SCIENTIA
Series A: *Mathematical Sciences*, Vol. 25 (2014), 47–53
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

On Strongly Socle-Regular $QTAG$-Modules

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Abstract. A module M over an associative ring with unity is a $QTAG$-module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules. Recently the socles of fully invariant submodules have been studied and this led to the notion of socle-regular $QTAG$-modules. In this paper, we study the socles of characteristic submodules of $QTAG$-modules and define strongly socle-regular $QTAG$-modules. We also discuss some interesting properties of these modules.

1. Introduction and Preliminaries

Recently [**5**], the authors study the socles of fully invariant submodules and some of their properties. This paper builds on that approach for characteristic submodules and it leads to the concept of strong socle-regular $QTAG$-modules.

Throughout this paper, all rings will be associative with unity and modules M are unital $QTAG$-modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique composition series, $d(M)$ denotes its composition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup\left\{d\left(\frac{yR}{xR}\right) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ and it is h-reduced if it does not contain any h-divisible submodule. In other words it is free from the elements of infinite height. A characteristic submodule N of a $QTAG$-module M is a submodule that is invariant under each automorphism of M.

A $QTAG$-module M is called separable, if every finitely generated submodule of M can be embedded in a summand of M. A submodule $B \subseteq M$ is a basic submodule

2000 *Mathematics Subject Classification.* Primary 16 K 20 .
Key words and phrases. QTAG-module, transitive modules, characteristic submodules, socles, socle-regular $QTAG$-modules.

of M, if B is h-pure in M, $B = \oplus B_i$, where each B_i is the direct sum of uniserial modules of length i and M/B is h-divisible. A fully invariant submodule $L \subset M$ is a large submodule of M, if $L + B = M$ for every basic submodule B of M. A $QTAG$-module M is fully transitive if for $x, y \in M$, $U(x) \leqslant U(y)$, there is an endomorphism f of M such that $f(x) = f(y)$ and it is transitive if for any two elements $x, y \in M$, with $U(x) \leqslant U(y)$, there is an automorphism f of M such that $f(x) = f(y)$. Singh [**6**] proved that the results which hold for TAG-modules also hold good for $QTAG$-modules.

2. Properties of Strongly Socle-Regular $QTAG$-Modules

First we recall the following:

DEFINITION 2.1. *A h-reduced QTAG-module M is said to be* socle-regular *if for all fully invariant submodules N of M, there exists an ordinal σ such that Soc (N)=Soc $(H_\sigma(M))$. Hence σ depends on N.*

REMARK 2.1. *Clearly, the class of socle-regular QTAG-modules strictly contains the class of fully transitive QTAG-modules.*

Now we define the strongly socle-regular $QTAG$-modules as follows:

DEFINITION 2.2. *A QTAG-module M is* strongly socle-regular *if for all the characteristic submodules N of M, there exists an ordinal α such that $Soc(N) = Soc(H_\alpha(M))$.*

A strongly socle-regular module is socle-regular but the converse is not true in general. This motivates us to investigate strongly socle-regular modules and characterize them.
We start with a simple lemma.

LEMMA 2.1. *If N is a h-divisible QTAG-module then its characteristic submodules are of the form $H^k(N)$, $k < \omega$, and all of them are fully invariant. Moreover, if K is a characteristic submodule of the h-reduced QTAG-module M, then $N \oplus K$ is a characteristic submodule of $N \oplus M$. If K is not fully invariant in M, then $N \oplus K$ is not fully invariant in $N \oplus M$.*

PROOF. For a characteristic submodule L of N, $Soc(L) = 0$ or $Soc(N)$, therefore L is of the form $H^k(N)$ only. Every endomorphism of $N \oplus M$ may be represented by a matrix $\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}$ where f_i's are also endomorphisms. Since a h-divisible module can not be mapped onto h-reduced module, this matrix becomes a lower triangular matrix, therefore $N \oplus K$ is characteristic. Now if K is not fully invariant, then there exists an endomorphism f of M such that $f(K) \nsubseteq K$. This f can be extended to an endomorphism $\bar{f}$ of $N \oplus M$ by defining $\bar{f}(N) = 0$, thus $\bar{f}(N \oplus K) \nsubseteq N \oplus K$ implying that $N \oplus K$ is not fully invariant. Hence the result follows. □

