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A table of definite integrals from the marriage of power and Fourier series

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Abstract. In this paper we show an unusual way to obtain values for definite integrals of the forms $\int_{-\pi}^{\pi} f(x)dx$, $\int_{-\pi}^{\pi} f(x)\cos(nx)dx$ and $\int_{-\pi}^{\pi} f(x)\sin(nx)dx$ where n is a positive integer. Our method is indirect in that we do not start with the integral. The integral is obtained as the end result of a process that did not visualize the integral at the start. We begin with an analytic function, and obtain our integrals by comparing the coefficients of related power series and Fourier series. A table of 36 definite integrals results. Eleven of these can be reproduced by Mathematica or found in familiar tables, while 25 others cannot. These may be new integral evaluations.

1. Introduction

We will evaluate some rather complex integrals of the forms $\int_{-\pi}^{\pi} f(x)dx$, $\int_{-\pi}^{\pi} f(x)\cos(nx)dx$ and $\int_{-\pi}^{\pi} f(x)\sin(nx)dx$ where n is a positive integer. Our method is as follows. We begin with the familiar Fourier series of the function $f(x)$ with period 2π,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \tag{1.1}$$

It is well known [3] that the coefficients are given by,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \tag{1.2}$$

and for $n = 1, 2, 3,$ we have the remaining coefficients,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \tag{1.3}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \tag{1.4}$$

Now suppose we are given a known Taylors series,

$$g(z) = \sum_{n=0}^{\infty} c_n z^n \tag{1.5}$$

Let $z = re^{i\theta}$ and get

$$\begin{aligned} g(re^{i\theta}) &= \sum_{n=0}^{\infty} c_n r^n e^{in\theta} \\ &= c_0 + \sum_{n=1}^{\infty} c_n r^n \cos(n\theta) + i\sum_{n=1}^{\infty} c_n r^n \sin(n\theta) \end{aligned} \tag{1.6}$$

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Now, suppose we can conveniently decompose $g(re^{i\theta})$ into real and imaginary parts as

$$g(re^{i\theta}) = u(r,\theta) + iv(r,\theta) \tag{1.7}$$

Then comparing real and imaginary parts in (6) and (7) we get

$$u(r,\theta) = c_0 + \sum_{n=1}^{\infty} c_n r^n \cos(n\theta) \tag{1.8}$$

$$v(r,\theta) = \sum_{n=1}^{\infty} c_n r^n \sin(n\theta) \tag{1.9}$$

We see that u, viewed as a function of θ, (and holding r fixed), (8) is a Fourier cosine series for $u(r,\theta)$ and (9) is a Fourier sine series for $v(r,\theta)$. Thus we can write integrals for the coefficients $c_n r^n$ using (1) through (4) to get

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(r,\theta)d\theta, \tag{1.10}$$

and for $n = 1, 2, 3$,...

$$c_n r^n = \frac{1}{\pi}\int_{-\pi}^{\pi} u(r,\theta)\cos(n\theta)d\theta \text{ and,} \tag{1.11}$$

$$c_n r^n = \frac{1}{\pi}\int_{-\pi}^{\pi} v(r,\theta)\sin(n\theta)d\theta \tag{1.12}$$

We can use (10) through (12) to determine the values for some definite integrals.

2. A table of definite integrals

The simple procedure outlined above produces some surprisingly complex definite integrals. We demonstrate this in four tables. Table 1 shows the functions and series used to calculate the 36 definite integrals in Tables 2 to 4. The final columns in Tables 2 to 4 show the result of the authors search to find these integrals in either the noted volumes [1,2] or through the software program Mathematica. The integrals numbered 1, 6, 13, 19, 25, 28, and 31 were found to be known in this way and are marked “Known”. Four integrals were obviously true because the integrand is an odd function and are marked “Obvious”. The authors could not find closed forms in the literature for the remaining 25 integrals which are marked “May be new”. We did check these numerically by Mathematica for specific values of the parameters.

