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On *Ulm* Support in *QTAG*-Modules

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Abstract. A right module M over an associative ring with unity is a $QTAG$-module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules. Mehdi and Naji introduced the notion of transitivity for QTAG-modules. Motivated by the transitivity and full transitivity we study full transitive pairs of QTAG-modules and obtain several characterizations. Here we examine how the formation of direct sums of QTAG-modules affects transitivity and full transitivity. We extend this concept by defining *Ulm* supports of QTAG-modules and consequently derive more results about the interrelationships of the various transitivities.

1. Introduction

All the rings R considered here are associative with unity and right modules M are unital $QTAG$-modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique decomposition series, $d(M)$ denotes its decomposition length. For a uniform element $x \in M,\ e(x) = d(xR)$ and $H_M(x) = \sup\left\{ d\left(\frac{yR}{xR}\right) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ and it is h-reduced if it does not contain any h-divisible submodule. In other words it is free from the elements of infinite height.

The cardinality of the minimal generating set of M is denoted by $g(M)$. For all ordinals $\alpha,\ f_M(\alpha)$ is the α^{th}-Ulm invariant of M and it is equal to

$$g(Soc(H_\alpha(M))/Soc(H_{\alpha+1}(M))).$$

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For a $QTAG$-module M, there is a chain of submodules $M^0 \supset M^1 \supset M^2 \cdots \supset M^\tau = 0$, for some ordinal τ. $M^{\sigma+1} = (M^\sigma)^1$, where M^σ is the σ^{th}-Ulm submodule of M.

Transitive and fully transitive $QTAG$-modules are defined with the help of U-sequences. Ulm invariants and Ulm sequences play an important role in the study of $QTAG$-modules. Using these concepts transitive and fully transitive modules were defined in [**6**]. A $QTAG$-module M is fully transitive if for $x, y \in M$, $U(x) \leqslant U(y)$, there is an endomorphism f of M such that $f(x) = f(y)$ and it is transitive if for any two elements $x, y \in M$, with $U(x) \leqslant U(y)$, there is an automorphism f of M such that $f(x) = f(y)$. A $QTAG$-module M is strongly transitive if for $x, y \in M$, $U(x) = U(y)$, there exists an endomorphism f of M such that $f(x) = y$.

Singh [**7**] proved that the results which hold for TAG-modules also hold good for $QTAG$-modules.

2. Main Results.

Motivated by the transitivity and full transitivity of $QTAG$-modules we study fully transitive pairs of $QTAG$-modules and obtain several characterizations. Here we examine how the formation of direct sums of $QTAG$-modules affects transitivity and full transitivity. We also extend this concept by defining Ulm support of $QTAG$-modules and consequently, derive more results about the inter-relationships of the various transitivities.

Definition 2.1. *Let M_1 and M_2 be QTAG-modules. Then $\{M_1, M_2\}$ is a* fully transitive pair *if for every $x \in M_i$, $y \in M_j$ $(i, j \in \{1, 2\})$ satisfying $U_{M_i}(x) \leqslant U_{M_j}(y)$, there exists a homomorphism f form M_i to M_j such that $f(x) = y$.*

Remark 2.1. *$\{M_1, M_2\}$ is a fully transitive pair whenever M_1 and M_2 are direct summands of a fully transitive module.*

Now we are able to prove the following:

Proposition 2.1. *Let $\{M_i\}_{i \in I}$ be a collection of QTAG-modules such that for each $i, j \in I$, $\{M_i, M_j\}$ is a fully transitive pair. Then the direct sum $\bigoplus_{i \in I} M_i$ is fully transitive.*

Proof. For the finite set $I = \{1, ..., n\}$, put $M = M_1 \oplus ... \oplus M_n$ and suppose $x, y \in M$ such that $U_M(x) \leqslant U_M(y)$. We have to define an endomorphism f of M such that $f(x) = y$. To apply induction on the exponent, consider y such that $d(yR) = 1$. Now we may assume that $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ such that $H_M(x) = H_{M_1}(x_1)$. Since $e(y) = 1$, $U_{M_1}(x_1) \leqslant U_M(y) \leqslant U_{M_i}(y_i)$ for all i. By assumption, there exist homomorphisms f_i's from M_1 to M_i such that $f_i(x_1) = y_i$, $1 \leqslant i \leqslant n$. Now, the $n \times n$ matrix with first row $(f_1, ..., f_n)$ and other rows zero represents an endomorphism f of M mapping x to y.

