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Evaluation of some definite integrals through arctangent function and infinite products

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Abstract. Some integrals involving elementary functions and special functions such as exponential integral, psi function, etc., are evaluated through arctangent function, infinite series and infinite products.

1. Introduction

Some integrals involving various combinations of elementary functions in Ref[**6**] are evaluated through following infinite series of arctangent functions that found in chapter 2 of Ramanujan notebooks, part-I. Let a and x be real[**1**, p. 39-40]. Then

$$\sum_{k=-\infty}^{\infty} (-1)^k \tan^{-1} \frac{x}{k\pi + a} = \tan^{-1} (\sinh x \csc a), \tag{1.1}$$

$$\sum_{k=-\infty}^{\infty} \tan^{-1} \frac{x}{k\pi + a} = \tan^{-1} (\tanh x \cot a), \tag{1.2}$$

$$\sum_{k=0}^{\infty} (-1)^k \tan^{-1} \frac{x}{2k+1} = \tan^{-1} \left(\tanh \frac{\pi x}{4}\right) = \frac{\pi}{4} - \tan^{-1} \left(e^{-x\pi/2}\right). \tag{1.3}$$

In Ref[**5**], integrals involving elementary functions and psi function are evaluated through following infinite series for psi function[**3**, p. 893] and infinite products for gamma function [**3**, p. 886]. For real x, y and $x \neq 0, -1, -2, \ldots$

$$\sum_{k=0}^{\infty} \frac{2yi}{y^2 + (x+k)^2} = \psi(x+iy) - \psi(x-iy). \tag{1.4}$$

$$\prod_{k=0}^{\infty} \left(1 + \frac{y^2}{(x+k)^2}\right) = \frac{\Gamma(x)^2}{\Gamma(x+iy)\Gamma(x-iy)} = \left|\frac{\Gamma(x)}{\Gamma(x-iy)}\right|^2. \tag{1.5}$$

Where ψ and Γ are psi and gamma functions respectively.

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The chapter 37 of Ramanujan notebooks, part-V contains many entries related to infinite series. The entries from 27 to 30 contain following infinite series for the integral $\phi(x) = \int_0^x \frac{tan^{-1}}{t} dt$[**2**, 457-461]. For every real x,

$$\sum_{k=0}^{\infty}(-1)^k(2k+1)\log\left(1+\frac{x^2}{(2k+1)^2}\right) = \frac{4}{\pi}\left[G-\phi(e^{-\frac{\pi x}{2}})\right] - 2x\tan^{-1}\left(e^{-\frac{\pi x}{2}}\right). \tag{1.6}$$

$$\sum_{n=0}^{\infty}\frac{\sin(4n+2)x}{(2n+1)^2} = \phi(\tan x) - x\log(\tan x). \qquad 0 < x < \pi/2. \tag{1.7}$$

Where G is Catalan constant. The classical table of integrals by Gradshteyn and Ryzhik[**3**]contains solutions of many integrals that involving various combinations of elementary and special functions in closed form. Some of the integrals are given in terms of infinite series of elementary functions or special functions. For instance[**3**, p. 369], $a > 0$

$$\int_1^{\infty}\frac{dx}{x\sinh ax}dx = -2\sum_{k=0}^{\infty}Ei\left[-(2k+1)a\right]. \tag{1.8}$$

$$\int_1^{\infty}\frac{dx}{x\cosh ax}dx = 2\sum_{k=0}^{\infty}(-1)^{k+1}Ei\left[-(2k+1)a\right]. \tag{1.9}$$

Where Ei is the exponential integral function. In present study, integrals that involving various combinations of elementary functions and special functions (such as exponential integrals, psi function etc.,) are evaluated through above mentioned series of arctangent function, infinite products and infinite series as studied in Ref[**5, 6**], either in closed form or infinite series. The integrals given in this paper are not available in the classical tables of integrals[**3**]. Also, they cannot be evaluated through symbolic language such as Mathematica.

