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On g-scattered spaces

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Abstract. This paper is devoted to investigate scatteredness on generalized topological spaces. The concept of g-scattered spaces is introduced. Their characterizations, properties and mapping theorems are obtained.

1. Introduction

The theory of generalized topological spaces, which was founded by Császár in recent years, is one of the most important development of general topology (see [**4, 5, 6, 7, 8**]). To make progress in applications of generalized topologies, some researchers have investigated generalized separation axioms [**7, 13**], generalized extremally disconnectedness [**9**], generalized hyperconnectedness [**10**], weak continuity and contra continuity on generalized topological spaces [**14, 16**], Baireness on generalized topological spaces [**15**].

A scattered space is defined as a topological space in which every nonempty subspace has its isolated points. All ordinal spaces are scattered. Scattered spaces are a class of important topological spaces. They have been researched deeply (see [**1, 2, 3, 12, 17, 18**]).

The purpose of this paper is to study scatteredness on generalized topological spaces. The concept of g-scattered spaces is introduced. Their characterizations, properties and mapping theorems are investigated.

2. Preliminaries

We recall some basic concepts and results.

Let X be a nonempty set and let 2^X be the power set of X. $g \subset 2^X$ is called a generalized topology [**4**] (briefly, GT) on X, if

(1) $\emptyset \in g$

(2) $G_i \in g$ for each $i \in I \neq \emptyset$ implies $\bigcup_{i \in I} G_i \in g$

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The pair (X, g) is called a generalized topological space (briefly, GTS). The elements of g are called g-open [**4**] subsets of X and the complements are called g-closed subsets of X.

Let (X, g) be a GTS and let $x \in X$ and $A \subset X$. We denote $g' = \{A : X - A \in g\}$. The family of all g-open (resp. g-closed) subsets of (X, g) is also denoted by g_X (resp. g'_X), i.e., $g_X = g$ (resp. $g'_X = g'$). The family of all g-open (resp. g-closed) subsets of (X, g) containing x is denoted by $g(x)$ or $g_X(x)$ (resp. $g'(x)$ or $g'_X(x)$). The closure of A and the interior of A in (X, g) are respectively defined as follows:

$$cl_g(A) = \bigcap\{F : F \in g' \text{ and } A \subset F\},$$

$$int_g(A) = \bigcup\{U : U \in g \text{ and } U \subset A\}.$$

The operators $cl_g(\cdot)$ and $int_g(\cdot)$ are studied in [**4**] where it is observed that $cl_g(\cdot)$ and $int_g(\cdot)$ are idempotent (i.e. $cl_g(cl_g(A)) = cl_g(A)$, $int_g(int_g(A)) = int_g(A)$ for $A \subset X$) and monotonic (i.e. $cl_g(A) \subset cl_g(B)$, $int_g(A) \subset int_g(B)$ for $A \subset B \subset X$).

A GTS (X, g) is called strong [**6**], if $X \in g$. Clearly,

$$(X, g) \text{ is strong} \iff cl_g(\emptyset) = \emptyset \iff \emptyset \in g' \iff X \in g.$$

In this paper, spaces always mean GTS's on which no separation axiom is assumed, and all mappings are onto. N denotes the set of all natural numbers. We simply use cA and iA instead of $cl_g(A)$ and $int_g(A)$, respectively.

Let (X, g) be a GTS and let $A \subset S \subset X$. Then (S, g_S) is called a subspace of (X, g), where $g_S = \{U \cap S : U \in g\}$ is a GT on S. We denote the closure of A and the interior of A in the subspace (S, g_S) by $c_S A$ and $i_S A$, respectively.

Lemma 2.1 ([**4**]). *Let (X, g) be a GTS and let $A \subset X$. Then*
(1) $cA = X - i(X - A)$.
(2) $iA = X - c(X - A)$.