Now assume $e(y) > 1$, then $x', y' \in M$ such that $U_M(x') \leqslant U_M(y')$, where $d\left(\dfrac{xR}{x'R}\right) = d\left(\dfrac{yR}{y'R}\right) = 1$. Since $e(y') < e(y)$, there is an endomorphism ϕ of M such that $\phi(x') = y'$. Then $y - \phi(x) \in Soc(M)$ and $U_M(x) \leqslant U_M(y - \phi(x))$; hence there exists an endomorphism ψ of M such that $\psi(x) = y - \phi(x)$. Now $\phi + \psi$ maps x to y, and it is the required endomorphism. □

The following corollary is an immediate consequence of the above proposition:

Corollary 2.1. *Let M be a fully transitive QTAG-module and β any ordinal. Then all direct summands of $\bigoplus_{\beta} M$ are fully transitive.*

Proof. Since M is fully transitive, $\{M, M\}$ is a fully transitive pair. By Proposition 2.1, $\bigoplus_{\beta} M$ is fully transitive, for any ordinal β. Since the direct summands of fully transitive modules are fully transitive, we are done. □

Lemma 2.1. *Let $M = M_1 \oplus M_2$ be a fully transitive QTAG-module and $x_i, y_i \in M_i$ $(i = \{1, 2\})$. If $U_{M_1}(x_1) \leqslant U_{M_2}(y_2 - x_2)$ and $U_{M_2}(y_2) \leqslant U_{M_1}(y_1 - x_1)$, then there is an automorphism of M mapping (x_1, x_2) to (y_1, y_2).*

Proof. Since $\{M_1, M_2\}$ is a fully transitive pair, there exist homomorphisms f_1 from M_1 to M_2 and f_2 from M_2 to M_1 such that $f_1(x_1) = y_2 - x_2$ and $f_2(y_2) = y_1 - x_1$. The matrix $\begin{pmatrix} 1 + f_2 f_1 & f_1 \\ f_2 & 1 \end{pmatrix}$ represents an automorphism ϕ of $M_1 \oplus M_2$ such that $\phi(x_1, x_2) = (y_1, y_2)$. □

For a *QTAG*-module M and an ordinal σ, $f_M(\sigma)$ denote σ^{th}-*Ulm* invariant of M, defined by Mehdi et.al [**5**].

Definition 2.2. *If M is a h-reduced QTAG-module of length ρ, the Ulm support, denoted by supp(M), of M is the set of all ordinals $\sigma < \rho$ for which $f_M(\sigma)$ is non-zero.*

Remark 2.2. *If M_1 and M_2 are QTAG-modules with supp$(M_1) \subseteq$ supp(M_2), then every U-sequence relative to M_1 is also a U-sequence relative to M_2. In particular, for every $x \in M_1$, there is an element $y \in M_2$ such that $U_{M_1}(x) \leqslant U_{M_2}(y)$.*

Lemma 2.2. *Let $M = M_1 \oplus M_2$ be a fully transitive QTAG-module and supp$(H_\omega(M_1)) \subseteq$ supp$(H_\omega(M_2))$. If $x \in H_\omega(M)$, there is an automorphism of M mapping x to an element $(y, z) \in M_1 \oplus M_2$ with $U_M(x) = U_{M_2}(z)$.*

