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On $\aleph_0$-sn-networks

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ABSTRACT. In this paper, we discuss spaces with $\aleph_0$-sn-networks and give some characterizations of $\aleph_0$-sn-metric spaces. For a space X, we prove that X has a point-countable $\aleph_0$-sn-network if and only if X is an sequentially quotient, countable-to-one image of a metric space. and X is an $\aleph_0$-sn-metric space if and only if X is a sequentially quotient, σ, countable-to-one image of a metric space.

1. Introduction

How to study of images of metric spaces under certain mapping? It is an important question in general topology [**9**]. The Liu and Lin [**14**] introduced the concept of $\aleph_0$-weak base and prove that a space X is a quotient, countable-to-one image of a metric space if and only if X has a point-countable $\aleph_0$-weak base. Note that $\aleph_0$-sn-networks is a generalization of $\aleph_0$-weak bases, an interesting question is how characterize $\aleph_0$-sn-networks and $\aleph_0$-sn-metric spaces. In this paper, we discuss spaces with $\aleph_0$-sn-networks and give some characterizations of $\aleph_0$-sn-metric spaces. For a space X, we prove that X has a point-countable $\aleph_0$-sn-network if and only if X is an sequentially quotient, countable-to-one image of a metric space. and X is an $\aleph_0$-sn-metric space if and only if X is a sequentially quotient, σ, countable-to-one image of a metric space. Throughout this paper, all spaces are regular T_1, all mappings are continuous and onto. N denotes the set of positive integer numbers. Sequence $\{x_n : n \in N\}$, sequence $\{P_n : n \in N\}$ of subsets and sequence $\{\mathcal{P}_n : n \in N\}$ of collections of subsets are abbreviated to $\{x_n\}$, $\{P_n\}$ and $\{\mathcal{P}_n\}$, respectively. For terms which are not defined here, please refer to [**1, 3, 10, 15**].

DEFINITION 1.1 ([**14**]). Let $\mathcal{B}$ be a family of subsets of a space X. $\mathcal{B}$ is called to be an $\aleph_0$-weak base for X if $\mathcal{B}=\cup\{\mathcal{B}_x(n) : x \in X, n \in N\}$ and satisfies the following (a) and (b)

(a)For each $x \in X$ and each $n \in N$, $\mathcal{B}_x(n)$ is a network at x, which is closed under finite intersections and $x \in \cap\mathcal{B}_x(n)$.

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(b) A subset U of X is open if and only if whenever $x \in U$ and $n \in N$ there exists a $B_x \in \mathcal{B}_x(n)$ such that $B_x(n) \subset U$.

X is called $\aleph_0$-weakly first-countable in the sense of Sirois-Dumais [**15**] if X has an $\aleph_0$-weak base $\mathcal{B} = \cup\{\mathcal{B}_x(n) : x \in X, n \in N\}$, where $\mathcal{B}_x(n)$ is countable for each $x \in X$ and each $n \in N$.

Definition 1.2. Let $\mathcal{B}$ be a family of subsets of a space X. $\mathcal{B}$ is called to be an $\aleph_0$-*sn*-network for X if $\mathcal{B} = \cup\{\mathcal{B}_x(n) : x \in X, n \in N\}$ and satisfies the following (a) and (b).

(a)For each $x \in X$ and each $n \in N$, $\mathcal{B}_x(n)$ is a network at x, which is closed under finite intersections and $x \in \cap\mathcal{B}_x(n)$.

(b) If L is a sequence converging to $x \notin L$ in X, then there is a subsequence L' of L and $n_0 \in N$ such that L' is eventually in $B_x(n_0)$ for any $B_x(n_0) \in \mathcal{B}_x(n_0)$.

X is called $\aleph_0$-*sn*-weakly first-countable (in the sense of Sirois-Dumais [**15**]) if X has an $\aleph_0$-*sn*-network $\mathcal{B} = \cup\{\mathcal{B}_x(n) : x \in X, n \in N\}$, where $\mathcal{B}_x(n)$ is countable for each $x \in X$ and $n \in N$. X is called $\aleph_0$-*sn*-metric space if X has a σ-locally finite $\aleph_0$-*sn*-network. If $\mathcal{B}_x(n) = \mathcal{B}_x(1)$ for each $n \in N$ in the definition of $\aleph_0$-*sn*-networks, then $\mathcal{B}$ is called to be an *sn*-network [**4**] for X. X is called *sn*-first countable (in the sense of S. Lin and P. Yan [**13**]) if $\mathcal{B}_x(1)$ is countable for each $x \in X$.