Lemma 2.2 ([**9**]). *Let (X, g) be a GTS and let $A \subset X$. Then $x \in cA$ if and only if $V \cap A \neq \emptyset$ for any $V \in g(x)$.*

Lemma 2.3. *Let (X, g) be a GTS and let $A \subset S \subset X$. Then $c_S A = cA \cap S$.*

Proof. This is obvious. □

Definition 2.1. Let (X, g) be a GTS and let $x \in A \subset X$.
(1) x is called a g-isolated point of A in X, if there exists $U \in g(x)$ such that $U \cap A = \{x\}$.
(2) x is called a g-limit point of A in X, if $U \cap (A - \{x\}) \neq \emptyset$ for any $U \in g(x)$.

The set of all g-isolated points of A in X is denoted by $I_g(A)$, short for $I(A)$. The set of all g-limit points of A in X is denoted by $d_g(A)$, short for $d(A)$, which is called the g-derived set of A in X.

PROPOSITION 2.1. *Let (X, g) be a GTS and let $A, B \subset X$.*
(1) $I(A) \subset A$.
(2) $I(A) = A - d(A)$.
(3) *a)* $A = I(A) \cup (d(A) \cap A)$;
b) $d(A) \cap A = A - I(A)$.
(4) *a)* $I(A) \cap I(B) \subset I(A \cap B)$;
b) $I(A \cup B) \subset I(A) \cup I(B)$.

PROOF. (1) This is obvious.
(2) Let $x \in I(A)$. Then there exists $U \in \tau(x)$ such that $U \cap A = \{x\}$. This implies $U \cap (A - \{x\}) = \emptyset$. Then $x \notin d(A)$. Thus $x \in A - d(A)$ and so $I(A) \subset A - d(A)$. Conversely, let $x \in A - d(A)$. Since $x \notin d(A)$, there exists $U \in \tau(x)$ such that $U \cap (A - \{x\}) = \emptyset$. Note that $U \cap A = \{x\}$. Then $x \in I(A)$ and so $I(A) \supset A - d(A)$. Hence $I(A) = A - d(A)$.
(3) *a)* For any $x \in A$ and $U \in \tau(x)$, $U \cap A = \{x\}$ or $U \cap \{A - \{x\}\} \neq \emptyset$, then $x \in I(A) \cup d(A)$ and $A \subset I(A) \cup d(A)$. Thus $A \subset (I(A) \cup d(A)) \cap A = I(A) \cup (d(A) \cap A)$. And $A \supset (I(A) \cup d(A)) \cap A$. Hence $A = I(A) \cup (d(A) \cap A)$.
b) This holds by *a)*.
(4) This is obvious. □

PROPOSITION 2.2. *Let (X, g_1) and (X, g_2) be two GTS's with $g_1 \subset g_2$. Then $I_{g_1}(A) \subset I_{g_2}(A)$ for any $A \subset X$.*

PROOF. This is obvious. □

3. g-scattered spaces

3.1. The concept of g-scattered spaces. Recall that a topological space (X, τ) is called scattered, if every nonempty subset has its isolated points.

DEFINITION 3.1. Let (X, g) be a GTS. X is called g-scattered, if $I_g(A) \neq \emptyset$ for any $A \in 2^X - \{\emptyset\}$.

It is clear that every scattered space is g-scattered. But the following example illustrates that the converse is not true.

EXAMPLE 3.2. Let $X = N$, $\mathcal{B} = \{\{1\}\} \cup \{\{i, i+1\} : i \in N\}$ and $g = \{G : G = \cup B' \; for some B' \subset B\} \cup \{\emptyset\}$. Then (X, g) is a GTS.

Since $\{1, 2\} \cap \{2, 3\} = \{2\} \notin g$, g is not a topology on X. Then (X, g) is not scattered.
Let $A \in 2^X - \{\emptyset\}$.
If $1 \in A$, then $\{1\} \in g(1)$ and $\{1\} \cap A = \{1\}$. So $1 \in I(A)$. This implies $I(A) \neq \emptyset$.
If $1 \notin A$, then $\{a-1, a\} \in g(a)$ and $\{a-1, a\} \cap A = \{a\}$, where $a = min\ A$. So $a \in I(A)$. This implies $I(A) \neq \emptyset$.
Hence (X, g) is g-scattered.