2. Evaluation of integrals through arctangent function

2.1. Combinations of powers, exponential and rational functions. In this subsection, the following type of integral is expressed in finite number of terms of sine and cosine integral function using integrals involving arctangent function.

$$\int_0^{\infty}\frac{e^{-px}x^{2m}}{(b^2+1^2x^2)\dots(b^2+(2m+1)^2x^2)}dx.$$

Entry **3.946.1** from Ref[**3**, p. 494] states that for $p > 0$

$$\int_0^{\infty}e^{-px}\sin^{2m+1}yx\frac{dx}{x} = \frac{(-1)^m}{2^{2m}}\sum_{k=0}^{m}(-1)^k\binom{2m+1}{k}\tan^{-1}\left[(2m-2k+1)\frac{y}{p}\right]. \tag{2.1}$$

Multiply e^{-yb} on both sides of (2.1) and integrating on $[0,\infty)$ with respect to p, then

$$\int_0^\infty e^{-px}\int_0^\infty e^{-yb}\sin^{2m+1}yxdy\frac{dx}{x} = \frac{(-1)^m}{2^{2m}}\sum_{k=0}^{m}(-1)^k\binom{2m+1}{k}\int_0^\infty e^{-yb}\tan^{-1}\left[(2m-2k+1)\frac{y}{p}\right]dy. \tag{2.2}$$

Entry **3.895.4** from Ref[**3**, p. 481] for Re $b>0$ can be written as follows

$$\int_0^\infty e^{-by}\sin^{2m+1}xydy = \frac{(2m+1)!x^{2m+1}}{(b^2+1^2x^2)(b^2+3^2x^2)\dots(b^2+(2m+1)^2x^2)}. \tag{2.3}$$

Also, integral **4.55.3** from Ref[**3**, p. 601] states that for Re $b>0$

$$\int_0^\infty \tan^{-1}\left(\frac{x}{a}\right)e^{-bx} = \frac{1}{b}\left(ci(ab)\sin(ab)-si(ab)\cos(ab)\right). \tag{2.4}$$

Using (2.3) and (2.4) in (2.2), yields

$$\int_0^\infty \frac{e^{-px}x^{2m}}{(b^2+1^2x^2)\dots(b^2+(2m+1)^2x^2)}dx = \frac{(-1)^m}{2^{2m}b}\sum_{k=0}^{m}\frac{(-1)^k}{k!(2m+1-k)!}\times$$
$$\left[ci\left(\frac{pb}{2m-2k+1}\right)\sin\left(\frac{pb}{2m-2k+1}\right)-si\left(\frac{pb}{2m-2k+1}\right)\cos\left(\frac{pb}{2m-2k+1}\right)\right].$$

Similarly, consider the following integral **3.946.2** from Ref[**3**, p. 494] for $p>0$

$$\int_0^\infty e^{-px}\sin^{2m}yx\frac{dx}{x} = -\frac{1}{2^{2m}}\binom{2m}{m}\log p+\frac{(-1)^{m+1}}{2^{2m}}\times \sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}\log\left[p^2+(2m-2k)^2y^2\right]. \tag{2.5}$$

Multiply by e^{-yb} on both sides of (2.5) and integrating with respect to y on $[0,\infty)$, then

$$\int_0^\infty e^{-px}\int_0^\infty e^{-yb}\sin^{2m}yxdy\frac{dx}{x} = -\frac{1}{2^{2m}b}\binom{2m}{m}\log p+\frac{(-1)^{m+1}}{2^{2m}}\times \sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}\int_0^\infty e^{-yb}\log\left[p^2+(2m-2k)^2y^2\right]dy. \tag{2.6}$$

Entry **3.895.1** from Ref[**3**, p. 481] states that for Re $b>0$

$$\int_0^\infty e^{-yb}\sin^{2m}yxdy = \frac{2m!x^{2m+2}}{b(b^2+2^2x^2)(b^2+4^2x^2)\dots(b^2+(2m)^2x^2)}. \tag{2.7}$$

Also, entry **4.338.1** from Ref[**3**, p. 568] states that for Re $b>0$ and Re $a>0$

$$\int_0^\infty e^{-bt}\log\left(a^2+t^2\right) = \frac{2}{b}\left[\log a-ci(ab)\cos(ab)-si(ab)\sin(ab)\right]. \tag{2.8}$$

