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On the Inverse of the Taylor Operator

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Abstract. We introduce a new expansion of a function inspired by the Taylor series expansion and we examine which functions admit such expansion. Via this expansion the solution of the difference equation $y(x+1) - y(x) = g(x)$ can be written down explicitly. Some examples involving the Gamma function are given.

1. Introduction

Let $f(x)$ be a complex-valued function whose domain contains the semi-infinite interval (a, ∞) of the real line, for some $a \in \mathbb{R}$. Three very classical operators which can apply to such functions f are the derivative operator $D := d/dx$, the *shift* operator

$$(Tf)(x) := f(x+1) \tag{1.1}$$

and the difference operator $\Delta := T - I$ (where I is the identity operator), so that

$$(\Delta f)(x) = f(x+1) - f(x) \tag{1.2}$$

(thus, 1-periodic functions, i.e. functions of period 1 are in the kernel of Δ). Two basic properties of Δ are the product rule

$$\Delta(fg) = f\Delta g + g\Delta f + (\Delta f)(\Delta g) \tag{1.3}$$

and the binomial rule

$$\Delta^k = (T - I)^k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} T^j, \tag{1.4}$$

where, of course,

$$(T^j f)(x) = f(x+j).$$

Obviously, the operators Δ and T commute. Also, Δ and D (and T and D) commute in the sense that, if $f'(x)$ exists for all $x \in (a, \infty)$, then

$$\Delta D f = D \Delta f. \tag{1.5}$$

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If f is analytic in $(x-\rho,\, x+\rho)$, with $\rho > 1$ (even $\rho = 1$ is sometimes sufficient), then application of Taylor's formula gives

$$f(x+1) = \sum_{k=0}^{\infty} \frac{(D^k f)(x)}{k!}. \tag{1.6}$$

Symbolically, (1.6) can be written as

$$T = e^D \qquad \text{or} \qquad I + \Delta = e^D \tag{1.7}$$

and e^D could be called the *Taylor operator*. It is tempting to "solve" the last equation for D and get

$$D = \text{the logarithm of } \; (I + \Delta). \tag{1.8}$$

However, it is not clear what is the meaning of (1.8). For example, the logarithm is not a single-valued function (actually, it is ∞-valued).

Recall that $\ln(1+z)$, where, as usual, $\ln(\cdot)$ denotes the principal branch of the logarithm (i.e. $-\pi < \Im\{\ln(\cdot)\} \leqslant \pi$), can be expanded as

$$\ln(1+z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} z^k, \tag{1.9}$$

where (we will need this fact later) this expansion is valid, if and only if $z \in \overline{U} := \{z \in \mathbb{C} : |z| \leqslant 1\}$. If $z = -1$, then both sides equal $-\infty$, while, for $z \in \partial\overline{U} \setminus \{-1\}$, the validity of (1.9) follows from a well-known theorem of Abel (see, e.g., [**5**], Th. 5.4.4).

The logarithmic expansion (1.9) suggests one precise way to interpret (1.8), leading to the definition:

Definition 1. We say that a function $f : (a, \infty) \to \mathbb{C}$, where $a \in \mathbb{R}$ belongs to the class $\mathcal{K}_a$ if

$$(Df)(x) = (Lf)(x) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (\Delta^k f)(x) \qquad \text{for all } \; x \in (a, \infty). \tag{1.10}$$

We also set $\mathcal{K}_{-\infty} := \bigcap_{a \in \mathbb{R}} \mathcal{K}_a$.

Let us mention that we do not claim that (1.10) is the only possible interpretation of (1.8). However, if f (or, rather Df) can be expanded according to (1.10), then the expansion itself might have computational and/or aesthetical value. In general, functional expansions of mathematical analysis (from Taylor expansions to discrete or continuous eigenfunction expansions, to Mittag-Leffler expansions, just to name a few) often lead to interesting formulas with high applicability, since they provide another, usually very different way of viewing the function in hand.

In the next section we examine the content of the class $\mathcal{K}_a$. We do not give a complete characterization of $\mathcal{K}_a$, but we show that it is a considerably rich class of functions. Then, in Section 3 we indicate how the expansion (1.10) helps to write down the solution of the difference equation $y(x+1) - y(x) = g(x)$. As an illustration we rederive some expansions for the Gamma and Digamma functions.