3.2. Characterizations of g-scattered spaces.

DEFINITION 3.3 ([**10**]). Let (X, g) be a GTS. $A \subset X$ is called g-dense in X, if $cA = X$.

Let (X, g) be a GTS. The family of all g-dense subsets of X is denoted by $\mathcal{D}$. For the subspace (Y, g_Y), the family of all g_Y-dense subsets of Y is denoted by $\mathcal{D}(Y)$, i.e. $\mathcal{D}(Y) = \{A \subset Y : c_Y A = Y\}$. Obviously, $\mathcal{D}(X) = \mathcal{D}$.

LEMMA 3.1. *Let (X, g) be a GTS and let $A \subset X$. Then A is g-dense in X if and only if $U \cap A \neq \emptyset$ for any $U \in g - \{\emptyset\}$.*

PROOF. *Necessity.* Let A be g-dense in X. Suppose $U \cap A = \emptyset$ for some $U \in g - \{\emptyset\}$. Pick $x \in U$. Clearly, $U \in g(x)$ and $x \in X = cA$. Then $U \cap A \neq \emptyset$, a contradiction.

Sufficiency. Suppose $cA \neq X$. Then $X - cA \neq \emptyset$. Put $U = X - cA$. So $U \in g - \{\emptyset\}$ and $U \cap A = (X - cA) \cap A = \emptyset$. This is a contradiction. □

THEOREM 3.4. *Let (X, g) be a GTS. The following are equivalent.*

(1) *X is g-scattered.*
(2) *For each $A \in 2^X - \{\emptyset\}$, $A \not\subset d(A)$.*
(3) *If $A \in g' - \{\emptyset\}$, then $I(A) \neq \emptyset$.*
(4) *$I(A) \in \mathcal{D}(A)$ for any $A \in 2^X - \{\emptyset\}$;*
(5) *For any $A \in 2^X - \{\emptyset\}$, $D \in \mathcal{D}(A)$ if and only if $D \supset I(A)$;*
(6) *$d(A) = d(I(A))$ for any $A \in 2^X - \{\emptyset\}$;*

PROOF. (1) $\Leftrightarrow$ (2) holds by Proposition 2.5(2).

(1) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Let $A \in 2^X - \{\emptyset\}$. Since $cA \in g' - \{\emptyset\}$, by (3), $I(cA) \neq \emptyset$. Pick $x \in I(cA)$. Then $U \cap cA = \{x\}$ for some $U \in g(x)$.

Suppose $U \cap A = \emptyset$. We have $X - U \supset A$. Then $X - U \supset cA$. So $U \cap cA = \emptyset$, a contradiction. Thus $U \cap A \neq \emptyset$.

Since $U \cap A \subset U \cap cA = \{x\}$, we have $U \cap A = \{x\}$. So $x \in I(A)$. This implies $I(A) \neq \emptyset$.

Hence X is g-scattered.

(1) $\Rightarrow$ (4) Let $A \in 2^X - \{\emptyset\}$. For any $V \in g_A - \{\emptyset\}$, $V = W \cap A$ for some $W \in g$. Since (X, g) is g-scattered, $I(V) \neq \emptyset$. Pick $x \in I(V)$, $U \cap V = \{x\}$ for some $U \in g(x)$. So $(U \cap W) \cap A = U \cap (W \cap A) = U \cap V = \{x\}$. Note that $U \cap W \in g(x)$. Then $x \in I(A)$. This implies $x \in V \cap I(A)$ and then $V \cap I(A) \neq \emptyset$. By Lemma 4.7, $c_A I(A) = A$. Thus, $I(A) \in \mathcal{D}(A)$.