Using (2.7) and (2.8) in (2.6), gives

$$\int_0^\infty \frac{e^{-px}x^{2m+1}}{(b^2+2^2x^2)\dots(b^2+(2m)^2x^2)}dx = -\frac{\log p}{2^{2m-1}}\left[\sum_{k=0}^{m-1}\frac{(-1)^{m+k}}{k!(2m-k)!}+\frac{1}{2(m!)^2}\right]-\frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}\frac{(-1)^k}{k!(2m-k)!}\times \left[ci\left(\frac{pb}{2m-2k}\right)\cos\left(\frac{pb}{2m-2k}\right)+si\left(\frac{pb}{2m-2k}\right)\sin\left(\frac{pb}{2m-2k}\right)\right].$$

2.2. Combinations of arctangent and cosine functions. In this subsection, integrals involving arctangent and cosine functions are evaluated through infinite series.

Consider the following entry **4.581** from Ref[**3**, p. 603] for Re $p > 0$

$$\int_0^\infty \tan^{-1}\left(\frac{x}{p}\right)\cos x\frac{dx}{x} = -\frac{\pi}{2}Ei(-p). \tag{2.9}$$

Then, for $|p| < \pi$

$$\int_0^\infty \left(\tan^{-1}\frac{x}{k\pi+p}-\tan^{-1}\frac{x}{k\pi-p}\right)\cos x\frac{dx}{x} = -\frac{\pi}{2}\left(Ei(-p-k\pi)-Ei(p-k\pi)\right). \tag{2.10}$$

Taking summation on both sides for $k = 1, 2, 3, \dots$, then

$$\int_0^\infty \sum_{k=1}^\infty \left(\tan^{-1}\frac{x}{k\pi+p}-\tan^{-1}\frac{x}{k\pi-p}\right)\cos x\frac{dx}{x} = -\frac{\pi}{2}\sum_{k=1}^\infty\left(Ei(-p-k\pi)-Ei(p-k\pi)\right).$$

Using the identity given in (1.2), gives

$$\int_0^\infty \tan^{-1}(\tanh x\cot p)\cos x\frac{dx}{x} = -\frac{\pi}{2}\left[Ei(-p)+\sum_{k=1}^\infty\left(Ei(-p-k\pi)-Ei(p-k\pi)\right)\right]. \tag{2.11}$$

Differentiating with respect to p, gives

$$\int_0^\infty \frac{\cos x}{\coth x\sin^2 p+\tanh x\cos^2 p}\frac{dx}{x} = \frac{\pi}{2}\left[\frac{e^{-p}}{p}+\sum_{k=1}^\infty\left(\frac{e^{-p-k\pi}}{p+k\pi}-\frac{e^{p-k\pi}}{p-k\pi}\right)\right]. \tag{2.12}$$

Similarly, using the identity (1.1), it can be easily found that

$$\int_0^\infty \tan^{-1}(\sinh x \csc p)\cos x \frac{dx}{x} = -\frac{\pi}{2}\left[Ei(-p) + \sum_{k=1}^{\infty}(-1)^k\left(Ei(-p-k\pi) - Ei(p-k\pi)\right)\right]. \tag{2.13}$$

Differentiating with respect to p, yields

$$\int_0^\infty \frac{\sinh x \cos x}{\sin^2 p + \sinh^2 x}\frac{dx}{x} = \frac{\pi}{2}\sec p\left[\frac{e^{-p}}{p} + \sum_{k=1}^{\infty}(-1)^k\left(\frac{e^{-p-k\pi}}{p+k\pi} - \frac{e^{p-k\pi}}{p-k\pi}\right)\right]. \tag{2.14}$$

Multiply by $(-1)^k$ on both sides of (2.9), and replacing p by $2k+1$ and taking summation for $k = 1, 2, 3\ldots$ and using identity (1.3), gives

$$\int_0^\infty \tan^{-1}\left(\tanh\frac{\pi x}{4}\right)\cos x\frac{dx}{x} = -\frac{\pi}{2}\sum_{k=0}^{\infty}(-1)^k Ei(-2k-1). \tag{2.15}$$

3. Evaluation of integrals through infinite products

The integrals involving special functions and elementary functions are evaluated through infinite series (1.4) and infinite products (1.5) and (1.6).