2. Understanding the Class $\mathcal{K}_a$

For a fixed real a, it is clear that $\mathcal{K}_a$ is a vector space of functions defined on (a, ∞). Furthermore, it follows immediately from Definition 1 that

$$a \leqslant b \Longrightarrow \mathcal{K}_a \subset \mathcal{K}_b \tag{2.1}$$

Also,

$$f(x) \in \mathcal{K}_a \Longleftrightarrow f_c(x) := f(x+c) \in \mathcal{K}_{a-c}. \tag{2.2}$$

Another obvious fact is that the constant functions are in $\mathcal{K}_{-\infty}$, i.e. in $\mathcal{K}_a$ for every $a \in \mathbb{R}$. On the other hand, if $f(x)$ is a nonconstant 1-periodic function, then (1.10) is definitely not true, i.e. $f \notin \mathcal{K}_a$, for all a.

Let

$$\phi_w(x) := e^{wx}, \tag{2.3}$$

where w is a complex number. We can say that these functions are eigenfunctions of the operators Δ and D (and T). For example,

$$\Delta \phi_w = (e^w - 1)\phi_w, \quad \text{hence} \quad \Delta^k \phi_w = (e^w - 1)^k \phi_w, \quad k = 0, 1, 2, \dots. \tag{2.4}$$

Proposition 1. For any $a \in \mathbb{R}$ the function $\phi_w(x) = e^{wx}$ is in $\mathcal{K}_a$, if and only if

$$|e^w - 1| \leqslant 1, \qquad \text{with} \quad -\pi/2 \leqslant \Im\{w\} \leqslant \pi/2. \tag{2.5}$$

Furthermore, condition (2.5) is equivalent to

$$w \in \Omega := \{u + iv \in \mathbb{C} : u \leqslant \ln(2\cos v), \quad -\pi/2 \leqslant v \leqslant \pi/2\}. \tag{2.6}$$

In particular, e^{ux} is in $\mathcal{K}_a$ if and only if $u \leqslant \ln 2$, while e^{ivx} is in $\mathcal{K}_a$ if and only if $-\pi/3 \leqslant v \leqslant \pi/3$.

Proof. Applying the operator L of (1.10) to ϕ_w yields

$$(L\phi_w)(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (\Delta^k \phi_w)(x) = e^{wx} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (e^w - 1)^k. \tag{2.7}$$

The last series converges if and only if

$$w \in \tilde{\Omega} := \{w \in \mathbb{C} : |e^w - 1| \leqslant 1\} \tag{2.8}$$

(notice that $e^w - 1 \neq -1$ for every $w \in \mathbb{C}$, thus we do not need to make any exception). It is an elementary exercise (see the Appendix) to show that

$$\tilde{\Omega} = \bigcup_{n \in \mathbb{Z}} (\Omega + 2ni\pi), \tag{2.9}$$

where Ω is given by (2.6) and $\Omega + 2ni\pi$ denotes the shift of Ω by $2ni\pi$.

To finish the proof we must determine for which w in $\tilde{\Omega}$ (1.10) becomes equality. In view of (2.7) equality (1.10) is valid (when $f = \phi_w$, hence $Df = D\phi_w = we^{wx}$) if and only if

$$w = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (e^w - 1)^k. \tag{2.10}$$

By invoking (1.9) we see that (2.10) is equivalent to

$$w = \ln(e^w), \tag{2.11}$$

where $\ln(\cdot)$ is the principal branch of the logarithm. Hence, (2.11), (2.10) and consequently (1.10) are true, if and only if $-\pi < \Im\{w\} \leqslant \pi$. Since we also have $w \in \tilde{\Omega}$, the proof is completed. □

A direct consequence of the above proposition is that linear combinations of exponentials e^{wx}, with $w \in \Omega$, belong to $\mathcal{K}_{-\infty}$. For example,

$$\sin(vx),\ \cos(vx) \ \text{ are in } \ \mathcal{K}_{-\infty} \qquad \text{for } \ v \in [-\pi/3, \pi/3].$$

In general, functions f which can be expressed in the form

$$f(x) = \int_{\Omega} H(u,v) e^{wx} du dv \qquad (w = u + iv),$$

have good chances to be in $\mathcal{K}_a$, even if $H(u,v)$ is a distribution. In fact, we believe that all elements of $\mathcal{K}_a$ admit such a representation. In this spirit we present the following partial result.