Theorem 2.1. *Let N be a h-divisible QTAG-module and K a reduced QTAG-module. If $N \oplus K$ is strongly socle-regular, then both N and K are strongly socle-regular. Moreover, if K is strongly socle-regular, then $N \oplus K$ is also strongly socle-regular.*

Proof. By Lemma 2.1, N is strongly socle-regular. Again if Q is a characteristic submodule in K, then $N \oplus Q$ is characteristic in $N \oplus K$. Now

$$Soc(N \oplus Q) = Soc(H_\alpha(N \oplus K))$$

for some ordinal α. Thus $Soc(Q) = Soc(H_\alpha(K))$.

Conversely, if Q is a characteristic submodule of $N \oplus K$, $Q = (N \cap Q) \oplus (Q \cap K)$ and $N \cap Q$ and $K \cap Q$ are characteristic in N and K respectively. Since K is strongly socle-regular $Soc(Q \cap N) = Soc(N)$ by Lemma 2.1, and $Soc(Q \cap K) = Soc(H_\alpha(K))$. Therefore,

$$Soc(Q) = Soc(N) \oplus Soc(H_\alpha(K)) = Soc(H_\alpha(N \oplus K))$$

implying that $N \oplus K$ is also strongly socle-regular. □

To study the socles of characteristic submodules, we define the following:

Definition 2.3. *For a submodule N of M, put $\sigma = min\{H(x) \mid x \in Soc(N)\}$ and denote $\sigma = inf(Soc(N))$. Here $Soc(N) \subseteq Soc(H_\sigma(M))$.*

Remark 2.2. *If K is a submodule of M containing N, $inf(Soc(N))$ may be calculated with respect to N and M respectively. To differentiate we write $inf(Soc(N))_K$ and $inf(Soc(N))_M$ respectively, but if K is an isotype submodule of M, then $inf(Soc(N))_K = inf(Soc(N))_M$. However if K is not an isotype submodule of M, then $inf(Soc(N))_K \leqslant inf(Soc(N))_M$.*

Now we prove the following:

Proposition 2.1.

- (i) *If N is a submodule of the h-reduced QTAG-module M such that $Soc(H_k(M)) \subseteq Soc(N)$ for some integer k, then $\inf(Soc(N))$ is finite.*
- (ii) *If N is a characteristic submodule of M and $\inf(Soc(N)) = k$, $k < \omega$, then $Soc(N) = Soc(H_k(M))$.*

Proof. (i) Let $\sigma = \inf(Soc(N))$. Now $\sigma \leqslant \min\{H_M(x) \mid x \in Soc(H_k(M))\}$. If $\sigma \geqslant \omega$, then $Soc(H_k(M)) \subseteq H_\omega(M) = H_\omega(H_k(M))$. Thus $Soc(H_k(M)) \subseteq H_\omega(H_k(M))$. This means $H_k(M)$ is h-divisible (if not zero) which is not possible because M is h-reduced. Thus $\inf(Soc(N)) < \omega$.

(ii) Since $\inf(Soc(N)) = k$, $Soc(N) \subseteq Soc(H_k(M))$. Let x be a uniform element of $Soc(N)$ such that $H_M(x) = k$, then there exists $y \in M$ such that $d\left(\dfrac{yR}{xR}\right) = k$. Since every element of exponent one and finite height can be embedded in a direct summand, by [**3**], yR is a summand of M. Therefore $M = yR \oplus M'$, for some submodule M' of M. If z is an arbitrary uniform element of $Soc(H_k(M) \setminus Soc(H_{k+1}(M))$, then there exists $u \in M$ such that $d\left(\frac{uR}{zR}\right) = k$ and hence $M = uR \oplus M''$. Clearly M' and M'' are isomorphic since $yR \cong uR$, both being of uniserial modules of exponent $k+1$ and uniserial modules of finite exponent have cancellation property. We may define a homomorphism $f : M \to M$ such that $f : y \to u$, mapping M' to M'', $f(x) = z$ and f is an automorphism. Since $Soc(N)$ is characteristic in M, $z \in Soc(N)$ and so $Soc(H_k(M)) \setminus Soc(H_{k+1}(M)) \subseteq Soc(N)$. However if $v \in Soc(H_{k+1}(M))$, then $z + v$ has height exactly n and exponent 1, so that $z + v \in Soc(N)$. This implies that $v \in Soc(N)$ and hence $Soc(H_k(M)) \subseteq Soc(N)$ and the result follows. □

