SCIENTIA
Series A: *Mathematical Sciences*, Vol. 25 (2014), 17–25
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

Companions of Hermite-Hadamard Inequality for Convex Functions (I)

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ABSTRACT. Companions of Hermite-Hadamard inequalities for convex functions defined on the positive axis in the case when the integral has the weight $\frac{1}{t^3}$, $t > 0$ are given. Applications for special means are provided as well.

1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(t)\,dt \leqslant \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f:[a,b]\to\mathbb{R}$, is well known in the literature as the Hermite-Hadamard inequality.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, for which we would like to refer the reader to the papers [**1**] – [**60**] and the references therein.

In this paper we establish some companions of Hermite-Hadamard inequalities for convex functions defined on the positive axis in the case when the integral has the weight $\frac{1}{t^3}$, $t > 0$. Applications for special means are provided as well.

2. The Results

The following result holds:

THEOREM 1. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be a convex function on $[a,b]$, then we have the inequalities*

$$\frac{A\left(\frac{f(a)}{a},\frac{f(b)}{b}\right)}{G^2(a,b)} \geqslant \frac{1}{b-a}\int_a^b \frac{1}{t^3} f(t)\,dt \geqslant \frac{f(H(a,b))}{H(a,b)\,G^2(a,b)}, \tag{2.1}$$

2000 *Mathematics Subject Classification.* Primary 26D15, 25D10.
Key words and phrases. Convex functions, Hermite-Hadamard inequality, Special means.
The authors would like to thank the anonymous referee for valuable comments that have been implemented in the final version of the paper.

where

$$H(p,q) := \frac{2}{\frac{1}{p}+\frac{1}{q}},\ G(p,q) := \sqrt{pq} \text{ and } A(p,q) := \frac{p+q}{2}$$

are the Harmonic, Geometric and Arithmetic means, respectively.

If the function f is concave, then the inequalities (2.1) reverse.

Proof. Let $x := \frac{1}{b} < \frac{1}{a} := y$ and consider the function $\varphi : [x,y] \to \mathbb{R}$ defined by

$$\varphi(t) := tf\left(\frac{1}{t}\right).$$

If $t_1, t_2 \in [x,y]$ and $\alpha, \beta \geqslant 0$ with $\alpha+\beta = 1$ then by the convexity of f we have

$$\begin{aligned}
\varphi(\alpha t_1 + \beta t_2) &:= (\alpha t_1 + \beta t_2) f\left(\frac{1}{\alpha t_1 + \beta t_2}\right) \\
&= (\alpha t_1 + \beta t_2) f\left(\frac{\alpha+\beta}{\alpha t_1 + \beta t_2}\right) \\
&= (\alpha t_1 + \beta t_2) f\left(\frac{\alpha t_1 \cdot \frac{1}{t_1} + \beta t_2 \cdot \frac{1}{t_2}}{\alpha t_1 + \beta t_2}\right) \\
&\leqslant (\alpha t_1 + \beta t_2) \frac{\alpha t_1 f\left(\frac{1}{t_1}\right) + \beta t_2 f\left(\frac{1}{t_2}\right)}{\alpha t_1 + \beta t_2} \\
&= \alpha t_1 f\left(\frac{1}{t_1}\right) + \beta t_2 f\left(\frac{1}{t_2}\right) = \alpha\varphi(t_1) + \beta\varphi(t_2),
\end{aligned}$$

which shows that the function φ is convex on $[x,y]$.

Now, if we write the Hermite-Hadamard inequality for the function φ on the interval $[x,y]$, namely

$$\frac{\varphi(x)+\varphi(y)}{2} \geqslant \frac{1}{y-x}\int_x^y \varphi(t)\,dt \geqslant \varphi\left(\frac{x+y}{2}\right),$$

then we have

$$\frac{1}{2}\left[xf\left(\frac{1}{x}\right) + yf\left(\frac{1}{y}\right)\right] \geqslant \frac{1}{y-x}\int_x^y tf\left(\frac{1}{t}\right)dt \geqslant \frac{x+y}{2} f\left(\frac{2}{x+y}\right)$$

that is equivalent with

$$\frac{1}{2}\left[\frac{f(b)}{b} + \frac{f(a)}{a}\right] \geqslant \frac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}} tf\left(\frac{1}{t}\right)dt \geqslant \frac{a+b}{2ab} f\left(\frac{2ab}{a+b}\right), \tag{2.2}$$

since $x = \frac{1}{b} < \frac{1}{a} = y$, which is an inequality of interest in itself.