3.1. Combinations of logarithmic of gamma function, powers and trigonometric functions. Consider the integral **6.232.1** from Ref[**3**, p. 633] for $a > 0$ and $b > 0$

$$\int_0^\infty Ei(-ax)\sin bx dx = -\frac{1}{2b}\log\left(1 + \frac{b^2}{a^2}\right). \tag{3.1}$$

Replacing a by $a+k$ and taking summation for $k = 0, 1, 2, \ldots$, gives

$$\int_0^\infty \sum_{k=0}^{\infty} Ei(-(a+k)x)\sin bx dx = -\frac{1}{b}\log\left|\frac{\Gamma(a)}{\Gamma(a-ib)}\right|. \tag{3.2}$$

Then, by Fourier sine transform[**3**, p. 1113]

$$\int_0^\infty \frac{\sin bx}{x}\log\left|\frac{\Gamma(a)}{\Gamma(a-ix)}\right| dx = -\frac{\pi}{2}\sum_{k=0}^{\infty} Ei(-(a+k)b). \tag{3.3}$$

Differentiating (3.3) with respect to b and using integral representation of exponential integral **8.211.1** from Ref[**3**, p. 875], gives

$$\int_0^\infty \cos bx \log\left|\frac{\Gamma(a)}{\Gamma(a-ix)}\right| dx = -\frac{\pi}{2}\sum_{k=0}^{\infty}\frac{e^{-(a+k)b}}{a+k}. \tag{3.4}$$

Again differentiating (3.4) with respect to b, gives

$$\int_0^\infty x\sin bx\log\left|\frac{\Gamma(a)}{\Gamma(a-ix)}\right|dx=-\frac{\pi}{2}\frac{e^{-ab}}{1-e^{-b}}. \tag{3.5}$$

Differentiating (3.5) $2n$ times with respect to b , then

$$\int_0^\infty x^{2n+1}\sin bx\log\left|\frac{\Gamma(a)}{\Gamma(a-ix)}\right|dx=(-1)^{n+1}\frac{\pi}{2}\frac{d^{2n}}{db^{2n}}\left[\frac{e^{-ab}}{1-e^{-b}}\right]. \tag{3.6}$$

Differentiating (3.5) $2n-1$ times with respect to b, then

$$\int_0^\infty x^{2n}\cos bx\log\left|\frac{\Gamma(a)}{\Gamma(a-ix)}\right|dx=(-1)^{n}\frac{\pi}{2}\frac{d^{2n-1}}{db^{2n-1}}\left[\frac{e^{-ab}}{1-e^{-b}}\right]. \tag{3.7}$$

3.2. Combinations of powers, trigonometric and hyperbolic functions. Consider the formula **3.946.2** from Ref[**3**, p. 494] for $p>0$

$$\begin{aligned}\int_0^\infty e^{-px}\sin^{2m}ax\frac{dx}{x}&=-\frac{1}{2^{2m}}\binom{2m}{m}\log p\\&+\frac{(-1)^{m+1}}{2^{2m}}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}\log\left[p^2+(2m-2k)^2a^2\right].\end{aligned} \tag{3.8}$$

This can be easily found that

$$\begin{aligned}&\int_0^\infty e^{-px}\left(\sin^{2m}ax-\sin^{2m}bx\right)\frac{dx}{x}\\&=\frac{(-1)^{m+1}}{2^{2m}}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}\log\left[\frac{1+(2m-2k)^2a^2/p^2}{1+(2m-2k)^2b^2/p^2}\right].\end{aligned} \tag{3.9}$$