Corollary 2.1. *If M is a separable QTAG-module, then M is strongly socle-regular.*

Proof. This is immediate since the hypothesis of separability implies that for any characteristic submodule N of M, $\inf(Soc(N))$ is finite. □

Remark 2.3. *The fully transitive modules are socle-regular but they need not be strongly socle-regular.*

The following result establishes the relation between transitive modules and strongly socle-regular modules.

Theorem 2.2. *If M is a transitive QTAG-module, then M is strongly socle-regular. In particular, totally projective QTAG-modules are strongly socle-regular.*

Proof. Suppose M is a transitive $QTAG$-module and let N be any characteristic submodule of M. If $\inf(Soc(N)) = \alpha$, then $Soc(N) \subseteq Soc(H_\alpha(M))$.For the reverse inequality, let x be an element of $Soc(N)$ of height α and let y be an arbitrary uniform element of $Soc(H_\alpha(M))$. The Ulm sequence of y is $U_M(y) = (\beta, \infty, \ldots)$ for some $\beta \geqslant \alpha$. If $\beta = \alpha$, then there is an automorphism θ of M such that $\theta(x) = y$, as M is transitive. Since $Soc(N)$ is characteristic in M, $y \in Soc(N)$. If however, $\beta > \alpha$, then $H_M(x+y) = H_M(x)$, hence $U_M(x+y) = U_M(x)$. Again by transitivity of M there is an automorphism ϕ of M with $\phi(x) = x+y$ and so $y = \phi(x) - x \in Soc(N)$, as $Soc(N)$ is characteristic in M. Therefore $Soc(H_\alpha(M)) \subseteq Soc(N)$ and the result follows. □

Remark 2.4. *Since totally projective QTAG-modules are transitive, they are strongly socle-regular.*

Now we investigate the conditions under which the property of being strongly socle-regular is shared by the submodules and quotient modules.

Proposition 2.2.

(*i*) *If M is strongly socle-regular, then so also is $H_\alpha(M)$ for all ordinals α.*
(*ii*) *If M is strongly socle-regular and L is a characteristic submodule of M such that $H_\omega(L) = H_\omega(M)$, then L is strongly socle-regular.*
(*iii*) *M is strongly socle-regular if and only if $H_n(M)$ is strongly socle-regular for a positive integer n. In particular, if N is a submodule of M and either M/N is finitely generated or $M = N \oplus B$, where B is bounded, then M is strongly-socle-regular if and only if N is strongly socle-regular.*

(*iv*) *If $H_\omega(M)$ is strongly socle-regular and $M/H_\omega(M)$ is a direct sum of uniserial modules, then M is strongly socle-regular.*

(*v*) *Suppose that α is an ordinal less than ω^2. If $H_\alpha(M)$ is strongly-socle-regular and $M/H_\alpha(M)$ is totally projective then M is strongly socle-regular.*

Proof. Part (*i*) follows immediately from the fact that a characteristic submodule of a characteristic module is again a characteristic submodule.

To establish part (*ii*), let N be a characteristic submodule of L. Clearly N is characteristic in M and hence as M is strongly socle-regular, $Soc(N) = Soc(H_\alpha(M))$ for some ordinal α. If $\alpha \geqslant \omega$, then inductively we get $H_\alpha(L) = H_\alpha(M)$ and so $Soc(N) = Soc(H_\alpha(L))$ as required. However, if $Soc(N) = Soc(H_n(M))$ for some integer n then $Soc(N) \supseteq Soc(H_n(L))$ and then it follows from Proposition 2.1 (*i*), that $\inf(Soc(N))$ is finite. Again by Proposition 2.1 (*ii*), $Soc(N) = Soc(H_m(L))$ for some integer m and L is strongly socle-regular.