(4) $\Rightarrow$ (5) Let $D \supset I(A)$. By (4), $A = c_A I(A) \subset c_A D$. Thus $D \in \mathcal{D}(A)$.

Conversely, suppose $D \not\supset I(A)$ for some $D \in \mathcal{D}(A)$. Then $I(A) - D \neq \emptyset$. Pick $x \in I(A) - D$. Then $U \cap A = \{x\}$ for some $U \in g(x)$. Note that $U \cap A \in g_A - \{\emptyset\}$ and $D \in \mathcal{D}(A)$. By Lemma 3.6, $D \cap (U \cap A) \neq \emptyset$. But $D \cap (U \cap A) = D \cap \{x\} = \emptyset$. This a contradiction.

(5) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (6) Since $A \supset I(A)$, we have $d(A) \supset d(I(A))$. It suffices to show $d(A) \subset d(I(A))$.

Suppose $d(A) \not\subset d(I(A))$. Then $d(A) - d(I(A)) \neq \emptyset$. Pick $x \in d(A) - d(I(A))$. By Proposition 2.5(2), $I(A) = A - d(A)$. Since $x \in d(A)$, $x \notin I(A)$.

Since $x \notin d(I(A))$, there exists $U \in g(x)$ such that $U \cap (I(A) - \{x\}) = \emptyset$. Note that $x \notin I(A)$. Then $(U \cap A) \cap I(A) = U \cap I(A) = \emptyset$ with $U \cap A \in g_A$.

By (4), $I(A) \in \mathcal{D}(A)$. Then $V \cap I(A) \neq \emptyset$ for any $V \in g_A$. This is a contradiction. Hence $d(A) = d(I(A))$.

(6) $\Rightarrow$ (1) Suppose $I(A) = \emptyset$ for some $A \in 2^X - \{\emptyset\}$. By (6), $d(A) = d(I(A)) = d(\emptyset) = \emptyset$. By Proposition 2.5(3), $A = I(A) \cup (d(A) \cap A) = \emptyset$. This is a contradiction. □

Definition 3.5. Let X be a GTS. Put $X^0 = X$ and

$$X^1 = \{x \in X : x \text{ is not } g\text{-isolated in } X\}.$$

Let α be any ordinal number. If X^β is already defined for all ordinal $\beta < \alpha$, then we put

$$X^\alpha = \begin{cases} (X^\beta)^1, & \text{if } \alpha = \beta + 1 \text{ and } \beta \text{ is an ordinal number,} \\ \bigcap\limits_{\beta<\alpha} X^\beta, & \text{if } \alpha \text{ is a limit ordinal number.} \end{cases} \tag{3.1}$$

Remark 3.1. (1) $X^1 = X - I(X) = X \cap d(X)$.

(2) $X^\alpha \supset X^\beta$ whenever $\alpha \leqslant \beta$.

(3) $X^\alpha = X^{\alpha-1} - I(X^{\alpha-1}) = X^{\alpha-1} \cap d(X^{\alpha-1})$ for any successor ordinal number α.

(4) If α is a successor ordinal number and $X^\alpha = \emptyset$, then $X = \bigcup\limits_{\beta \leqslant \alpha-1} I(X^\beta)$.

Lemma 3.2. *$X^\delta = X^{\delta+1}$ for some ordinal number δ.*

Proof. Put $|X| = k$. Then $X^{k+1} = X^{k+2}$. Pick $\delta = k + 1$. Then $X^\delta = X^{\delta+1}$. □

Proposition 3.1. *Let (X, g) be a GTS. The following properties hold.*

(1) $X^\alpha \in g'$ for any ordinal number α.

(2) If $Y \subset X$, then $Y^\alpha \subset X^\alpha$ for any ordinal number α.

Proof. (1) We use induction on α.

1) $\alpha = 1$. Let $x \in I(X)$. Then $U_x \cap X = \{x\}$ for some $U_x \in g$. This implies $\{x\} = U_x \in g$. Thus $I(X) = \bigcup\limits_{x \in I(X)} \{x\} \in g$. Thus $X^1 = X - I(X) \in g'$.