Multiply by p on both sides and after simplification, gives

$$\begin{aligned}&\int_0^\infty \left(\sin^{2m}ax-\sin^{2m}bx\right)\frac{d(e^{-px})}{x}\\&=\frac{(-1)^{m}}{2^{2m}}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}p\log\left[\frac{1+(2m-2k)^2a^2/p^2}{1+(2m-2k)^2b^2/p^2}\right].\end{aligned} \tag{3.10}$$

Multiply by $(-1)^i$, replace p by $p(2i+1)$ and taking summation for $i=0,1,2,\ldots$,gives

$$\begin{aligned}\int_0^\infty \left(\sin^{2m}ax-\sin^{2m}bx\right)\frac{1}{x}d\left(\frac{e^{-px}}{1+e^{-2px}}\right)&=\frac{(-1)^{m}}{2^{2m}}\sum_{k=0}^{m-1}(-1)^k\binom{2m}{k}\\&\times\sum_{i=0}^{\infty}(-1)^i(2i+1)\log\left[\frac{1+(2m-2k)^2a^2/p^2(2i+1)^2}{1+(2m-2k)^2b^2/p^2(2i+1)^2}\right].\end{aligned} \tag{3.11}$$

Using the infinite product identity (1.6), yields

$$\begin{aligned}(3.12)\qquad &\int_0^\infty \left(\sin^{2m} ax - \sin^{2m} bx\right) \tanh px \sec hpx \frac{dx}{x}\\ &= \frac{(-1)^{m+1}}{2^{2m-1}\pi} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \left[\frac{1}{\pi} \left\{ \phi(e^{-\pi \frac{a}{p}(m-k)}) - \phi(e^{-\pi \frac{b}{p}(m-k)}) \right\} \right.\\ &\left. + (m-k)\frac{a}{p} \tan^{-1}\left(e^{-\pi \frac{a}{p}(m-k)}\right) - (m-k)\frac{b}{p} \tan^{-1}\left(e^{-\pi \frac{b}{p}(m-k)}\right) \right].\end{aligned}$$

Consider the entry **3.948.3** following integral from Ref[**3**, p. 495] for Re $p > 0$

$$\int_0^\infty e^{-px} (\cos ax - \cos bx) \frac{dx}{x^2} = \frac{p}{2} \log \left(\frac{1 + a^2/p^2}{1 + b^2/p^2} \right) + b \tan^{-1} \frac{b}{p} - a \tan^{-1} \frac{a}{p}.$$

Multiply by $(-1)^i$, replacing p by $p(2i+1)$ and taking summation for $i = 0, 1, 2, \ldots$,gives

$$\begin{aligned}\int_0^\infty \frac{2}{\cosh px} (\cos ax - \cos bx) \frac{dx}{x^2} &= \frac{1}{2} \sum_{i=0}^\infty (-1)^i p(2i+1) \log \left(\frac{1 + \frac{a^2}{p^2(2i+1)^2}}{1 + \frac{b^2}{p^2(2i+1)^2}} \right)\\ &+ \sum_{i=0}^\infty (-1)^i \left(b \tan^{-1} \frac{b}{p(2i+1)} - a \tan^{-1} \frac{a}{p(2i+1)} \right).\end{aligned}$$

Using the identities (1.6) and (1.3), gives

$$\int_0^\infty \frac{\cos ax - \cos bx}{\cosh px} \frac{dx}{x^2} = \frac{1}{\pi} \left(\phi(e^{-b\pi/2p}) - \phi(e^{-a\pi/2p}) \right) + (b-a)\frac{\pi}{8}.$$

3.3. Combinations of powers and logarithmic functions. Consider the following integral from Ref [**4**] for $a > 0$ and $b > 0$

$$\int_0^\infty \frac{e^{-zx}}{1 - e^{-x}} (\cos ax - \cos bx) \frac{dx}{x} = \log \left| \frac{\Gamma(z - ia)}{\Gamma(z - ib)} \right|. \tag{3.13}$$