Table 1: Functions

No.	$g(z) = u + iv$	$u(r,\theta)$ $v(r,\theta)$	**Taylor Series**
(1)-(3)	$\frac{1}{a-z}$	$\frac{a-r\cos\theta}{a^2+r^2-2ar\cos\theta}$ $\frac{r\sin\theta}{a^2+r^2-2ar\cos\theta}$	$\sum_{n=1}^{\infty} \frac{z^n}{a^{n+1}}$
(4)-(6)	$\cos(z)$	$\cos(r\cos\theta)\cosh(r\sin\theta)$ $-\sin(r\cos\theta)\sinh(r\sin\theta)$	$\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$
(7)-(9)	$\sin(z)$	$\sin(r\cos\theta)\cosh(r\sin\theta)$ $\cos(r\cos\theta)\sinh(r\sin\theta)$	$\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$
(10)-(12)	$\arctan(z)$	$\frac{1}{2}\arctan\left(\frac{2r\cos\theta}{r^2-1}\right)$ $\frac{1}{4}\log\left(\frac{1+r^2+2r\sin\theta}{1+r^2-2r\sin\theta}\right)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^{2n-1}}{2n-1}$
(13)-(15)	e^z	$e^{r\cos\theta}\cos(r\sin\theta)$ $e^{r\cos\theta}\sin(r\sin\theta)$	$\sum_{n=0}^{\infty} \frac{z^n}{n!}$
(16)-(18)	$\tan(z)$	$\frac{\sin(2r\cos\theta)}{\cos(2r\cos\theta)+\cosh(2r\sin\theta)}$ $\frac{\sinh(2r\cos\theta)}{\cos(2r\cos\theta)+\cosh(2r\sin\theta)}$	$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}z^{2n-1}}{(2n)!}$
(19)-(21)	$\log(1+z)$	$\frac{1}{2}\log(1+r^2+2r\cos\theta)$ $\arctan\left(\frac{r\sin\theta}{1+r\cos\theta}\right)$	$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n}$
(22)-(24)	$\sinh(z)$	$\sinh(r\cos\theta)\cos(r\sin\theta)$ $\cosh(r\cos\theta)\sin(r\sin\theta)$	$\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$
(25)-(27)	$\frac{e^z}{1-z}$	$\frac{e^{r\cos\theta}(\cos(r\sin\theta)(1-r\cos\theta)-r\sin\theta\sin(r\sin\theta))}{1+r^2-2r\cos\theta}$ $\frac{e^{r\cos\theta}(\cos(r\sin\theta)r\sin\theta+(1-r\cos\theta)\sin(r\sin\theta))}{1+r^2-2r\cos\theta}$	$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{k!}\right)z^n$
(28)-(30)	$(e^z)^2$	$e^{2r\cos\theta}\cos(2r\sin\theta)$ $e^{2r\cos\theta}\sin(2r\sin\theta)$	$\sum_{n=0}^{\infty}\frac{2^n}{n!}z^n$
(31)-(33)	$\left(\frac{1}{1-z}\right)^2$	$\frac{r^2\cos(2\theta)-2r\cos\theta+1}{(1+r^2-2r\cos\theta)^2}$ $\frac{2r\sin\theta-r^2\sin(2\theta)}{(1+r^2-2r\cos\theta)^2}$	$\sum_{n=0}^{\infty}(1+n)z^n$
(34)-(36)	$\sqrt{z+1}$	$\cos\left(\frac{1}{2}\arctan\frac{r\sin\theta}{1+r\cos\theta}\right)\sqrt[4]{1+r^2+2r\cos\theta}$ $-\sin\left(\frac{1}{2}\arctan\frac{r\sin\theta}{1+r\cos\theta}\right)\sqrt[4]{1+r^2+2r\cos\theta}$	$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n-3)!}{2^{2n-2}n!(n-2)!}z^n$

Table 2: Integrals (1) - (12)