Proof. Put $x = (a, b)$, and consider an automorphism ϕ of M such that $\phi(x) = (a_1, b_1)$ and $H_{M_1}(a_1') \neq H_{M_2}(b_1')$, whenever $a_1'R \neq 0$. Here $a_1' \in M_1$ and $b_1' \in M_2$, such that $d\left(\dfrac{a_1R}{a_1'R}\right) = d\left(\dfrac{b_1R}{b_1'R}\right) = i$. Since *supp*$(H_\omega(M_1)) \subseteq$ *supp*$(H_\omega(M_2))$, we may choose $b_2 \in H_\omega(M_2)$ such that $U_{M_1}(a_1) = U_{M_2}(b_2)$. By full transitivity, there exists a homomorphism $f_1 : M_1 \to M_2$ such that $f_1(a_1) = b_2$. Therefore the composite automorphism $\begin{pmatrix} 1 & f_1 \\ 0 & 1 \end{pmatrix} \phi$ of $M_1 \oplus M_2$ maps x onto $(a_1, b_1 + b_2)$. Since $H(b_1') \neq H(b_2')$, where $d\left(\dfrac{b_1R}{b_1'R}\right) = d\left(\dfrac{b_2R}{b_2'R}\right) = i$ and $b_1'R \neq 0$, we compute $U_{M_2}(b_1 + b_2) = U_{M_2}(b_1) \wedge U_{M_2}(b_2) = U_{M_2}(b_1) \wedge U_{M_1}(a_1) = U_M(x)$. Here $U_{M_2}(b_1) \wedge U_{M_2}(b_2) = U_{M_2}(b_2)$ if $U_{M_2}(b_2) \leqslant U_{M_2}(b_1)$ and it is $U_{M_2}(b_1)$ if $U_{M_2}(b_1) < U_{M_2}(b_2)$. If we put $z = (b_1 + b_2)$, we get the required result.

Now we have to ensure the existence of the elements a_1, b_1 and the automorphism ϕ of M. We apply induction on the maximum m of the set $\mathcal{S}_M(a, b) = \{i < \omega : H_{M_1}(a') = H_{M_2}(b') \neq \infty\}$, where $a' \in M_1$ and $b' \in M_2$ such that $d\left(\dfrac{aR}{a'R}\right) = d\left(\dfrac{bR}{b'R}\right) = i$.

If $\mathcal{S}_M(a, b)$ is empty, then $\phi = I_M$. If the maximum of $\mathcal{S}_M(a, b)$ is $m = 0$, then either $H(a') > H(b')$ or $H(a') < H(b')$, here $d\left(\dfrac{aR}{a'R}\right) = d\left(\dfrac{bR}{b'R}\right) = 1$. Suppose $H_{M_1}(a') > H_{M_2}(b')$. Therefore $H_{M_1}(a') > H_{M_1}(a) + 1$, hence $a'R = a_1'R$ for some $a_1 \in M$and $H(a_1) > H(a)$, here $d\left(\dfrac{a_1R}{a_1'R}\right) = 1$. Put $b_1 = b$ and we have $H(a_1'') = H(b_1'')$ if $a_1''R = 0$, here $d\left(\dfrac{a_1R}{a_1''R}\right) = d\left(\dfrac{bR}{b''R}\right) = i$. Since $U_{M_1}(a_1 - a) = (H(a), \infty, ...) \geqslant U_{M_2}(b_1)$ and $\{M_1, M_2\}$ is a fully transitive pair, there is a homomorphism $f_2 : M_2 \to M_1$ such that $f_2(b_1) = a_1 - a$. Therefore the automorphism $\begin{pmatrix} 1 & 0 \\ f_2 & 1 \end{pmatrix}$ of $M_1 \oplus M_2$ maps (a, b) to (a_1, b_1) as required. If $H(a') < H(b')$, we obtain a suitable automorphism of the form $\begin{pmatrix} 1 & f_1 \\ 0 & 1 \end{pmatrix}$ and we are done.

Now assume that $\mathcal{S}_M(a, b)$ is non-empty and has maximum $m > 0$. If $d\left(\dfrac{aR}{a'R}\right) = d\left(\dfrac{bR}{b'R}\right) = 1$, then $\mathcal{S}_M(a', b')$ has maximum $< m$. Inductively there exists an automorphism ψ of M such that $\psi(a', b') = (a_2, b_2)$ and $H(a_2') \neq H(b_2')$, here $d\left(\dfrac{a_2R}{a_2'R}\right) = d\left(\dfrac{b_2R}{b_2'R}\right) = i$, whenever $a_2'R \neq 0$. Again we put $x' = (c, d) = \psi(x)$. Since $x'' = (a_2, b_2)$, $d\left(\dfrac{x'R}{x''R}\right) = 1$, $\mathcal{S}_M(c, d)$ is empty or has maximum 0. By the previous argument there exists an automorphism ϕ of M such that $\phi(c, d) = \phi\psi(x) = (a_1, b_1)$ and $H(a_i) \neq H(b_i)$, whenever $a_iR \neq 0$. Here $d\left(\dfrac{a_1R}{a_iR}\right) = d\left(\dfrac{b_1R}{b_iR}\right) = i$ and the result follows. □