Proposition 2. Let $H(u,v)$ be a Borel measurable complex-valued function defined on Ω and μ a Borel measure on Ω, where Ω is the subset of $\mathbb{C}$ defined in (2.6). If

$$\int_{\Omega} \left[-\ln\left(1 - |e^w - 1|\right) \vee |H(u,v)|(1 + |w|)\right] e^{ux} d\mu < \infty \quad \text{for } \ x > a \tag{2.12}$$

(where $s \vee t := \max\{s,t\}$), then the function

$$f(x) = \int_{\Omega} H(u,v) e^{wx} d\mu, \qquad x > a, \tag{2.13}$$

belongs to the class $\mathcal{K}_a$.

Proof. Condition (2.12) implies that the integrals

$$\int_{\Omega} H(u,v) e^{wx} d\mu \qquad \text{and} \qquad \int_{\Omega} H(u,v) w e^{wx} d\mu$$

converge absolutely for $x > a$. Hence, $f(x)$ of (2.12) is a well-defined function on (a, ∞) and

$$f'(x) = \int_{\Omega} H(u,v) w e^{wx} d\mu, \qquad x > a, \tag{2.14}$$

Now, by (1.10)

$$(Lf)(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \int_{\Omega} H(u,v)(e^w - 1)^k e^{wx} d\mu.$$

Due to (2.12) we can apply the Fubini Theorem in the right-hand side above in order to interchange sum and integral. The result is (recalling (1.9))

$$(Lf)(x) = \int_{\Omega} H(u,v) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (e^w - 1)^k e^{wx} d\mu = \int_{\Omega} H(u,v) w e^{wx} d\mu.$$

Thus, by (2.14) $f'(x) = (Lf)(x)$ for $x > a$. □

By taking μ to be supported on a piecewise smooth curve α of the complex plane $\mathbb{C}$ lying entirely in Ω, the line (or contour) integral

$$f(x) = \int_{\alpha} H(w)e^{wx}dw, \tag{2.15}$$

becomes a special case of the integral in (2.13). Notice that the function $H(w)$ need not be analytic. For example, if $\alpha = (-\infty, 0]$, then (2.15) takes the form of a Laplace transform, i.e.

$$f(x) = \int_0^{\infty} H(t)e^{-tx}dt.$$

Proposition 3. All polynomials are in $\mathcal{K}_{-\infty}$.

Proof. One can use induction on n to show that the monomial x^n is in $\mathcal{K}_{-\infty}$ for all $n \in \mathbb{N}$. Here we prefer to give an alternative proof.

For a fixed positive integer n, let $\mathcal{P}_n$ be the $(n+1)$-dimensional vector space of polynomials over $\mathbb{C}$ of degree $\leqslant n$. Notice that the operators Δ and D leave $\mathcal{P}_n$ invariant. Let us denote by Δ_n and D_n the restrictions of Δ and D on $\mathcal{P}_n$. Since polynomials have Taylor expansions (after all they are analytic everywhere), formula (1.7) is valid for polynomials, therefore

$$I + \Delta_n = e^{D_n}. \tag{2.16}$$

A key issue in our argument is that Δ_n and D_n are *nilpotent*. Recall that an operator A is nilpotent if there is an integer $k \geqslant 1$ such that $A^k = O$. The smallest such k is called the *nilpotency index* of A. It is clear that both Δ_n and D_n have nilpotency index $n+1$. Since the spectrum of a nilpotent operator consists only of 0, it follows that $\ln(I + \Delta_n)$ is well-defined and, actually,

$$\ln(I + \Delta_n) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \Delta_n^k = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \Delta_n^k. \tag{2.17}$$