$\aleph_0$-*sn*-network is a generalization of $\aleph_0$-weak base and *sn*-network. It is easy to see that $\aleph_0$-weak base doesn't imply *sn*-network. For example, S_ω has a countable $\aleph_0$-weak base but it does not have a countable *sn*-network (since Frechet space with a countable *sn*-network has a countable base). *sn*-network does not imply $\aleph_0$-weak base, for example, $\beta\omega$ (Stone-Cech compactification of ω) is $\aleph_0$-*sn*-weakly first-countable since every convergent sequence is finite. but it is not $\aleph_0$-weakly first-countable since it is not a sequential space (a sequential space in which every convergent sequence is finite is a discrete space).

Definition 1.3 ([**13**]). Let X be a space, $P \subset X$ is called a sequential neighborhood of x in X, if each sequence converging to x in X is eventually in P.

Definition 1.4 ([**8**]). Let $\mathcal{F}$ be a cover of a space X. $\mathcal{F}$ is a *cs*-network of X, if every convergent sequence S converging to a point $x \in U$ with U open in X, then S is eventually in $F \subset U$ for some $F \in \mathcal{F}$.

Definition 1.5 ([**2**]). Let $f : X \longrightarrow Y$ be a sequentially quotient mapping if whenever $\{y_n\}$ is a convergent sequence in Y, there is a convergent sequence $\{x_k\}$ in X with each $x_k \in f^{-1}(y_{n_k})$.

Definition 1.6 ([**12**]). Let $f : X \longrightarrow Y$ be an s-mapping if $f^-1(y)$ is separable for each $y \in Y$; f is called a countable-to-one mapping if $f^-1(y)$ is countable for each $y \in Y$.

2. Main Results

Theorem 2.1. Let X be an $\aleph_0$-*sn*-weakly first-countable space and $\mathcal{P}$ be a point-countable *cs*-network for X. If $\mathcal{P}$ is closed under finite intersections, then there exists a subfamily $\mathcal{B}$ of $\mathcal{P}$ such that $\mathcal{B}$ is an $\aleph_0$-*sn*-network for X.

PROOF. Let $\cup\{\mathcal{B}_x(n) : x \in X, n \in N\}$ be an $\aleph_0$-sn-network of X, here each $\mathcal{B}_x(n) = \{B_x(n,m) : m \in N\}$ with $B_x(n,m+1) \subset B_x(n,m)$ for each $m \in N$. For each $n \in N$, let $\mathcal{P}_x(n) = \{P \in \mathcal{P} : B_x(n,m) \subset P\ for\ some\ m \in N\}$. Then $\mathcal{P}_x(n)$ is closed under finite intersections. To prove $\mathcal{B}$ is an $\aleph_0$-sn-network for X, we need to prove the following claim.

Claim: $\mathcal{P}_x(n)$ is a network of x for each $x \in X$ and each $n \in N$.

If there are $x \in X$ and $n \in N$ such that $\mathcal{P}_x(n)$ is not a network of x, then there is a neighborhood U of x in X such that $P \not\subset U$ for each $P \in \mathcal{P}_x(n)$. Let $\{P \in \mathcal{P} : x \in P \subset U\} = \{P_k : k \in N\}$. Then $B(n,m) \not\subset U$ for any $m,k \in N$. Pick $x_{mk} \in B(n,m) \setminus P_k$ for each $m \geqslant k$. Let $y_i{=}x_{mk}$, where $i = k+m(m-1)/2$. Then the sequence $\{y_i\}$ converges to x in X because $\{B_x(n,m) : m \in N\}$ is a decreasing network of x in X. Since $\mathcal{P}$ is a cs-network of X, there is $k,j \in N$ such that $\{y_i : i \geqslant j\} \subset P_k$. Pick $i \geqslant j$ such that $y_i = x_{mk}$ for some $m \geqslant k$, then $x_{mk} \in P_k$, a contradiction.

Put $\mathcal{B} = \cup\{\mathcal{P}_x(n) : x \in X, n \in N\}$. Then $\mathcal{B}$ is countable.

Now we prove that $\mathcal{B}$ is an $\aleph_0$-sn-network. Let L be a sequence converging to $x \notin L$ in X. Then there is a subsequence $L^{'}$ of L and $n_0 \in N$ such that $L^{'}$ is eventually in $B_x(n_0,m)$ for any $m \in N$. But $B_x(n_0,m){\subset}P_x(n_0)$ for some $m \in N$, $L^{'}$ is eventually in $P_x(n_0)$ for any $P_x(n_0) \in \mathcal{P}_x(n_0)$. So $\mathcal{B}$ is an $\aleph_0$-sn-network for X. □

THEOREM 2.2. The following are equivalent for a space X.