Replace a by $\alpha - \beta$ and b by $\alpha + \beta$, then

$$\int_0^\infty \frac{e^{-zx}}{1 - e^{-x}} (\cos(\alpha - \beta)x - \cos(\alpha + \beta)x) \frac{dx}{x} = \frac{1}{2} \log \left| \frac{\Gamma(z - i\alpha + i\beta)}{\Gamma(z - i\alpha - i\beta)} \right|.$$

$$\int_0^\infty \frac{e^{-zx}}{1 - e^{-x}} \sin \alpha x \sin \beta x \frac{dx}{x} = \frac{1}{2} \log \left| \frac{\Gamma(z - i\alpha + i\beta)}{\Gamma(z - i\alpha - i\beta)} \right|. \tag{3.14}$$

Then by Fourier sine transform,

$$\int_0^\infty \log \left| \frac{\Gamma(z - i\alpha + ix)}{\Gamma(z - i\alpha - ix)} \right| \sin \beta x dx = \frac{\pi e^{-z\beta}}{1 - e^{-\beta}} \frac{\sin \alpha\beta}{\beta}. \tag{3.15}$$

Let $y = z - i\alpha$. Then for Im(y)> 0,

$$\int_0^\infty \log \left| \frac{\Gamma(y + ix)}{\Gamma(y - ix)} \right| \sin \beta x dx = \frac{\pi}{2\beta} \frac{e^{-\bar{y}\beta} - e^{-y\beta}}{1 - e^{-\beta}}. \tag{3.16}$$

Differentiating (3.15) $2m$ times with respect to β , then

$$\int_0^\infty x^{2m} \log \left| \frac{\Gamma(y+ix)}{\Gamma(y-ix)} \right| \sin \beta x = (-1)^m \frac{\pi}{2} \frac{d^{2m}}{d\beta^{2m}} \left[\frac{1}{\beta} \frac{e^{-\bar{y}\beta} - e^{-y\beta}}{1-e^{-\beta}} \right]. \tag{3.17}$$

Differentiating (3.15) $2m+1$ times with respect to β, then

$$\int_0^\infty x^{2m+1} \log \left| \frac{\Gamma(y+ix)}{\Gamma(y-ix)} \right| \cos \beta x = (-1)^{m+1} \frac{\pi}{2} \frac{d^{2m+1}}{d\beta^{2m+1}} \left[\frac{1}{\beta} \frac{e^{-\bar{y}\beta} - e^{-y\beta}}{1-e^{-\beta}} \right]. \tag{3.18}$$

Differentiating (3.16) with respect to y, yields

$$\int_0^\infty [\psi(y+ix) - \psi(y-ix)] \sin \beta x = -\frac{\pi}{2} \left(\frac{e^{-\bar{y}\beta} - e^{-y\beta}}{1-e^{-\beta}} \right). \tag{3.19}$$

Differentiating (3.19) n times with respect to y, then

$$\int_0^\infty [\zeta(y+ix, n+1) - \zeta(y-ix, n+1)] \sin \beta x = -\frac{\pi}{2} \frac{\beta^n}{n!} \left(\frac{e^{-\bar{y}\beta} - e^{-y\beta}}{1-e^{-\beta}} \right).$$

Using Fourier sine transform[**3**, p. 1113], yields

$$\int_0^\infty x^n \frac{e^{-zx}}{1-e^{-x}} \sin \alpha x \sin \beta x dx = \frac{2i}{n!} \left[\zeta(y+i\beta, n+1) - \zeta(y-i\beta, n+1) \right].$$

3.4. Combinations of powers, logarithmic gamma and Bessel functions. Entry **6.776** from Ref[**3**, p. 740] states that for Re $a > 0$ and $b > 0$

$$\int_0^\infty x \log(1 + a^2/x^2) J_0(bx) dx = \frac{2}{b^2} - \frac{2a}{b} K_1(ab). \tag{3.20}$$

Where J_0 and K_1 are Bessel function of first kind and modified Bessel function of second kind respectively. Replacing a by $1/(a+k)$ and taking summation for $k = 0, 1, 2, \ldots$, then

$$\int_0^\infty x \sum_{k=0}^\infty \log \left(1 + \frac{1/x^2}{(a+k)^2} \right) J_0(bx) dx = \frac{2}{b^2} - \frac{2}{b(a+k)} K_1 \left(\frac{b}{a+k} \right). \tag{3.21}$$