Also, the functional calculus of operators yields

$$e^{\ln(I+\Delta_n)} = I + \Delta_n. \tag{2.18}$$

Thus, by combining (2.16) and (2.18) we obtain that

$$e^{\ln(I+\Delta_n)} = e^{D_n}. \tag{2.19}$$

Finally, since by (2.17) $\ln(I + \Delta_n)$ is nilpotent, Corollary A1 of the Appendix applied to (2.19) implies that $D_n = \ln(I + \Delta_n)$, which means that polynomials in $\mathcal{P}_n$ satisfy (1.10). Since n is arbitrary, the proof is completed. □

Proposition 4. Let $a < 0$ and suppose that $f(x)$ have the form (2.13), namely

$$f(x) = \int_{\Omega} H(u, v)e^{wx}d\mu, \qquad x > a,$$

where $H(u, v)$ satisfies condition (2.12). Then, for $n \in \mathbb{N}$, any n-th antiderivative $F(x)$ of $f(x)$ (i.e. $(D^n F)(x) = f(x)$) is in the class $\mathcal{K}_a$.

Proof. By Proposition 3 it is enough to show that one n-th antiderivative of $f(x)$ is in $\mathcal{K}_a$, since any two n-th antiderivatives differ by a polynomial of degree $\leqslant n-1$.

Since $a < 0$, (2.12) is valid for $x = 0$, i.e.

$$\int_{\Omega} [-\ln(1 - |e^w - 1|) \vee |H(u,v)|(1 + |w|)]\, d\mu < \infty. \tag{2.20}$$

For $n = 1, 2, ...$, the n-th Taylor polynomial of e^x about $x = 0$ is

$$e_n(x) := \sum_{k=0}^{n-1} \frac{x^k}{k!}.$$

Notice that (for completeness we have set $e_0(x) \equiv 0$)

$$e_n'(x) = e_{n-1}(x) \qquad \text{for } \; n = 1, 2, \dots. \tag{2.21}$$

Now, the quantity

$$\frac{e^{wx} - e_n(wx)}{w^n}$$

is an entire function of w; in particular, it is continuous at $w = 0$. Thus, under (2.12) and (2.20) the function

$$F(x) := \int_{\Omega} H(u,v) \frac{e^{wx} - e_n(wx)}{w^n}\, d\mu, \tag{2.22}$$

is well-defined and, furthermore,

$$(D^n F)(x) = f(x).$$

We need to show that $F(x)$ belongs to the class $\mathcal{K}_a$. If we apply the operator L of (1.10) to F, we obtain

$$(LF)(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \int_{\Omega} H(u,v) \Delta^k \left[\frac{e^{wx} - e_n(wx)}{w^n}\right] d\mu,$$

Due to (2.12) and (2.20) we can apply the Fubini Theorem in the right-hand side above in order to interchange sum and integral and get

(2.23)

$$(LF)(x) = \int_{\Omega} H(u,v) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \Delta^k \left[\frac{e^{wx} - e_n(wx)}{w^n}\right] d\mu = \int_{\Omega} H(u,v) L \left[\frac{e^{wx} - e_n(wx)}{w^n}\right] d\mu.$$

However, by Propositions 1 and 3 we know that e^{wx} and $e_n(wx)$ are in $\mathcal{K}_a$. This means that (recall (2.21))

$$L[e^{wx}] = \frac{d}{dx}[e^{wx}] \qquad \text{and} \qquad L[e_n(wx)] = \frac{d}{dx}[e_n(wx)].$$

Substituting in (2.23) yields

$$(LF)(x) = \int_{\Omega} H(u,v) \frac{d}{dx} \left[\frac{e^{wx} - e_n(wx)}{w^n}\right] d\mu. \tag{2.24}$$

Since (recalling (2.21))

$$\frac{d}{dx} \left[\frac{e^{wx} - e_n(wx)}{w^n}\right] = \frac{e^{wx} - e_{n-1}(wx)}{w^{n-1}},$$

it follows that the derivative can be taken out of the integral in the right-hand side of (2.24). Therefore, $(LF)(x) = (DF)(x)$. □

