)) = \gamma(\phi(x_1 * x_2)) = \psi(x_1) * \psi(x_2) = \gamma(a) * \gamma(b).$$

Hence γ is a DIG-homomorphism. The uniqueness of γ follows directly from the fact that ϕ is DIG-epimorphism. □

THEOREM 4.15. *Let X, Y and Z be commutative DIG-semigroups and $g : X \to Z$ be a DIG-homomorphism and $h : Y \to Z$ be a DIG-monomorphism with $Im(g) \subseteq Im(h)$ then there exists a unique DIG-homomorphism $f : X \to Y$ satisfying $h \circ f = g$*

PROOF. For each $x \in X,\ g(x) \in Im(g) \subseteq Im(h)$. Since h is a DIG-monomorphism there exists unique $b \in Y$ such that $g(a) = h(b)$. Define a map $f : X \to Y$ by $f(a) = b$. Then $h \circ f = g$. Let $c, d \in X$. Then $h(f(c * d)) = g(c * d) = g(c) * g(d) = h(f(c)) * h(f(d)) = h(f(c) * f(d))$. Since h is a DIG-monomorphism, we have $f(c*d) = f(c) * f(d)$. Similarly we can prove that $f(c \odot d) = f(c) \odot f(d)$. The uniqueness follows from the fact that h is monomorphism. □

DEFINITION 4.16. *Let θ be a binary relation on X. Then*

(1) *θ is said to be compatible if $(x, y) \in \theta$ and $(u, v) \in \theta$ then $(x * u, y * v) \in \theta$ and $(x \odot u, y \odot v) \in \theta$ for all $x, y, u, v \in X$.*

(2) *A compatible equivalence relation on X is called a congruence relation on X.*

Let D be a deductive system of X. For any $x, y \in X$, we define a relation " $\sim_D$ " on X as follows.

$$x \sim_D y \text{ if and only if } x * y \in D \text{ and } y * x \in D.$$

PROPOSITION 4.17. *Let D be a deductive system of X. Then $\sim_D$ is a congruence relation on X.*

PROOF. Let D be a deductive system of X. Since $1 \in D$, the relation $\sim_D$ is reflexive. Clearly, $\sim_D$ is symmetric. We prove transitivity of $\sim_D$:
Let $(x, y) \in \sim_D$ and $(y, z) \in \sim_D$. Then $x*y, y*x, y*z, z*y \in D$. Since $(y*z)*(x*(y*z)) = 1 \in D$ and $y*z \in D$, we get that $x*(y*z) \in D$. Consider $x*(y*z) = (x*y)*(x*z)$, then $(x * y) * (x * z) \in D$ and $x * y \in D$ imply that $x * z \in D$. Similarly, we can prove $z*x \in D$, thus $(x, z) \in \sim_D$. Let us prove the compatibility of $\sim_D$. Assume $(x, y) \in \sim_D$ and $(u, v) \in \sim_D$. Then $x * y, y * x, u * v, v * u \in D$ and

$$(x * u) * (x * v) = x * (u * v) \in D$$

$$(x * v) * (x * u) = x * (v * u) \in D$$

Therefore, $(x*u, x*v) \in \sim_D$. Further, by Lemma 2.4, we have $(y*x) \leqslant (x*v)*(y*v)$ and $x * y \leqslant (y * v) * (x * v)$
By Theorem 2.7, $(x*v)*(y*v) \in D$ and $(y*v)*(x*v) \in D$. That is $(x*v, y*v) \in \sim_D$. By using transitivity of $\sim_D$, we get that $(x * u, y * v) \in \sim_D$. Since D is a deductive system of X and $\sim_D$ is transitive, we can prove that $(x \odot u, y \odot v) \in \sim_D$. Thus $\sim_D$ is a congruence relation on X. □

Let D be a deductive system of X. Denote the equivalence class containing x by $[x]_D$ and the set of equivalence classes in X by X/D i.e., $[x]_D = \{y \in X \mid y \sim_D x\}$ and $X/D = \{[x]_D \mid x \in X\}$. Clearly $[1]_D = D$ and $[x]_D = [y]_D$ if and only if $x \sim_D y$.