2) Suppose $X^\beta \in g'$ for any $\beta < \alpha$. We will prove $X^\alpha \in g'$ in the following two cases.

a) α is a successor ordinal number.

Let $x \in I(X^{\alpha-1})$. Then $U_x \cap X^{\alpha-1} = \{x\}$ for some $U_x \in \tau(x)$. By Remark 3.9, $X^\alpha = X^{\alpha-1} - I(X^{\alpha-1})$. So

$$X^\alpha = X^{\alpha-1} - \bigcup_{x \in I(X^{\alpha-1})} \{x\} = (X - \bigcup_{x \in I(X^{\alpha-1})} U_x) \cap X^{\alpha-1}.$$

By induction hypothesis, $X^{\alpha-1} \in g'$. Thus $X^\alpha \in g'$.

b) α is a limit ordinal number.

By induction hypothesis, $X^\beta \in g'$ for any $\beta < \alpha$. Thus $X^\alpha = \bigcap_{\beta<\alpha} X^\beta \in g'$

(2) Let $Y \subset X$. We will prove $Y^\alpha \subset X^\alpha$ for any ordinal number α.

1)

$$Y^1 = Y \cap d(Y) \subset X \cap d(X) = X^1.$$

This shows $Y^\alpha \subset X^\alpha$ when $\alpha = 1$.

2) Suppose $Y^\beta \subset X^\beta$ for any $\beta < \alpha$. We consider the following two cases.

a) α is a successor ordinal number.

By induction hypothesis, $Y^{\alpha-1} \subset X^{\alpha-1}$. By Remark 3.9,

$$Y^\alpha = Y^{\alpha-1} \cap d(Y^{\alpha-1}) \subset X^{\alpha-1} \cap d(X^{\alpha-1}) = X^\alpha$$

b) α is a limit ordinal number.

By induction hypothesis, $Y^\beta \subset X^\beta$ for any $\beta < \alpha$. Thus

$$Y^\alpha = \bigcap_{\beta<\alpha} Y^\beta \subset \bigcap_{\beta<\alpha} X^\beta = X^\alpha.$$

By 1) and 2), $Y^\alpha \subset X^\alpha$.

□

Definition 3.6. Let (X, g) be a GTS.

(1) An ordinal number γ is called the derived length of X, if $\gamma = min\{\alpha : X^\alpha = \emptyset\}$. γ is denoted by $\delta(X)$.

(2) X is called to have a derived length, if there is an ordinal number α such that $X^\alpha = \emptyset$.

Theorem 3.7. *Let (X, g) be a GTS. Then X is g-scattered if and only if X has a derived length.*

Proof. *Sufficiency*. Suppose that X is not g-scattered. Then $I(A) = \emptyset$ for some $A \in 2^X - \{\emptyset\}$.

Claim $A \subset X^\alpha$ for any ordinal number α.

(1) Let $x \in A$ and $U \in g(x)$. Since $I(A) = \emptyset$, $U \cap A \neq \{x\}$. Note that $x \in U \cap A$. Then $|U \cap A| \geqslant 2$ and so $U \cap (A - \{x\}) \neq \emptyset$. Now $U \cap (X - \{x\}) \supset U \cap (A - \{x\})$. Then $U \cap (X - \{x\}) \neq \emptyset$. This implies $x \in d(X) \cap X$. By Remark 3.9, $x \in X^1$

Thus $A \subset X - I(X) = X^1$, i.e., $A \subset X^\alpha$ when $\alpha = 1$.

(2) Suppose $A \subset X^\beta$ for any $\beta < \alpha$. We will prove $A \subset X^\alpha$ in the following cases.

a) α is a successor ordinal number.