As an application let us take $f(x) = x^{-\beta}$, with $\Re\{\beta\} > 0$. Since

$$x^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-xt} dt,$$

Proposition 2 implies that $f(x)$ is in $\mathcal{K}_0$ (notice that f is not in $\mathcal{K}_a$, when $a < 0$). Thus, if $\epsilon > 0$, then, by (2.2) we have that $f_\epsilon(x) = f(x+\epsilon)$ is in $\mathcal{K}_{-\epsilon}$ and

$$(x+\epsilon)^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-\epsilon t} e^{-xt} dt$$

Hence, Proposition 4 applied to $f_\epsilon(x)$ implies that all its antiderivatives are in $\mathcal{K}_{-\epsilon}$. It follows that the functions $(x+\epsilon)^\beta$ are in $\mathcal{K}_{-\epsilon}$ for all $\beta \in \mathbb{C}$ and, also, the same is true for $\ln(x+\epsilon)$ and its antiderivatives. Finally, again by (2.2), we can conclude that

$$x^\beta, \quad \beta \in \mathbb{C}, \qquad \text{and also} \qquad \ln x \quad \text{and its antiderivatives} \quad \text{are in } \mathcal{K}_0. \tag{2.25}$$

3. The Difference Equation $\Delta y = g$

The difference equation

$$(\Delta y)(x) = y(x+1) - y(x) = g(x), \qquad g(x) \text{ given}, \tag{3.1}$$

was first studied by Krull, in his pioneer work [**7**] and subsequently by other researchers (see, e.g., [**8**], [**3**], [**9**], and [**6**]).

One naive idea for solving (3.1) is to try to express g as $g(x) = (\Delta g_\star)(x)$, so that (3.1) becomes

$$(\Delta y)(x) = (\Delta g_\star)(x), \tag{3.2}$$

and then "cancel" the Δ's to obtain

$$y(x) = g_\star(x) + p(x), \tag{3.3}$$

where $p(x)$ is an arbitrary 1-periodic function. It is remarkable that formula (1.10) helps to implement this idea. First we choose an antiderivative $G(x)$ of $g(x)$. Suppose $G \in \mathcal{K}_a$, for some a. Then, by (1.10)

$$g(x) = (DG)(x) = \Delta \left[\sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} (\Delta^{k-1} G)(x) \right],$$

and, hence, we have an explicit expression for $g_\star(x)$, namely

$$g_\star(x) = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} (\Delta^{k-1} G)(x), \qquad x > a. \tag{3.4}$$

3.1. Example. The Gamma Function. The Gamma function satisfies various functional equations, but the most well-known is, probably, the equation

$$\Gamma(x+1) = x\Gamma(x).$$

If we take logarithms of both sides, then the above equation becomes

$$\ln\Gamma(x+1) - \ln\Gamma(x) = \ln x, \tag{3.5}$$

hence, $\ln\Gamma(x)$ satisfies the difference equation

$$(\Delta y)(x) = \ln x. \tag{3.6}$$

In view of (2.25), the antiderivative $G(x) = (x\ln x - x)$ of $\ln x$ is in $\mathcal{K}_0$. Thus, by setting $G(x) = x\ln x - x$ in (3.4) we get that (recall (3.3))

$$\ln\Gamma(x) = p(x) + (x\ln x - x) - \frac{1}{2}\Delta[x\ln x - x] + \sum_{k=2}^{\infty}\frac{(-1)^k}{k+1}\,\Delta^k[x\ln x - x], \tag{3.7}$$

where $p(x)$ is some 1-periodic function. Notice that

$$\Delta[x\ln x - x] = \Delta[x\ln x] - 1 = (x+1)\ln(x+1) - x\ln x - 1 = \ln x + o(1), \quad x\to\infty \tag{3.8}$$

and, also, that

$$\Delta^k[x\ln x - x] = \Delta^k[x\ln x] \qquad \text{for} \;\; k \geqslant 2. \tag{3.9}$$

Claim. As $x\to\infty$

$$\sum_{k=2}^{\infty}\frac{(-1)^k}{k+1}\,\Delta^k[x\ln x] = o(1). \tag{3.10}$$

Proof. For $x > 0$ we have

$$\ln x = \int_1^x \frac{d\xi}{\xi} = \int_1^x\int_0^\infty e^{-\xi t}dtd\xi = \int_0^\infty\int_1^x e^{-\xi t}d\xi dt = \int_0^\infty \frac{e^{-t}-e^{-xt}}{t}\,dt.$$