LEMMA 4.18. *If θ is a congruence relation on X, then $[1]_\theta = \{x \in X \mid (x, 1) \in \theta\}$ is a deductive system of X.*

PROOF. Let θ be a congruence relation on X. Clearly, $1 \in [1]_\theta$. Suppose $x \in X, y \in [1]_\theta$. Then $(y, 1) \in \theta$ and hence

$$(x \odot y, 1) = (x \odot y, x \odot 1) \in \theta \text{ and } (y \odot x, 1) = (y \odot x, 1 \odot x) \in \theta.$$

Thus $x \odot y \in [1]_\theta$ and $y \odot x \in [1]_\theta$. Suppose $x \in [1]_\theta$ and $x * y \in [1]_\theta$. Then $(x, 1) \in \theta$ and hence $(x*y, y) = (x*y, 1*y) \in \theta$. On the other hand, $x*y \in [1]_\theta$ gives $(x*y, 1) \in \theta$. We obtain $(y, 1) \in \theta$ proving $y \in [1]_\theta$. □

THEOREM 4.19. *If D is a deductive system of X, then the relation θ_D defined by*

$$(x, y) \in \theta_D \textit{ if and only if } x * y \in D \textit{ and } y * x \in D$$

is a congruence of X such that $[1]_{\theta_D} = D$

PROOF. Clearly, by Proposition 4.17, θ_D is a congruence on X. If $x \in D$, then $1 * x = x \in D$ and $x * 1 = 1 \in D$ which means $(x, 1) \in \theta_D$, i.e. $x \in [1]_{\theta_D}$. Conversely, if $x \in [1]_{\theta_D}$, then $(x, 1) \in \theta_D$ and hence $x = 1 * x \in D$. Thus $[1]_{\theta_D} = D$. □

THEOREM 4.20. *If D is a deductive system of X, then $(X/D, \circledcirc, \circledast, [1]_D)$ is a DIG-semigroup under the operations*

$$[x]_D \circledcirc [y]_D = [x \odot y]_D \textit{ and } [x]_D \circledast [y]_D = [x * y]_D.$$

PROOF. Since $\sim_D$ is a congruence relation, the operation $\circledast$ is well-defined. Clearly, $(X/D, \circledast, [1]_D)$ is a distributive implication groupoid. Let $[x]_D = [u]_D$ and $[y]_D = [v]_D$. Then since D is a deductive system, we have $(x \odot u) * (x \odot v) = x \odot (u * v) \in D$ and $(x \odot v) * (x \odot u) = x \odot (v * u) \in D$. Then $(x \odot u) \sim_D (x \odot v)$. On the other hand, $(x \odot v) * (y \odot v) = (x * y) \odot v \in D$ and $(y \odot v) * (x \odot v) = (y * x) \odot v \in D$. Hence $x \odot v \sim_D y \odot v$ that is $[x \odot u]_D = [y \odot v]_D$. This shows that $\circledcirc$ is well defined. Clearly, $(X/D, \circledcirc)$ is a semigroup. For every $[x]_D, [y]_D, [z]_D \in X/D$, we have

$$\begin{aligned}
[x]_D \circledcirc ([y]_D \circledast [z]_D) &= [x]_D \circledcirc [y * z]_D \\
&= [x \odot (y * z)]_D \\
&= [(x \odot y) * (x \odot z)]_D \\
&= [x \odot y]_D \circledast [x \odot z]_D \\
&= ([x]_D \circledcirc [y]_D) \circledast ([x]_D \circledcirc [z]_D)
\end{aligned}$$

and

$$\begin{aligned}
([x]_D \circledast [y]_D) \circledcirc [z]_D &= [x * y]_D \circledcirc [z]_D \\
&= [(x * y) \odot z]_D \\
&= [(x \odot z) * (y \odot z)]_D \\
&= [x \odot z]_D \circledast [y \odot z]_D \\
&= ([x]_D \circledcirc [z]_D) \circledast ([y]_D \circledcirc [z]_D)
\end{aligned}$$

Hence $(X/D, \circledcirc, \circledast, [1]_D)$ is a DIG-semigroup. □

THEOREM 4.21. *If X is commutative and D is a deductive system of X, then $(X/D, \circledcirc, \circledast, [1]_D)$ is an HS-algebra under the operations*

$$[x]_D \circledcirc [y]_D = [x \odot y]_D \textit{ and } [x]_D \circledast [y]_D = [x * y]_D.$$

EXAMPLE 4.22. *Let $X = \{1, a, b, c, d\}$. The operations '$\odot$' and '$*$' are defined by*

$\odot$	*1*	*a*	*b*	*c*	*d*
1	*1*	*1*	*1*	*1*	*1*
a	*1*	*1*	*1*	*1*	*1*
b	*1*	*1*	*1*	*1*	*1*
c	*1*	*1*	*1*	*1*	*1*
d	*1*	*1*	*1*	*1*	*d*