Replacing a by α ,then

$$\int_0^\infty x \sum_{k=0}^\infty \log \left(1 + \frac{1/x^2}{(\alpha+k)^2} \right) J_0(bx) dx = \frac{2}{b^2} - \frac{2}{b(\alpha+k)} K_1 \left(\frac{b}{\alpha+k} \right). \tag{3.22}$$

Subtracting (3.21) from (3.22) and using the identity (1.6), gives

$$\int_0^\infty x \log \left| \frac{\Gamma(a-i/x)}{\Gamma(\alpha-i/x)} \right| J_0(bx) dx = -\frac{1}{b} \sum_{k=0}^\infty \left(\frac{K_1(b/(\alpha+k))}{(\alpha+k)} - \frac{K_1(b/(a+k))}{(a+k)} \right). \tag{3.23}$$

But entry **8.446** from Ref[**3**, p. 909] shows that

$$K_1(z) = \frac{1}{z} + \sum_{j=0}^{\infty} \frac{(z/2)^{2j+1}}{j!(n+j!)} \left[\log\frac{z}{2} - \psi(j+1) - \frac{1}{2(j+1)} \right]. \tag{3.24}$$

Using (3.24) in (3.23), gives

$$\int_0^\infty x \log\left|\frac{\Gamma(a-i/x)}{\Gamma(\alpha-i/x)}\right| J_0(bx)dx = -\frac{1}{b}\sum_{j=0}^{\infty} \frac{(b/2)^{2j+1}}{j!(n+j)!} \times \left[\sum_{k=0}^{\infty} \frac{1}{(\alpha+k)^{2j+2}} \left(\frac{\log(b/2)}{\alpha+k} - \psi(j+1) - \frac{1}{2(j+1)} \right) - \sum_{k=0}^{\infty} \frac{1}{(a+k)^{2j+2}} \left(\frac{\log(b/2)}{a+k} - \psi(j+1) - \frac{1}{2(j+1)} \right) \right].$$

After simplification, yields

$$\begin{aligned} &\int_0^\infty x \log\left|\frac{\Gamma(a-i/x)}{\Gamma(\alpha-i/x)}\right| J_0(bx)dx \\ &= -\frac{1}{b}\sum_{j=0}^{\infty} \frac{(b/2)^{2j+1}}{j!(n+j)!} \left[-\zeta'(2j+2,a) + \zeta'(2j+2,\alpha) \right. \\ &\left. + (\zeta(2j+2,\alpha) - \zeta(2j+2,a)) \left(\log(b/2) - \psi(j+1) - \frac{1}{2(j+1)} \right) \right]. \end{aligned} \tag{3.25}$$

4. Integrals involving exponential, sine and cosine integral functions

DEFINITION 4.1. Let $\Re(x) > 0$ and n be a natural number. Define $\phi_n(x)$ as follows

$$\phi_n(x) = \int_0^x \frac{\phi_{n-1}(t)}{t} dt, \qquad n = 2, 3, \ldots \tag{4.1}$$

where $\phi_1(x) = \tan^{-1} x$.

The function ϕ_n satisfies the following properties

(1) For all x, $\phi_n(-x) = -\phi_n(x)$.
(2) If $|x| \le 1$, then

$$\phi_n(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)^n}.$$

(3) For $n = 0, 1, 2, \ldots$

$$\phi_{2n+1}(1) = \frac{\pi^{2n+1}}{2^{2n+2}(2n)!} |E_{2n}|. \tag{4.2}$$

where E_{2n} is Euler number of order $2n$.