Let $x \in A$ and $U \in g(x)$. By the proof above, $U \cap (A - \{x\}) \neq \emptyset$. By induction hypothesis, $A \subset X^{\alpha-1}$. Then $U \cap (X^{\alpha-1} - \{x\}) \neq \emptyset$. This implies $x \in d(X^{\alpha-1}) \cap X^{\alpha-1}$. By Remark 3.9, $x \in X^\alpha$.

Hence $A \subset X^\alpha$.

b) α is a limit ordinal number.

By induction hypothesis, $A \subset X^\beta$ for any $\beta < \alpha$. Then $A \subset \bigcap_{\beta<\alpha} X^\beta = X^\alpha$.

Since X has a derived length, $X^\delta = \emptyset$ for some δ. By **Claim**, $A \subset X^\alpha$, we have $A = \emptyset$. This is a contradiction.

Necessity. Suppose that X has no derived length. By Lemma 3.10, there exists an ordinal number δ such that $X^\delta = X^{\delta+1}$. By Remark 3.9, $X^{\delta+1} = X^\delta - I(X^\delta)$. Then $I(X^\delta) = \emptyset$. Note that X has no derived length. Then $X^\delta \neq \emptyset$. It follows that X is not g-scattered, a contradiction. □

4. Some properties of g-scattered spaces

In this section we give some properties of g-scattered spaces.

4.1. Simple properties of g-scattered spaces.

THEOREM 4.1. *Let (X, g_1) and (X, g_2) be two GTS's with $g_1 \subset g_2$. If (X, g_1) is g_1-scattered, then (X, g_2) is g_2-scattered.*

PROOF. This holds by Proposition 2.6. □

THEOREM 4.2. *Let (X, g) be a GTS and let $Y \in 2^X - \{\emptyset\}$. If (X, g) is g-scattered, then (Y, g_Y) is g_Y-scattered.*

PROOF. Let $A \in 2^Y - \{\emptyset\}$. Since (X, g) is g-scattered, $I(A) \neq \emptyset$. Pick $x \in I(A)$. Then $U \cap A = \{x\}$ for some $U \in g(x)$. Note that $U \cap Y \in g_Y(x)$. Now $(U \cap Y) \cap A = U \cap (A \cap Y) = U \cap A = \{x\}$. Then $x \in I_{g_Y}(A)$ and so $I_{g_Y}(A) \neq \emptyset$. Thus, (Y, g_Y) is g_Y-scattered. □

4.2. g-scatteredness and GT-sums. Let $\{(X_\alpha, g_\alpha) : \alpha \in \Gamma\}$ be a family of pairwise disjoint strong GTS's, i.e., $X_\alpha \cap X_\beta = \emptyset$ whenever $\alpha \neq \beta$. Put $X = \bigcup_{\alpha\in\Gamma} X_\alpha$ and

$$g = \{A \subset X : A \cap X_\alpha \in g_\alpha \text{ for each } \alpha \in \Gamma\}.$$

Then (X, g) is a GTS, which is denoted by $\bigoplus_{\alpha\in\Gamma} X_\alpha$, and called the generalized topological sum (briefly, GT-sum) of $\{(X_\alpha, g_\alpha) : \alpha \in \Gamma\}$.

THEOREM 4.3. *Let (X, g) be the sum of $\{(X_\alpha, g_\alpha) : \alpha \in \Gamma\}$. Then (X, g) is g-scattered if and only if (X_α, g_α) is g_α-scattered for each $\alpha \in \Gamma$.*

PROOF. *Sufficiency.* Let $A \in 2^X - \{\emptyset\}$. Since $A = \bigcup_{\alpha\in\Gamma} (A \cap X_\alpha)$, $A \cap X_\beta \neq \emptyset$ for some $\beta \in \Gamma$. By X_β is g_β-scattered, $I_{X_\beta}(A \cap X_\beta) \neq \emptyset$. Pick $x \in I_{X_\beta}(A \cap X_\beta)$. Then there exists $U \in g_\beta(x)$ such that $U \cap (X_\beta \cap A) = \{x\} = U \cap A$. Since

$$U \cap X_\alpha = \begin{cases} U \in g_\beta, & \alpha = \beta, \\ \emptyset \in g_\alpha, & \alpha \neq \beta, \end{cases}$$

we have $U \in g(x)$. This implies $x \in I(A)$ and then $I(A) \neq \emptyset$. Thus (X, g) is g-scattered.