Proposition 2.2. *Let $M = M_1 \oplus M_2$ be a fully transitive QTAG-module and $supp(H_\omega(M_1)) \subseteq supp(H_\omega(M_2))$. If M_2 is transitive, then M is transitive.*

Proof. Suppose $x, y \in H_\omega(M)$ have the same U-sequences in M. By Lemma 2.2, there exist automorphisms f_1, f_2 of M such that $f_1(x) = (x_1, x_2)$ and $f_2(y) = (y_1, y_2)$, therefore $U_{M_2}(x_2) = U_M(x) = U_M(y) = U_{M_2}(y_2)$. Since $U_M(x_1 - y_1, x_2 - y_2) \geqslant U_M(f_1(x))$, we have $U_{M_2}(x_2) \leqslant U_{M_1}(y_1 - x_1)$. As M_2 is transitive, there is an automorphism f_1' of M_2 such that $f_1'(x_2) = y_2$ and because M is fully transitive, there is a homomorphism f_2' from M_2 to M_1 such that $f_2'(x_2) = y_1 - x_1$. If we put $\psi = \begin{pmatrix} 1 & 0 \\ f_2' & f_1' \end{pmatrix}$, then it is an automorphism of M. Also $(f_2^{-1}\psi f_1)(x) = y$, thus $(f_2^{-1}\psi f_1)$ is the required automorphism and we are done. □

Corollary 2.2. *If $\{N_i\}$ is a collection of direct summands of any power of the $QTAG$-module M, such that M is transitive and fully transitive, then the direct sum $M \oplus (\oplus N_i)$ is transitive and fully transitive.*

Proof. By Proposition 2.1, $M \oplus (\oplus N_i)$ is fully transitive. Since $supp(H_\omega(N_i)) \subseteq supp(H_\omega(M))$, then by Proposition 2.2, the direct sum is also transitive. □

We observe that a fully transitive $QTAG$-module M with transitive direct summand N is itself transitive provided that $H_\omega(M)$ and $H_\omega(N)$ have the same *Ulm* support. We investigate the conditions under which transitivity and fully transitivity are equivalent.

Theorem 2.1. *Let M be a $QTAG$-module with a decomposition $M = M_1 \oplus M_2$ such that $H_\omega(M_1)$ and $H_\omega(M_2)$ have the same Ulm supports. Then M is fully transitive if and only if M is transitive.*

Proof. Suppose M is full transitive and let $x, y \in H_\omega(M)$ such that $U_M(x) = U_M(y)$. By Lemma 2.2, there are automorphisms f_1, f_2 of M such that $f_1(x) = (a, b), f_2(y) = (c, d)$, satisfying $U_M(x) = U_{M_1}(a)$ and $U_M(y) = U_{M_2}(d)$. Since $U_M(x) = U_M(y)$ we have $U_{M_1}(a) \leqslant U_{M_2}(b), U_{M_2}(d)$, hence $U_{M_1}(a) \leqslant U_{M_2}(d - b)$. Similarly, we obtain $U_{M_2}(d) \leqslant U_{M_1}(c - a)$. Then by Lemma 2.1, there exists an automorphism ψ of M such that $\psi(a, b) = (c, d)$. Now $f_2^{-1}\psi f_1(x) = y$ is the required automorphism of M. Thus M is transitive.

Conversely, suppose M is transitive. Consider $B = B' \oplus B'$, where B' is the basic submodule of M. Then $N = M \oplus B$ is transitive since B is separable. The structure of the modules B and $H_\omega(N) = H_\omega(M_1) \oplus H_\omega(M_2)$ implies that N has no *Ulm* invariants equal to one. Therefore N is fully transitive, whence M is fully transitive. □

The following corollary is the immediate consequence of the above result.

Corollary 2.3. *The following conditions are equivalent for a $QTAG$-module:*

(*i*) *For all ordinals β, $\bigoplus_\beta M$ is fully transitive.*

(*ii*) *For some $\beta > 0$, $\bigoplus_\beta M$ is fully transitive.*

(*iii*) *For all $\beta > 1$, $\bigoplus_\beta M$ is transitive.*

(*iv*) *For some* $\beta > 1$, $\bigoplus_{\beta} M$ *is transitive.*

Proof. The implications $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$ are trivial.