4.1. Combinations of exponential integral function and trigonometric functions. Entry **3.946.1** from Ref[**3**, p. 494] states that for $p > 0$

$$\int_0^\infty e^{-px} \sin^{2m+1} yx \frac{dx}{x} = \frac{(-1)^m}{2^{2m}} \sum_{k=0}^{m} (-1)^k \binom{2m+1}{k} \tan^{-1}\left[(2m-2k+1)\frac{y}{p}\right]. \tag{4.3}$$

Divide (4.3) by p and integrating on $[1,\infty)$ with respect to p. Then

$$\int_0^\infty \int_1^\infty \frac{e^{-px}}{p} \sin^{2m+1} yx \frac{dx}{x} dp = \frac{(-1)^m}{2^{2m}} \sum_{k=0}^{m} (-1)^k \binom{2m+1}{k} \int_1^\infty \frac{\tan^{-1}\left[(2m-2k+1)\frac{y}{p}\right]}{p} dp. \tag{4.4}$$

Using the integral representation of exponential integral $Ei(-x) = \int_{-\infty}^{x} \frac{e^t}{t} dt$ Ref[**3**, p. 875], gives

$$\int_0^\infty Ei(-x) \sin^{2m+1} yx \frac{dx}{x} = \frac{(-1)^{m+1}}{2^m} \sum_{k=0}^{m} (-1)^k \binom{2m+1}{k} \int_0^1 \frac{\tan^{-1}\left[(2m-2k+1)yu\right]}{u} du.$$

Then using (4.1) for $n = 2$, yields

$$\int_0^\infty Ei(-x) \sin^{2m+1} yx \frac{dx}{x} = \frac{(-1)^{m+1}}{2^m} \sum_{k=0}^{m} (-1)^k \binom{2m+1}{k} \phi_2\left((2m-2k+1)y\right). \tag{4.5}$$

Let $m = 0$ in (4.5). Then,

$$\int_0^\infty Ei(-x) \sin yx \frac{dx}{x} = -\phi_2(y). \tag{4.6}$$

Integrating (4.6) with respect to y and using the definition of sine integral $Si(x) = \int_0^x \frac{\sin u}{u} dx$ [**3**, p. 878] and using (4.2) $\phi_3(1) = \pi^3/32$. Then

$$\int_0^\infty \frac{Ei(-x)}{x} Si(x) dx = -\frac{\pi^3}{32}. \tag{4.7}$$

4.2. Combinations of Sine and cosine integrals with psi function. Entry **6.244.1** from Ref[**3**, p. 634] for $p > 0$ and $q > 0$

$$\int_0^\infty si(px) \frac{x dx}{q^2 + x^2} = \frac{\pi}{2} Ei(-pq). \tag{4.8}$$

Replace q by $a+k$ and taking summation on both sides for $k=0,1,2,\ldots$, gives

$$\int_0^\infty si(px)\sum_{k=0}^\infty \frac{2ixdx}{(a+k)^2+x^2} = i\pi\sum_{k=0}^\infty Ei(-p(a+k)).$$

Using the identity (1.4), gives

$$\int_0^\infty si(px)\left(\psi(a+ix)-\psi(a-ix)\right)dx = i\pi\sum_{k=0}^\infty Ei(-p(a+k)). \tag{4.9}$$

Similarly, using the following entry **6.245.1** from Ref[**3**, p. 634] for $p>0$ and $q>0$

$$\int_0^\infty ci(px)\frac{dx}{q^2+x^2} = \frac{\pi}{2q}Ei(-pq).$$

Gives the following identity

$$\int_0^\infty \frac{ci(px)}{x}\left(\psi(a+ix)-\psi(a-ix)\right)dx = i\pi\sum_{k=0}^\infty \frac{Ei(-p(a+k))}{a+k}. \tag{4.10}$$

Remark 4.2. Let $a=1/2$ and replace p by $p/2$ in (4.9) and (4.10). Then

$$\int_0^\infty si(2px)\tanh\pi x dx = \sum_{k=0}^\infty Ei(-p(2k+1)).$$

$$\int_0^\infty \frac{ci(2px)}{x}\tanh\pi x dx = 2\sum_{k=0}^\infty \frac{Ei(-p(2k+1))}{2k+1}.$$

5. Conclusion

The integrals involving various combinations of powers, exponentials, trigonometric functions with special functions (such as exponential, sine and cosine integral functions) are evaluated through arctangent function, infinite products and infinite series. The integrals given here are not available in the classical tables by Gradshteyn and Ryzhik [**3**]. Also, they cannot be expressed in closed form using a symbolic language.

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