Necessity. Obviously, $g_{X_\alpha} = g_\alpha$ for any $\alpha \in \Gamma$. By Theorem 3.4, every (X_α, g_α) is g_α-scattered.

□

4.3. g-scatteredness and g-irresolvableness.

DEFINITION 4.4. Let (X, g) be a GTS. X is called g-resolvable, if X has two disjoint g-dense subset. Otherwise, X is called g-irresolvable.

In the following we give an example on g-resolvable spaces.

EXAMPLE 4.5. Let $X = N$ and

$$g = \{\emptyset, \{1, 2, 3, \cdots, 100\}, \{1, 2, 3, \cdots, 1000\}\}.$$

Then (X, g) is a GTS.

Put $A = \{1, , 3, 5, \cdots\}$ and $B = \{2, 4, 6, \cdots\}$. Then $X = A \cup B$ and $A \cap B = \emptyset$.

Since $cA = \bigcap\{F : F \in g' \; and \; A \subset F\}$ and $\{F : F \in g' \; and \; A \subset F\} = \{X\}$, we have $cA = X$. Similarly, $cB = X$.

Hence (X, g) is g-resolvable.

LEMMA 4.1. *Let (X, g) be a GTS and let $A \subset X$. If $A \in D$, then $A \supset I(X)$.*

PROOF. If $A \not\supset I(X)$, then $I(X) - A \neq \emptyset$. Pick $x \in I(X) - A$. Then $U \cap X = U = \{x\}$ for some $U \in g(x)$. By $x \notin A$, $X - U = X - \{x\} \supset A$. Since $X - U \in g'$ and $cA = \bigcap\{F : F \in g' \; and \; A \subset F\}$, we have $cA \subset X - U$. Then $A \notin D$, a contradiction. □

THEOREM 4.6. *If (X, g) is g-scattered, then (X, g) is g-irresolvable.*

PROOF. For any $A, B \in 2^X - \{\emptyset\}$ with $cA = cB = X$ and $X = A \cup B$, by Lemma 4.6, $A, B \supset I(X)$. Then $A \cap B \supset I(X)$. Since X is g-scattered, $I(X) \neq \emptyset$. So $A \cap B \neq \emptyset$. Thus, X is g-irresolvable. □

The following example illustrates that the converse in Theorem 4.7 is not true.

EXAMPLE 4.7. Let $X = N$, $\mathcal{B} = \{\emptyset, \{1\}, \{2, 3\}\} \cup \{\{i, i+1, i+2\} : i \in X \; and \; i > 3\}$ and $g = \{G : G = \bigcup \mathcal{B}' \; for \; some \; \mathcal{B}' \subset \mathcal{B}\}$.

PROOF. Obviously, (X, g) is a GTS.

Since $\{1\} \in g(1)$ and $\{1\} \cap X = \{1\}$, then $I(X) \supset \{1\} \neq \emptyset$.

For any $A, B \in 2^X - \{\emptyset\}$ with $cA = cB = X$ and $X = A \cup B$, by Lemma 4.6, $A, B \supset I(X) \neq \emptyset$. Then (X, g) is g-irresolvable.

Put $S = \{2, 3\}$. By Proposition 2.5, $I(S) \subset S$. Then for any $x \in X - S$, $x \notin I(S)$.

For any $U \in g(2)$, $U \cap S = S \neq \{2\}$. Then $2 \notin I(S)$. Similarly, $3 \notin I(S)$. Then $I(S) = \emptyset$.