Thus,

$$x\ln x - x + 1 = \int_1^x\int_0^\infty \frac{e^{-t}-e^{-\xi t}}{t}\,dtd\xi = \int_0^\infty\int_1^x \frac{e^{-xt}+xte^{-t}-(1+t)e^{-t}}{t^2}\,dt, \tag{3.11}$$

where the interchange of the integrals is justified by Tonelli's Theorem (considering separately the cases $x < 1$ and $x > 1$). From the above it follows that

$$\Delta^k[x\ln x] = \int_0^\infty \Delta^k\left[\frac{e^{-xt}+xte^{-t}-(1+t)e^{-t}}{t^2}\right]dt = (-1)^k\int_0^\infty\frac{(1-e^{-t})^k}{t^2}\,e^{-xt}dt \tag{3.12}$$

for $k = 2, 3, \ldots$, and hence

$$\sum_{k=2}^{\infty}\frac{(-1)^k}{k+1}\,\Delta^k[x\ln x] = \sum_{k=2}^{\infty}\frac{1}{k+1}\int_0^\infty\frac{(1-e^{-t})^k}{t^2}\,e^{-xt}dt. \tag{3.13}$$

Tonelli's Theorem, again, allows us to pass the sum inside the integral, in the the right-hand side of (3.13). After that we can compute the series in closed form since (e.g., by (1.9)),

$$\sum_{k=2}^{\infty} \frac{z^k}{k+1} = -\frac{\ln(1-z)}{z} - 1 - \frac{z}{2}, \qquad |z| \leqslant 1.$$

Therefore, (3.13) becomes

$$\sum_{k=2}^{\infty} \frac{(-1)^k}{k+1}\, \Delta^k[x \ln x] = \int_0^{\infty} \frac{1}{t^2} \left(\frac{t}{1-e^{-t}} + \frac{e^{-t}-3}{2} \right) e^{-xt} dt, \tag{3.14}$$

where the integrand in the right-hand side is regular at $t = 0$. Thus, (3.10) follows from the above equation by dominated convergence. □

By using (3.8) and (3.10) in (3.7) we obtain

$$\ln \Gamma(x) = p(x) + x \ln x - x - \frac{\ln x}{2} + o(1) \qquad \text{as } \ x \to \infty.$$

On the other hand, by Stirling's formula (see, e.g., [**1**] or [**5**])

$$\ln \Gamma(x) = \frac{1}{2} \ln(2\pi) + x \ln x - x - \frac{\ln x}{2} + o(1) \qquad \text{as } \ x \to \infty.$$

Therefore, $p(x) \equiv (1/2)\ln(2\pi)$ and, in view of (3.8), (3.7) becomes

$$\ln \Gamma(x) = \frac{\ln(2\pi)}{2} + x \ln x - x + \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k+1}\, \Delta^k[x \ln x] \tag{3.15}$$

(a side remark here is that by using (3.14) in (3.15) we obtain an integral representation of $\ln \Gamma(x)$; this representation is not new). Now, the binomial rule (1.4) gives

$$\Delta^k[x \ln x] = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} T^j[x \ln x] = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (x+j) \ln(x+j). \tag{3.16}$$

Finally, by using (3.16) in (3.15) we get that, for $x > 0$,

$$\ln \Gamma(x) = \frac{1 + \ln(2\pi)}{2} - x + \sum_{k=0}^{\infty} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x+j) \ln(x+j). \tag{3.17}$$

This formula is essentially not new, as we will explain in a comment below. Exponentiating both sides of (3.17) we obtain the equivalent formula,

$$\Gamma(x) = \sqrt{2e\pi}\, e^{-x} \prod_{k=0}^{\infty} \left[\prod_{j=0}^{k} (x+j)^{(x+j)\binom{k}{j}(-1)^j} \right]^{\frac{1}{k+1}}, \tag{3.18}$$

i.e.