(1)X has a point-countable $\aleph_0$-sn-network;

(2)There is a metric space M and a sequentially quotient, countable-to-one mapping $f : M \to X$;

(3)There is a metric space M and a sequentially quotient, s-mapping $f : M \to X$ such that $|\partial f^{-1}(y)| \leqslant \omega$ for each $x \in X$.

PROOF. (1) $\Leftrightarrow$ (2) due to [**1**] and (2) $\Rightarrow$ (3) is obvious. We prove that (3) $\Rightarrow$ (1).

Let $\mathcal{B}$ be a point countable base for space M. For each non-isolated point $x \in X$, denotes $\partial f^{-}1(y)$ by $\{x_n : n \in N\}$. Let $\mathcal{B}_x(n) = \{B_x(n,m) : m \in N\} \subset \mathcal{B}$ be a countable local base at $x_n \in M$ such that $B_x(n,m+1) \subset B_x(n,m)$ for each $m,n \in N$. Put $P_x(n,m) = f(B_x(n,m))$ if x is a non-isolated point in X; $P_x(n,m) = \{x\}$ if x is an isolated point in X. $\mathcal{P}_x(n) = \{P_x(n,m) : m \in N\}$, $\mathcal{P} = \cup\{\mathcal{P}_x(n) : x \in X, n \in N\}$.

Since f is an s-mapping, $\mathcal{P}$ is point-countable. It is easy to see $\mathcal{P}_x(n)$ is a decreasing network at x for each $x \in X$ and $n \in N$. We shall prove that $\mathcal{P}$ is an $\aleph_0$-sn-network for X.

without loss of generality, we can assume that $x \notin L$, then x is a non-isolated point in X. Since f is sequentially quotient, there is a sequence S in M such that $f(S)$ is a subsequence of L and S converges to some $y \in f^{-}1(y)$. Obviously, $y \in \partial f^{-}1(y)$, so there is $n_0 \in N$ such that $y = x_{n_0}$, hence $f(S)$ is eventually in $P_x(n_0,m)$ for each $m \in N$, therefore $\mathcal{P}$ is an point-countable $\aleph_0$-sn-network for X. □

THEOREM 2.3. For a space X, the following are equivalent:

(1) X is a $\aleph_0$-sn-metric space;

(2) There is a metric space M and a sequentially quotient, σ, countable-to-one mapping $f : M \longrightarrow X$.

PROOF. (1) $\longrightarrow$ (2): Let $\mathcal{P} = \cup\{\mathcal{P}_x(n) : x \in X, n \in N\}$ be a σ-locally finite $\aleph_0$-sn-network for X, here each $\mathcal{P}_x(n) = \{P_x(n,m) : m \in N\}$ with $P_x(n,m+1) \subset P_x(n,m)$ for each $m \in N$. For each $n \in N$, put $\mathcal{P} = \cup\{\mathcal{P}_i : i \in N\}$, here $\mathcal{P}_i$ is locally finite and $\mathcal{P}_i \subset \mathcal{P}_{i+1}$ for each $i \in N$. For each $x \in X$, select $i(x,n,m) \in N$ such that $P_x(n,m) \in \mathcal{P}_{i(x,n,m)}$ and $i(x,n,m) < i(x,n,m+1)$. Put $B_x(n,m) = X$ whenever $i < i(x,n,1)$; $B_x(n,m) = P_x(n,m)$ whenever $i(x,n,m+1) \leqslant i < i(x,n,m+1)$. Put $\mathcal{B}_x(n) = \{B_x(n,i) : i \in N\}$ and $\mathcal{B} = \cup\{\mathcal{B}_x(n) : x \in X, n \in N\}$. Then $\mathcal{B}$ is an $\aleph_0$-sn-network for X satisfying $B_x(n,i+1) \subset B_x(n,i)$ and $B_x(n,i) \in \mathcal{P}_i$ for each $x \in X, n, i \in N$.

We rewrite $\mathcal{P}_i$=$\{B_\alpha : \alpha \in I_i\}$. Endow I_i with discrete topology for each $i \in N$.

We call two families $\{R_n : n \in N\}$ and $\{Q_m : m \in N\}$ of subsets of a space are cofinal (in the sense of [**14**]) if there are $n_0, m_0 \in N$ such that $R_{n_0+i} = Q_{m_0+i}$ for each $i \in N$. Put

$M = \{\alpha = (\alpha_i) \in \prod_{i\in N} I_i : \{B_{\alpha_i} : i \in N\}$ *is cofinal to* $\mathcal{B}_{x(\alpha)}(n)$ *for some* $x(\alpha) \in X, n \in N, \{B_{\alpha_i} : i \in N\}$ *is a network of* $x(\alpha)\}$.