No.	**Integral**	**Conditions**	**Closed Form Found in Literature**
(1)	$\int_{-\pi}^{\pi} \frac{a-r\cos\theta}{a^2+r^2-2ar\cos\theta} d\theta = \frac{2\pi}{a}$	$r \in [0,a)$	Known
(2)	$\int_{-\pi}^{\pi} \frac{a-r\cos\theta}{a^2+r^2-2ar\cos\theta} \cos(n\theta) d\theta = \frac{\pi r^n}{a^{n+1}}$	$r \in [0,a)$ $n \in \mathbb{N}$	*May be new*
(3)	$\int_{-\pi}^{\pi} \frac{r\sin\theta}{a^2+r^2-2ar\cos\theta} \sin(n\theta) d\theta = \frac{\pi r^n}{a^{n+1}}$	$r \in [0,a)$ $n \in \mathbb{N}$	*May be new*
(4)	$\int_{-\pi}^{\pi} \cos(r\cos\theta)\cosh(r\sin\theta) d\theta = 2\pi$	$r \in [0,\infty)$	*May be new*
(5)	$\int_{-\pi}^{\pi} \cos(r\cos\theta)\cosh(r\sin\theta)\cos(2n\theta) d\theta = \frac{(-1)^n \pi r^{2n}}{(2n)!}$	$r \in [0,\infty)$ $n \in \mathbb{N}$	*May be new*
(6)	$\int_{-\pi}^{\pi} \sin(r\cos\theta)\sinh(r\sin\theta)\sin(2n\theta) d\theta = \frac{(-1)^{n+1} \pi r^{2n}}{(2n)!}$	$r \in [0,\infty)$ $n \in \mathbb{N}$	Known
(7)	$\int_{-\pi}^{\pi} \sin(r\cos\theta)\cosh(r\sin\theta) d\theta = 0$	$r \in [0,\infty)$	Obvious
(8)	$\int_{-\pi}^{\pi} \sin(r\cos\theta)\cosh(r\sin\theta)\cos((2n+1)\theta) d\theta = \frac{(-1)^n \pi r^{2n+1}}{(2n+1)!}$	$r \in [0,\infty)$ $n \in \mathbb{N}$	*May be new*
(9)	$\int_{-\pi}^{\pi} \cos(r\cos\theta)\sinh(r\sin\theta)\sin((2n+1)\theta) d\theta = \frac{(-1)^n \pi r^{2n+1}}{(2n+1)!}$	$r \in [0,\infty)$ $n \in \mathbb{N}$	*May be new*
(10)	$\int_{-\pi}^{\pi} \arctan\left(\frac{2r\cos\theta}{r^2-1}\right) d\theta = 0$	$r \in (0,\infty)$	Obvious
(11)	$\int_{-\pi}^{\pi} \arctan\left(\frac{2r\cos\theta}{r^2-1}\right) \cos((2n-1)\theta) d\theta = \frac{2\pi(-1)^n r^{2n-1}}{2n-1}$	$r \in (0,1)$ $n \in \mathbb{N}$	*May be new*
(12)	$\int_{-\pi}^{\pi} \log\left(\frac{1+r^2+2r\sin\theta}{1+r^2-2r\sin\theta}\right) \sin((2n-1)\theta) d\theta = \frac{4\pi(-1)^{n-1} r^{2n-1}}{2n-1}$	$r \in (0,1)$ $n \in \mathbb{N}$	*May be new*

Table 3: Integrals (13) - (24)

No.	Integral	Conditions	Closed Form Found in Literature
(13)	$\int_{-\pi}^{\pi} e^{r\cos\theta}\cos(r\sin\theta)d\theta = 2\pi$	$r \in [0,\infty)$	Known
(14)	$\int_{-\pi}^{\pi} e^{r\cos\theta}\cos(r\sin\theta)\cos(n\theta)d\theta = \frac{\pi r^n}{n!}$	$r \in [0,\infty)$ $n \in \mathbb{N}$	*May be new*
(15)	$\int_{-\pi}^{\pi} e^{r\cos\theta}\sin(r\sin\theta)\sin(n\theta)d\theta = \frac{\pi r^n}{n!}$	$r \in [0,\infty)$ $n \in \mathbb{N}$	*May be new*
(16)	$\int_{-\pi}^{\pi} \frac{\sin(2r\cos\theta)d\theta}{\cos(2r\cos\theta)+\cosh(2r\sin\theta)} = 0$	$r \in \left[0, \frac{\pi}{2}\right]$	Obvious
(17)	$\int_{-\pi}^{\pi} \frac{\sin(2r\cos\theta)\cos((2n-1)\theta)d\theta}{\cos(2r\cos\theta)+\cosh(2r\sin\theta)} = \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}\pi r^{2n-1}}{(2n)!}$	$r \in \left[0, \frac{\pi}{2}\right]$ $n \in \mathbb{N}$	*May be new*
(18)	$\int_{-\pi}^{\pi} \frac{\sinh(2r\cos\theta)\sin((2n-1)\theta)d\theta}{\cos(2r\cos\theta)+\cosh(2r\sin\theta)} = \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}\pi r^{2n-1}}{(2n)!}$	$r \in \left[0, \frac{\pi}{2}\right]$ $n \in \mathbb{N}$	*May be new*
(19)	$\int_{-\pi}^{\pi} \log(1+r^2+2r\cos\theta)d\theta = 0$	$r \in [0,1]$	Known
(20)	$\int_{-\pi}^{\pi} \log(1+r^2+2r\cos\theta)\cos(n\theta)d\theta = \frac{(-1)^{n+1}2\pi r^n}{n}$	$r \in [0,1]$ $n \in \mathbb{N}$	*May be new*
(21)	$\int_{-\pi}^{\pi} \arctan\left(\frac{r\sin\theta}{1+r\cos\theta}\right)\sin(n\theta)d\theta = \frac{(-1)^{n+1}\pi r^n}{n}$	$r \in [0,1]$ $n \in \mathbb{N}$	*May be new*
(22)	$\int_{-\pi}^{\pi} \sinh(r\cos\theta)\cos(r\sin\theta)d\theta = 0$	$r \in [0,\infty)$	Obvious
(23)	$\int_{-\pi}^{\pi} \sinh(r\cos\theta)\cos(r\sin\theta)\cos((2n+1)\theta)d\theta = \frac{\pi r^{2n+1}}{(2n+1)!}$	$r \in [0,\infty)$ $n \in \mathbb{N}$	*May be new*
(24)	$\int_{-\pi}^{\pi} \cosh(r\cos\theta)\sin(r\sin\theta)\sin((2n+1)\theta)d\theta = \frac{\pi r^{2n+1}}{(2n+1)!}$	$r \in [0,\infty)$ $n \in \mathbb{N}$	*May be new*