However, if we make the change of variable $s = \frac{1}{t}$ in the integral from (2.2), then we get

$$\int_{\frac{1}{b}}^{\frac{1}{a}} tf\left(\frac{1}{t}\right)dt = \int_a^b \frac{1}{s^3} f(s)\,ds$$

and from (2.2) we deduce the desired result (2.1). □

The following reverses of (2.1) also hold:

THEOREM 2. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be a convex function on $[a,b]$, then we have the inequalities*

$$0 \leqslant \frac{A\left(\frac{f(a)}{a}, \frac{f(b)}{b}\right)}{G^2(a,b)} - \frac{1}{b-a}\int_a^b \frac{1}{t^3} f(t)\,dt \leqslant \frac{1}{8}\frac{b-a}{G^4(a,b)}\left[bf'_+(b) - af'_-(a) - f(b) + f(a)\right], \tag{2.3}$$

and

$$0 \leqslant \frac{1}{b-a}\int_a^b \frac{1}{t^3} f(t)\,dt - \frac{f(H(a,b))}{H(a,b)\,G^2(a,b)} \leqslant \frac{1}{8}\frac{b-a}{G^4(a,b)}\left[bf'_+(b) - af'_-(a) - f(b) + f(a)\right]. \tag{2.4}$$

PROOF. We use the following reverses of the Hermite-Hadamard inequalities

$$0 \leqslant \frac{\varphi(x) + \varphi(y)}{2} - \frac{1}{y-x}\int_x^y \varphi(t)\,dt \leqslant \frac{1}{8}(y-x)\left[\varphi'_-(y) - \varphi'_+(x)\right], \ [\mathbf{16}] \tag{2.5}$$

and

$$0 \leqslant \frac{1}{y-x}\int_x^y \varphi(t)\,dt - \varphi\left(\frac{x+y}{2}\right) \leqslant \frac{1}{8}(y-x)\left[\varphi'_-(y) - \varphi'_+(x)\right], \ [\mathbf{15}] \tag{2.6}$$

where φ is convex on $[x,y]$, $\varphi'_-(y)$, $\varphi'_+(x)$ are the lateral derivatives assumed to be finite and the constant $\frac{1}{8}$ is sharp in both inequalities.

Observe that if $\varphi(t) = tf\left(\frac{1}{t}\right)$, then

$$\varphi'(t) = f\left(\frac{1}{t}\right) - \frac{1}{t}f'\left(\frac{1}{t}\right).$$

By the inequality (2.5) we have

$$0 \leqslant \frac{1}{2}\left[xf\left(\frac{1}{x}\right) + yf\left(\frac{1}{y}\right)\right] - \frac{1}{y-x}\int_x^y tf\left(\frac{1}{t}\right)dt \leqslant \frac{1}{8}(y-x)\left[f\left(\frac{1}{y}\right) - \frac{1}{y}f'_-\left(\frac{1}{y}\right) - f\left(\frac{1}{x}\right) + \frac{1}{x}f'_+\left(\frac{1}{x}\right)\right]. \tag{2.7}$$

If we take $x = \frac{1}{b} < \frac{1}{a} = y$ in (2.7), then we get

$$0 \leqslant \frac{1}{2}\left[\frac{f(b)}{b} + \frac{f(a)}{a}\right] - \frac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}} tf\left(\frac{1}{t}\right)dt \leqslant \frac{1}{8}\frac{b-a}{ab}\left[f(a) - af'_-(a) - f(b) + bf'_+(b)\right], \tag{2.8}$$

which is equivalent to (2.3).

The inequality (2.4) follows in a similar way from (2.6) and the details are omitted. □

REMARK 1. We observe that the second inequality in (2.8) is equivalent to

$$\frac{1}{2}\left[\frac{f(b)\left(\frac{3a+b}{4}\right)+f(a)\left(\frac{a+3b}{4}\right)}{a^2b^2}\right]-\frac{1}{b-a}\int_a^b\frac{1}{t^3}f(t)\,dt \leqslant \frac{1}{8}\frac{b-a}{a^2b^2}\left[bf'_+(b)-af'_-(a)\right], \tag{2.9}$$

while the second inequality in (2.4) can be written as

$$\frac{1}{b-a}\int_a^b\frac{1}{t^3}f(t)\,dt+\frac{1}{8}\frac{b-a}{G^4(a,b)}\left[f(b)-f(a)\right]-\frac{f(H(a,b))}{H(a,b)\,G^2(a,b)} \leqslant \frac{1}{8}\frac{b-a}{G^4(a,b)}\left[bf'_+(b)-af'_-(a)\right]. \tag{2.10}$$

3. Applications for Special Means

Let us recall the following means :

(1) The arithmetic mean

$$A=A(a,b):=\frac{a+b}{2},\qquad a,b\geqslant 0;$$

(2) The geometric mean:

$$G=G(a,b):=\sqrt{ab},\qquad a,b\geqslant 0;$$

(3) The harmonic mean:

$$H=H(a,b):=\frac{2}{\frac{1}{a}+\frac{1}{b}},\qquad a,b\geqslant 0;$$

(4) The logarithmic mean:

$$L=L(a,b):=\begin{cases} a & \text{if } a=b \\ \dfrac{b-a}{\ln b-\ln a} & \text{if } a\neq b \end{cases}\qquad a,b>0;$$

(5) The identric mean:

$$I:=I(a,b)=\begin{cases} a & \text{if } a=b \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a\neq b \end{cases}\qquad a,b>0;$$

(6) The p-logarithmic mean:

$$L_p=L_p(a,b):=\begin{cases} \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a\neq b; \\ a & \text{if } a=b \end{cases}$$

where $p\in\mathbf{R}\backslash\{-1,0\}$ and $a,b>0$.