Define $f : M \to X$ as $f(\alpha_i) = x(\alpha)$. Since each $\mathcal{B}_x(n)$ is a network of x in X for each $n \in N$. It is easy to see that f is well-defined and onto, M is a metric space, and f is continuous. Note that each $\mathcal{P}_i$ is locally finite. So f is countable-to-one.

Put $D(\alpha_1, \alpha_2, ...\alpha_n) = \{\beta = (\beta_i) \in M : \beta_i = \alpha_i, i \leqslant n\}$ and $\mathcal{D} = \{D(\alpha_1, \alpha_2, ...\alpha_n) : \alpha_i \in I_i, i \leqslant n, n \in N\}$. It is routine to show $\mathcal{D}$ is a base for M and $f(D(\alpha_1, \alpha_2, ...\alpha_n)) = \cap\{B_{\alpha_i} : i \leqslant n\}$. We only need to prove that f is sequentially quotient mapping.

Let $\mathcal{B} = \cup\{\mathcal{B}_x(n) : x \in X, n \in N\}$ be a point-countable $\aleph_0$-sn-network. Let L be a sequence converging to $x \notin L$ in X. Then there is a subsequence L' of L and $n_0 \in N$ such that L' is eventually in $B_x(n_0, m)$ for any $m \in N$. For each $i \in N$, take $\alpha_i \in I_i$ with $B_{\alpha_i} = B_x(n_0, i)$. Let $\alpha = (\alpha_i)$, then $\alpha \in M$. For each $k \in N$, put $n_k = min\{m \in N : x_k \notin B_x(n_0, m)\}$. We put $z_k = (\beta_i(k)) \in \prod_{i\in N} I_i$ an follows: if $i < n_k$, pick $\beta_i(k) \in I_i$ with $B_{\beta_i(k)} = B_x(n_0, i)$; otherwise pick $\beta_i(k) \in I_i$ such that $B_{\beta_i(k)} = B_{x_k}(1, i - n_k + 1)$. Then $\{B_{\beta_i}(k) : i \in N\}$ is cofinal to $\mathcal{B}_{x_k}(1)$, thus $z_k \in M$ and $f(z_k) = x_k$. On the other hand, for each $i \in N$, there exists $k_0 \in N$ such that $x_k \in B_x(n_0, i)$ for any $k \geqslant k_0$ because L' is eventually in $B_x(n_0, i)$. Then $i < n_k$ when $k \geqslant k_0$ by the definition of n_k, so $\beta_i(k) = \alpha_i$. It means that $\{B_{\beta_i}(k) : i \in N\}$ converges to α_i in the discrete space I_i. Hence z_k converges to α in M. Therefore, f is a sequentially quotient mapping. $\square$

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References

1. A. V. Arhangel'skill, Mapping and spaces, Russian Math.Surveys, 21(1966), 115-162.
2. J. R. Boone and F. Siwiec, sequentially quotient mappings, Czech. Math. J., 26(1976), 174-182.

3. L. Foged, Characterizations of $\aleph$-space, Pacific J. Math., 110(1984), 59-63.
4. Y. Ge, On spaces with a σ-locally finite universal cs-network, Questions Answers in General Topology, 18(2000),93-96.
5. Y. Ge, sn-metric space, Acta Math. Sinica, 45(2002), 355-360. (In Chinese)
6. Y. Ge, Spaces with countable sn-networks, Comments. Math. Univ. Carolinae, 45(2004), 169-176.
7. Y. Ge, Networks in Ponomarev-systems, Series A: Mathematical Sciences, 11(2005), 25-29.
8. J.A.Guthrie, A characterization of $\aleph_0$-spaces, Gen. Top. Appl., 1(1971), 105-110.
9. Z. Li and Y. Ge, A question on images of metric spaces, Series A: Mathematical Sciences, 12(2006), 5-7.
10. S. Lin, Generalized metric spaces and mapping, Chinese Science Press, Beijing, 1995. (In Chinese)
11. S. Lin, A note on sequence-covering mappings, Acta Math. Hungar., 107(2005), 187-191.
12. S. Lin and Y. Tanaka, Point-countable k-networks, closed maps, and related results, Topology Appl., 59(1994), 79-86.
13. S. Lin and P. Yan, Sequence-covering maps of metric spaces, Topology Appl.,109(2001), 301-314.
14. C. Liu and S. Lin, Topology Appl., 154(2007), 449C454.
15. R. Sirois-Dumais, Quasi-and weakly quasi-first-countable spaces, Topology Appl., 11(1980), 223-230.

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