Hence (X, g) is not g-scattered. □

4.4. Mapping properties on g-scattered spaces.

THEOREM 4.8. *Let (X, g_X) be g_X-scattered, let (Y, g_Y) be a GTS, let $f : (X, g_X) \to (Y, g_Y)$ be closed bijection. Then the following properties hold:*

(i) $Y^\alpha \subset f(X^\alpha)$ for every ordinal number α.

(ii) $\delta(Y) \leqslant \delta(X)$.

(iii) Y is g_Y-scattered.

PROOF. Since (ii) and (iii) hold by (i) and Theorem 4.5, we only need to prove (i), i.e., $Y^\alpha \subset f(X^\alpha)$ for every ordinal number α.

We use induction on α.

(1) Since $Y^0 = Y = f(X) = f(X^0)$, then $Y^\alpha \subset f(X^\alpha)$ is true when $\alpha = 0$.

(2) Suppose that $Y^\beta \subset f(X^\beta)$ is true when $\beta < \alpha$. It suffices to show that $Y^\alpha \subset f(X^\alpha)$ in the following two cases.

1) $\alpha = \beta + 1$ for some ordinal number β.

Suppose $Y^\alpha \not\subset f(X^\alpha)$. Then $Y^\alpha - f(X^\alpha) \neq \emptyset$. Pick

$$y \in Y^\alpha - f(X^\alpha).$$

Since f is bijection, there is unique $x \in X$ such that $f(x) = y$. $y \notin f(X^\alpha)$, then $x \notin X^\alpha$.

X is g_X-scattered, then there is δ such that $X^\delta = \emptyset$. By Remark 4.2. $X = \bigcup_{\beta<\delta} I(X^\beta)$. Since $X^\alpha \supset I(X^\alpha)$ and $X^\alpha \supset X^\upsilon$ for every $\upsilon \geqslant \alpha$, then $X = (\bigcup_{\beta<\alpha} I(X^\beta)) \cup X^\alpha$. By Definition 4.1. $(\bigcup_{\beta<\alpha} I(X^\beta)) \cap X^\alpha = \emptyset$. There is $\gamma < \alpha$ such that $x \in I(X^\gamma)$, since $x \notin X^\alpha$.

There is $U \in g_X$ such that $U \cap X^\gamma = \{x\}$. Then $\{x\}$ is open in (X^γ, g_{X^γ}), and $X^\gamma - \{x\}$ is closed in (X^γ, g_{X^γ}).

f is closed in (X, g_X), then $f|_{X^\gamma}$ is closed in (X^γ, g_{X^γ}).

By induction hypothesis, $f(X^\gamma) = Y^\gamma$. $f|_{X^\gamma}$ is closed in (X^γ, g_{X^γ}), then $f(X^\gamma - \{x\}) = Y^\gamma - \{y\}$ is closed in (Y^γ, g_{Y^γ}), $\{y\}$ is open in (Y^γ, g_{Y^γ}).

There is $V \in g_Y$ such that $V \cap Y^\gamma = \{y\}$, then $y \in I(Y^\gamma)$. Since $\gamma < \alpha$ and $(\bigcup_{\beta<\alpha} I(Y^\beta)) \cap Y^\alpha = \emptyset$, then $y \notin Y^\alpha$, a contradiction.

2) α is a limit ordinal number.

$f(X^\alpha) = f(\bigcap_{\beta<\alpha} X^\beta) = \bigcap_{\beta<\alpha} f(X^\beta) \supset \bigcap_{\beta<\alpha} Y^\beta = Y^\alpha$. □

COROLLARY 4.1. *Let (X, g_X) be g_X-scattered, let (Y, g_Y) be a GTS, let $f : (X, g_X) \to (Y, g_Y)$ be open bijection. Then the following properties hold:*

(i) $Y^\alpha \subset f(X^\alpha)$ for every ordinal number α.

(ii) $\delta(Y) \leqslant \delta(X)$.

(iii) Y is g_Y-scattered.

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