To establish part (*iii*), if K is a characteristic submodule of M and $Soc(N) \nsubseteq H_n(M)$, then $\inf(Soc(K))$ is finite, k, say and then from Proposition 2.1 (*ii*), it follows that $Soc(N) = Soc(H_k(M))$. If $Soc(K) \subseteq H_n(M)$, then $Soc(K)$ is actually characteristic in $H_n(M)$ and so

$$Soc(K) = Soc(H_\alpha(H_n(M))) = Soc(H_{n+\alpha}(M))$$

for some α. To deduce particular cases, note that in either situation there exists an integer n such that $H_n(M) = H_n(N)$.

For the proof of (*iv*), Let N be a characteristic submodule of M. If $Soc(N) \nsubseteq Soc(H_\omega(M))$, then $\inf(Soc(N))$ is finite and by Proposition 2.1 (*ii*), $Soc(N) = Soc(H_k(M))$, for some $k < \omega$ and if $Soc(N) \subseteq Soc(H_\omega(M))$, $Soc(N)$ is characteristic in $H_\omega(M)$. Since $H_\omega(M)$ is strongly socle-regular, $Soc(N) = Soc(H_\alpha(H_\omega(M)))$ for some ordinal α and $Soc(N) = Soc(H_{\omega+\alpha}(M))$ and M is strongly socle-regular. Since an arbitrary automorphism ϕ of $H_\omega(M)$ is induced by an automorphism of M, $Soc(N)$ is characteristic in $H_\omega(M)$ and the result follows.

For the proof of (*v*), we use the transfinite induction. The initial cases follow from (*iii*) and (*iv*). So suppose that we have established the result for all ordinals less than α. There are two possibilities, either α is a successor or α is a limit ordinal of the form $\omega.n$. In the first case $\alpha = \beta + 1$, for some β. Let $X = H_\beta(M)$. Now $H_1(X) = H_\alpha(M)$ is strongly socle-regular. Hence by (*ii*) $H_\beta(M)$ is strongly socle-regular. Moreover, since $\beta < \alpha$, $M/H_\beta(M)$ is totally projective. Hence it follows from our inductive hypothesis that M is strongly socle-regular. In the second case, $\alpha = \beta + \omega$, for some β. Set $X = H_\beta(M)$ so that $H_\omega(X) = H_\alpha(M)$ is strongly socle-regular. Now $X/H_\omega(X) \cong H_\beta(M)/H_\alpha(M)$ and this is again totally projective hence it is a direct sum of uniserial modules. It now follows from (*iii*) that $X = H_\beta(M)$ is strongly socle-regular. However as $M/H_\beta(M)$ is totally projective and so it follows from the inductive hypothesis that M is strongly socle-regular. □

Remark 2.5. *For any large submodule L of M, $H_\omega(L) = H_\omega(M)$, therefore large submodules are strongly socle-regular.*

Proposition 2.3. *Let M be a QTAG-module where $H_\omega(M)$ is uniserial. Then M is strongly socle-regular. If there exists a QTAG-module M such that $H_\omega(M)$ is strongly socle-regular but M is not strongly socle-regular, then $H_\omega(M) = N \oplus K$, where $H_1(N) = H_1(K) = 0$. Moreover, the direct sum of two strongly socle-regular modules need not be strongly socle-regular.*

Proof. Let M be a $QTAG$-module with $H_\omega(M)$, uniserial and N a characteristic submodule of M. Then either $\inf(Soc(N))$ is finite or $Soc(N) \subseteq H_\omega(M)$. If $\inf(Soc(N))$ is finite, then $Soc(N) = Soc(H_k(M))$ for some finite $k < \omega$, by Proposition 2.1, otherwise $Soc(N) = Soc(H_\omega(M))$.

Let $M = N \oplus K$ be a $QTAG$-module, where $H_\omega(N) \simeq H_\omega(K)$ such that $H_{\omega+1}(N) = H_{\omega+1}(K) = 0$, $N/H_\omega(N)$ is a direct sum of uniserial modules and $K/H_\omega(K)$ is closed [**4**]. Here $M/H_\omega(M)$ is not a direct sum of uniserial modules but $H_\omega(M)$ is finitely generated. Thus $H_\omega(M)$ is strongly socle-regular. As proved in the last section, M is not even socle-regular. Therefore, M is not strongly socle-regular. The last statement is the immediate consequence of the above discussion. □

3. The class of Strongly Socle-Regular $QTAG$-Modules

This section deals with the properties of the strongly socle-regular $QTAG$-modules and we obtain a characterization of strongly socle-regular modules in terms of socle-regular modules.