Suppose (ii) holds, and $\beta(>1)$ is a fixed ordinal. Since M is a summand of a fully transitive module, it is fully transitive. Also by Corollary 2.1, $\bigoplus_{\beta} M$ is fully transitive. Since $\beta > 1$, then $\bigoplus_{\beta} M = M_1 \oplus M_2$ such that $supp(H_\omega(M_1) = supp(H_\omega(M_2) = supp(H_\omega(M)$. Hence, by Theorem 2.1, $\bigoplus_{\beta} M$ is transitive. Therefore (iii) holds.

Finally, suppose (iv) holds. Put $\bigoplus_{\beta} M = M_1 \oplus M_2$ and $\bigoplus_{\beta} M$ is transitive, then by Theorem 2.1, $\bigoplus_{\beta} M$ is fully transitive. Therefore M is fully transitive, and Corollary 2.1, yields condition (i). □

Corollary 2.4. *Let* $\{M_i\}_{i\in I}$ *be a collection of QTAG-modules. If there exists an ordinal* σ *such that* $M_i/H_\sigma(M_i)$ *is totally projective and* $\{H_\sigma(M_i), H_\sigma(M_j)\}$ *is a fully transitive pair for each* $i, j \in I$, *then* $\bigoplus_{i\in I} M_i$ *is fully transitive. Moreover, if there exists a partition* $I = I_1 \cup I_2$ *such that the modules* $\bigoplus_{i\in I_1} H_{\sigma+\omega}(M_i)$ *and* $\bigoplus_{i\in I_2} H_{\sigma+\omega}(M_i)$ *have equal Ulm supports, then* $\bigoplus_{i\in I} M_i$ *is transitive.*

Proof. Let $M = \bigoplus_{i\in I} M_i$. Then by Proposition 2.1, $H_\sigma(M) = \bigoplus_{i\in I} H_\sigma(M_i)$ is fully transitive. Since $\dfrac{M}{H_\sigma(M)} \cong \bigoplus_{i\in I} \dfrac{M_i}{H_\sigma(M_i)}$ is totally projective, M is fully transitive. Again by Theorem 2.1, $H_\sigma(M)$ is transitive and it follows that M is transitive. □

Lastly, we establish the relation among transitive, fully transitive and strongly transitive modules.

Theorem 2.2. *If* $M = M_1 \oplus M_2$ *and* $supp(H_\omega(M_1)) = supp(H_\omega(M_2))$, *then the following are equivalent:*

(*i*) M *is strongly transitive;*
(*ii*) M *is fully transitive;*
(*iv*) M *is transitive.*

Proof. The equivalence of (ii) and (iii) follows from Theorem 2.1 and the implication $(ii) \Rightarrow (i)$ is trivial.
Suppose (i) holds, and let B denote the basic submodule of M. Now put $N = M \oplus B \oplus B$. Since M is strongly transitive, then N is also strongly transitive. By the result, Let M be a strongly transitive $QTAG$-module such that M has atmost two Ulm invariants equal to 1. If M has exactly two Ulm invariants corresponding to successive ordinals, then M is fully transitive, in [**4**], so N is fully transitive, because no Ulm invariant of N is equal to one. As a summand of a fully transitive module, M is also fully transitive. Therefore (ii) holds. □

References

[1] Fuchs L., *Infinite Abelian Groups,* Vol. I, Academic Press, New York, 1970.

[2] Fuchs L., *Infinite Abelian Groups,* Vol. II, Academic Press, New York, 1973.

[3] Kaplansky, I., *Infinite Abelian Groups,* University of Michigan Press, Ann. Arbor, 1954 and 1969.

[4] Sikander F., Hasan A. and Mehdi A. *On different notions of transitivity for QTAG-Modules,* (Communicated)

[5] Mehdi A., Abbasi M. Y. and Mehdi F., *On some structure theorems of QTAG-modules of countable Ulm type*, South East Asian J. Math. & Math. Sci., **3**(3)(2005), 103–110.

[6] Naji S. A. R. K., *A study of different structures of QTAG-modules*, Ph.D. Thesis, A.M.U., Aligarh, 2011.

[7] Singh S., *Some decomposition theorems in abelian groups and their generalizations,* Ring Theory, Proc. of Ohio Univ. Conf. Marcel Dekker N.Y. 25, 183-189, (1976).

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