It is well known that L_p is monotonic nondecreasing over $p \in \mathbf{R}$ with $L_{-1} := L$ and $L_0 := I$.

In particular, we have the inequalities

$$H \leqslant G \leqslant L \leqslant I \leqslant A. \tag{3.1}$$

We can state the following proposition:

Proposition 1. *For any $0 < a < b$ we have*

$$G^2 \geqslant LH, \tag{3.2}$$

$$0 \leqslant AL - G^2 \leqslant \frac{1}{4}(b-a)^2 \frac{AL}{G^2} \tag{3.3}$$

and

$$0 \leqslant G^2 - HL \leqslant \frac{1}{4}(b-a)^2 \frac{AL}{G^2}. \tag{3.4}$$

Proof. If we write the inequality (2.1) for the convex function $f : [a,b] \to (0,\infty)$, $f(t) = t^2$ then we get

$$\frac{A(a,b)}{G^2(a,b)} \geqslant \frac{\ln b - \ln a}{b-a} \geqslant \frac{H(a,b)}{G^2(a,b)},$$

i.e.

$$LA \geqslant G^2 \geqslant LH. \tag{3.5}$$

The first inequality is trivial by (3.1) so we keep only the second inequality.

If we use the inequality (2.3) for $f : [a,b] \to (0,\infty)$, $f(t) = t^2$ then we get

$$0 \leqslant \frac{A(a,b)}{G^2(a,b)} - \frac{1}{L(a,b)} \leqslant \frac{1}{8}\frac{b-a}{G^4(a,b)}\left(b^2 - a^2\right) = \frac{1}{4}(b-a)^2 \frac{A(a,b)}{G^4(a,b)},$$

which is equivalent to (3.3).

If we use the inequality (2.4) for $f : [a,b] \to (0,\infty)$, $f(t) = t^2$, then we get

$$0 \leqslant \frac{1}{L(a,b)} - \frac{H(a,b)}{G^2(a,b)} \leqslant \frac{1}{4}(b-a)^2 \frac{A(a,b)}{G^4(a,b)},$$

which is equivalent to (3.4). □

We also have:

Proposition 2. *For any $0 < a < b$ and $p \in (-\infty,0) \cup (1,\infty) \setminus \{2,3\}$ we have*

$$\frac{A\left(a^{p-1}, b^{p-1}\right)}{G^2(a,b)} \geqslant L_{p-3}^{p-3}(a,b) \geqslant \frac{H^{p-1}(a,b)}{G^2(a,b)}, \tag{3.6}$$

$$0 \leqslant \frac{A\left(a^{p-1}, b^{p-1}\right)}{G^2(a,b)} - L_{p-3}^{p-3}(a,b) \leqslant \frac{1}{8}(p-1)\,p\frac{(b-a)^2}{G^4(a,b)} L_{p-1}^{p-1}(a,b) \tag{3.7}$$

and

$$0 \leqslant L_{p-3}^{p-3}(a,b) - \frac{H^{p-1}(a,b)}{G^2(a,b)} \leqslant \frac{1}{8}(p-1)\,p\frac{(b-a)^2}{G^4(a,b)} L_{p-1}^{p-1}(a,b). \tag{3.8}$$

Proof. Consider the function $f : [a, b] \to (0, \infty)$, $f(t) = t^p$ with $p \in (-\infty, 0) \cup (1, \infty) \setminus \{2, 3\}$, then f is convex on $[a, b]$ and if we apply the inequality (2.1), we get

$$\frac{A\left(a^{p-1}, b^{p-1}\right)}{G^2(a, b)} \geqslant \frac{1}{b-a} \int_a^b t^{p-3} dt \geqslant \frac{H^{p-1}(a, b)}{G^2(a, b)}. \tag{3.9}$$

Since

$$\frac{1}{b-a} \int_a^b t^{p-3} dt = L_{p-3}^{p-3}(a, b),$$

then we get from (3.9) the desired result (3.6).

By the inequality (2.3) we have

$$\begin{aligned} 0 &\leqslant \frac{A\left(a^{p-1}, b^{p-1}\right)}{G^2(a, b)} - L_{p-3}^{p-3}(a, b) \\ &\leqslant \frac{1}{8}(p-1) \frac{b-a}{G^4(a, b)} (b^p - a^p) = \frac{1}{8}(p-1) p \frac{(b-a)^2}{G^4(a, b)} L_{p-1}^{p-1}(a, b), \end{aligned}$$

which proves (3.7).

The inequality (3.8) follows by (2.4). □

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Received 10 03 2014, revised 27 06 2014

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