$*$	*1*	*a*	*b*	*c*	*d*
1	*1*	*a*	*b*	*c*	*d*
a	*1*	*1*	*b*	*b*	*d*
b	*1*	*a*	*1*	*a*	*d*
c	*1*	*1*	*1*	*1*	*d*
d	*1*	*1*	*b*	*b*	*1*

Then $(X, \odot, *, 1)$ *is a DIG-semigroup. We can observe that* $D = \{1, a, b, c\}$ *is a deductive system of* X *and*

$$\begin{aligned}\sim_D &= \{(x, y) \in X \times X \mid x \sim_D y\}\\ &= \{(1,1),(a,a),(b,b),(c,c),(d,d),(1,a),(1,b),(1,c),(a,1),(b,1),(c,1),\\ &(a,b),(b,a),(a,c),(c,a),(b,c),(c,b)\}\end{aligned}$$

is a congruence relation on X. *Then* $[1]_D = [a]_D = [b]_D = [c]_D = D = D_1$*(say) and* $[d]_D = \{d\} = D_2$. *Therefore* $X/D = \{D_1, D_2\}$ *with the following tables*

$\circledcirc$	D_1	D_2
D_1	D_1	D_1
D_2	D_1	D_2

$\circledast$	D_1	D_2
D_1	D_1	D_2
D_2	D_1	D_1

is a DIG-semigroup under the conditions $[x]_D \circledcirc [y]_D = [x \odot y]_D$ *and* $[x]_D \circledast [y]_D = [x * y]_D$.

Proposition 4.23. *If* D *and* E *are deductive systems of* X *and* $D \subset E$, *then*
(i) D *is also a deductive system of* E.
(ii) E/D *is a deductive system of* X/D.

Theorem 4.24. *Let* $\psi : X \to Y$ *be a DIG-homomorphism of commutative DIG-semigroups. Then for any deductive system* D *of* X, $D/(\ker(\psi) \cap D) \simeq \psi(D)$.

Proof. Let $A = \ker(\psi) \cap D$. Clearly A is a deductive system of D. Define a mapping $\sigma : D/A \to Y$ by $\sigma([x]_A) = \psi(x)$ for all $x \in D$. Then for any $[x]_A, [y]_A \in D/A$, we have

$$\begin{aligned}[x]_A = [y]_A &\Leftrightarrow x * y \in A, y * x \in A\\ &\Leftrightarrow \psi(x * y) = 1, \psi(y * x) = 1\\ &\Leftrightarrow \psi(x) \circledast \psi(y) = 1, \psi(y) \circledast \psi(x) = 1\\ &\Leftrightarrow \psi(x) = \psi(y)\\ &\Leftrightarrow \sigma([x]_A) = \sigma([y]_A).\end{aligned}$$

Hence σ is well-defined and one to one. For all $[x]_A, [y]_A \in D/A$, we have

$$\sigma([x]_A \circledast [y]_A) = \sigma([x * y]_A) = \psi(x * y) = \psi(x) * \psi(y) = \sigma([x]_A) * \sigma([y]_A)$$

$$\sigma([x]_A \circledcirc [y]_A) = \sigma([x \odot y]_A) = \psi(x \odot y) = \psi(x) \odot \psi(y) = \sigma([x]_A) \odot \sigma([y]_A)$$

Hence σ is a DIG-homomorphism of DIG-semigroups. Thus $Im(\sigma) = \{\sigma([x]_A) \mid x \in D\} = \{\psi(x) \mid x \in D\} = \psi(D)$. Therefore $D/(\ker(\psi) \cap D) \simeq \psi(D)$. □

Corollary 4.25. *If* $\psi : X \to Y$ *is a DIG-epimorphism of commutative DIG-semigroups, then* $X/\ker(\psi) \simeq Y$.

5. Conclusion

In this paper, we have introduced a new class of algebras related to distributive implication groupoids and semigroups, called a DIG-semigroup and also considered the concept of deductive systems and of unit divisors in DIG-semigroups. We have described left(resp. right) deductive system(LDS (resp. RDS) for short) generated by a nonempty subset in a DIG-semigroup as a simple form. We have given a description of the element of $< D \cup E >_l$ (resp. $< D \cup E >_r$) where D and E are left(resp. right) deductive system of a DIG-semigroup X. We have introduced the notion of DIG-homomorphisms between DIG-semigroups and investigated some of their properties. Also, we have constructed the quotient DIG-semigroup via deductive systems.

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