Proposition 3.1. *If M is strongly socle-regular QTAG-module, then so also is the direct sum of β copies of M, $A = \bigoplus_{\gamma<\beta} M_\gamma$ for any ordinal β.*

Proof. If N is an arbitrary characteristic submodule of A, then N is fully invariant in A. Since a $QTAG$-module M is socle-regular if and only if the direct sum of β copies of M, $\bigoplus_{\gamma<\beta} M_\gamma$ is socle-regular for any ordinal β, it follows that A is socle regular and so $Soc(N) = Soc(H_\alpha(A))$, for some ordinal α. Thus A is strongly socle-regular. □

Since separable modules are always socle-regular, addition of a separable summand does not affect the strong socle-regularity.

Proposition 3.2. *If M is the direct sum of two submodules N and K i.e $M = N \oplus K$ such that N is strongly socle-regular and K is separable then M is strongly socle-regular.*

Proof. Let L be a characteristic submodule of M. If $\inf(Soc(L))$ is finite then by Proposition 2.1 (ii), $Soc(L) = Soc(H_n(M))$, for some finite integer n. Otherwise $Soc(L) \subseteq H_\omega(M) = H_\omega(N)$, so that $Soc(L) \subseteq N$. Now let f be an arbitrary automorphism of N. Then f extends to an automorphism $\bar{f}$ of M by setting $\bar{f} = \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}$. Since $Soc(L)$ is characteristic in M, $\bar{f}(Soc(L)) \subseteq Soc(L)$. Hence $f(Soc(L)) \subseteq Soc(L)$ and $Soc(L)$ is characteristic in N also. Now the latter is strongly socle-regular, so

$Soc(L) = Soc(H_\beta(N))$ for some ordinal β; note that $\beta \geqslant \omega$ since $\inf(Soc(L))$ is infinite. However if β is infinite then $H_\beta(M) = H_\beta(N)$ and hence $Soc(L) = Soc(H_\beta(M))$. Thus M is strongly socle-regular. □

THEOREM 3.1. *Let $M = N + K$, where K is separable. If there exists an integer n such that $H_n(N) \cap H_n(K) = 0$ (in particular, if $N \cap K$ is finite), then N is strongly socle-regular implies that M is also strongly socle-regular.*

PROOF. Clearly $H_n(M) = H_n(N) + H_n(K)$. However, the hypothesis $H_n(N) \cap H_n(K) = 0$ makes the previous sum as direct *i.e.* $H_n(M) = H_n(N) \oplus H_n(K)$. Now in view of Proposition 2.2 (*i*), $H_n(N)$ is strongly socle-regular since N is, and one also has that $H_n(K)$ is separable. By Proposition 3.2, $H_n(M)$ is strongly socle-regular and hence it follows from Proposition 2.2 (*iii*) that M is strongly socle-regular. □

Now we are able to characterize the strongly socle-regular modules.

THEOREM 3.2. *A h-reduced QTAG-module M is socle-regular if the direct sum $M \oplus M$ is strongly socle-regular.*

PROOF. If $M \oplus M$ is strongly socle-regular, it immediately follows from the fact that a $QTAG$-module M is socle-regular if and only if the direct sum of β copies of M, $\bigoplus_{\gamma<\beta} M_\gamma$ is socle-regular for any ordinal β, it follows that M is socle-regular. □

References

[1] Fuchs L., Infinite Abelian Groups, Vol. I, Academic Press, New York, (1970).
[2] Fuchs L., Infinite Abelian Groups, Vol. II, Academic Press, New York, (1973).
[3] Khan M. Z., *Torsion modules behaving like torsion Abelian groups*, Tamkang J. Math., **9**(1), 15–20 (1978).
[4] Mehdi A. and Khan M. Z., *On closed modules*, Kyungpook Math. J., **24**(1)(1984), 45–50.
[5] Sikander F., Hasan A. and Mehdi A., *Socle-Regular QTAG-modules* (Communicated).
[6] Singh S., Some decomposition theorems in abelian groups and their generalizations , Ring Theory, Proc. of Ohio Univ. Conf. Marcel Dekker N.Y. 25, 183-189, (1976).

Received 03 12 2013, revised 18 03 2014

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