Table 4: Integrals (25) - (36)

No.	Integral	Conditions	Closed Form Found in Literature
(25)	$\int_{-\pi}^{\pi} \frac{e^{r\cos\theta}(\cos(r\sin\theta)(1-r\cos\theta)-r\sin\theta\sin(r\sin\theta))}{1+r^2-2r\cos\theta} d\theta = 2\pi$	$r \in [0,1)$	Known
(26)	$\int_{-\pi}^{\pi} \frac{e^{r\cos\theta}(\cos(r\sin\theta)(1-r\cos\theta)-r\sin\theta\sin(r\sin\theta))}{1+r^2-2r\cos\theta} \cos(n\theta) d\theta = \pi r^n \sum_{k=0}^{n} \frac{1}{k!}$	$r \in [0,1)$ $n \in \mathbb{N}$	*May be new*
(27)	$\int_{-\pi}^{\pi} \frac{e^{r\cos\theta}(\cos(r\sin\theta)r\sin\theta+(1-r\cos\theta)\sin(r\sin\theta))}{1+r^2-2r\cos\theta} \sin(n\theta) d\theta = \pi r^n \sum_{k=0}^{n} \frac{1}{k!}$	$r \in [0,1)$ $n \in \mathbb{N}$	*May be new*
(28)	$\int_{-\pi}^{\pi} e^{2r\cos\theta} \cos(2r\sin\theta) d\theta = 2\pi$	$r \in [0,\infty)$	Known
(29)	$\int_{-\pi}^{\pi} e^{2r\cos\theta} \cos(2r\sin\theta)\cos(n\theta) d\theta = \frac{2^n \pi r^n}{n!}$	$r \in [0,\infty)$ $n \in \mathbb{N}$	*May be new*
(30)	$\int_{-\pi}^{\pi} e^{2r\cos\theta} \sin(2r\sin\theta)\sin(n\theta) d\theta = \frac{2^n \pi r^n}{n!}$	$r \in [0,\infty)$ $n \in \mathbb{N}$	*May be new*
(31)	$\int_{-\pi}^{\pi} \frac{r^2\cos(2\theta)-2r\cos\theta+1}{(1+r^2-2r\cos\theta)^2} d\theta = 2\pi$	$r \in [0,1)$	Known
(32)	$\int_{-\pi}^{\pi} \frac{r^2\cos(2\theta)-2r\cos\theta+1}{(1+r^2-2r\cos\theta)^2} \cos(n\theta) d\theta = \pi r^n (n+1)$	$r \in [0,1)$ $n \in \mathbb{N}$	*May be new*
(33)	$\int_{-\pi}^{\pi} \frac{2r\sin\theta-r^2\sin(2\theta)}{(1+r^2-2r\cos\theta)^2} \sin(n\theta) d\theta = \pi r^n (n+1)$	$r \in [0,1)$ $n \in \mathbb{N}$	*May be new*
(34)	$\int_{-\pi}^{\pi} \cos\left(\frac{1}{2}\arctan\frac{r\sin\theta}{1+r\cos\theta}\right) \sqrt[4]{1+r^2+2r\cos\theta} d\theta = 2\pi$	$r \in [0,1]$	*May be new*
(35)	$\int_{-\pi}^{\pi} \cos\left(\frac{1}{2}\arctan\frac{r\sin\theta}{1+r\cos\theta}\right) \sqrt[4]{1+r^2+2r\cos\theta} \cos(n\theta) d\theta = \frac{\pi r^n (-1)^{n+1}(2n-3)!}{2^{2n-2} n!(n-2)!}$	$r \in [0,1]$ $n \in \mathbb{N}$	*May be new*
(36)	$\int_{-\pi}^{\pi} \sin\left(\frac{1}{2}\arctan\frac{r\sin\theta}{1+r\cos\theta}\right) \sqrt[4]{1+r^2+2r\cos\theta} \sin(n\theta) d\theta = \frac{\pi r^n (-1)^{n}(2n-3)!}{2^{2n-2} n!(n-2)!}$	$r \in [0,1]$ $n \in \mathbb{N}$	*May be new*

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