$$\Gamma(x) = \sqrt{2e\pi}\, e^{-x} x^x \left[\frac{x^x}{(x+1)^{x+1}} \right]^{1/2} \left[\frac{x^x (x+2)^{x+2}}{(x+1)^{2(x+1)}} \right]^{1/3} \left[\frac{x^x (x+2)^{3(x+2)}}{(x+1)^{3(x+1)} (x+3)^{x+3}} \right]^{1/4} \cdots .$$

For instance, if we set $x = 1$ in (3.18) and square both sides we get

$$\frac{2\pi}{e} = \prod_{k=0}^{\infty}\left[\prod_{j=0}^{k}(j+1)^{(j+1)\binom{k}{j}(-1)^{j-1}}\right]^{\frac{2}{k+1}}, \tag{3.19}$$

i.e.

$$\pi = \frac{e}{2}\cdot\left(\frac{2^2}{1}\right)^{2/2}\cdot\left(\frac{2^4}{1\cdot 3^3}\right)^{2/3}\cdot\left(\frac{2^6\cdot 4^4}{1\cdot 3^9}\right)^{2/4}\cdot\left(\frac{2^8\cdot 4^{16}}{3^{18}\cdot 5^5}\right)^{2/5}\cdot\left(\frac{2^{10}\cdot 4^{40}6^6}{3^{30}\cdot 5^{25}}\right)^{2/6}\cdots.$$

As far as we know, this formula was first obtained by J. Guillera and J. Sondow [**4**]. Substituting in (3.18) other values of x (such as $1/2$ and $3/2$) for which the value of $\Gamma(x)$ is known, yields similar "strange" expressions.

3.2. Some Remarks on the Digamma Function. Recall that the Digamma (or Psi) function is defined as

$$\psi(x) := \frac{d}{dx}\{\ln\Gamma(x)\} = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{3.20}$$

Thus, (3.5) implies that $\psi(x)$ satisfies the difference equation

$$(\Delta u)(x) = \frac{1}{x}. \tag{3.21}$$

We can obtain a formula for $\psi(x)$, by using again (3.3) and (3.4) in order to write down the solution of (3.21). In fact, it is a straightforward imitation of what we did above to obtain (3.15) as a solution of (3.6). The calculations are a bit simpler and eventually lead to

$$\psi(x) = \sum_{k=0}^{\infty}\frac{1}{k+1}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\ln(x+j), \qquad x > 0. \tag{3.22}$$

This formula had been first derived in 2006 by J. Guillera and J. Sondow [**4**] with the help of the Lerch Transcendent. Exponentiating both sides of (3.22) we obtain the equivalent formula,

$$e^{\psi(x)} = \prod_{k=0}^{\infty}\left[\prod_{j=0}^{k}(x+j)^{\binom{k}{j}(-1)^j}\right]^{\frac{1}{k+1}}. \tag{3.23}$$

For example, if we set $x = 1$ in (3.22) and recall that $\psi(1) = -\gamma$ (see, e.g., [**2**]), where γ is Euler's constant, then we obtain an expression for e^{γ} which was first discovered by J. Ser [**10**] in 1926 and later was rediscovered by J. Sondow.

Comment. By (3.12) the (simple) series $\sum_{k=0}^{\infty}$ in (3.17) (see also (3.15)) converges absolutely and uniformly in x on $[\delta, \infty)$, for any $\delta > 0$ (however, if we switch the order of the two summations $\sum_{k=0}^{\infty}$ and $\sum_{j=0}^{k}$, the resulting double series is divergent; the same is true for the series $\sum_{k=0}^{\infty}$ in (3.22)). It follows that one can obtain (3.17) by integrating (3.22), pass the integral inside the series, and use (3.19) to determine the constant of integration.

4. Appendix

Proof of (2.9). The boundary of the set $\tilde{\Omega}$ defined in (2.8) is

$$\partial\tilde{\Omega} = \{w \in \mathbb{C} : e^w - 1 = e^{i\theta}, \quad -\pi < \theta \leqslant \pi\}.$$

Consequently, $\partial\tilde{\Omega}$ can be described via the (implicit) parametric equations

$$e^u \cos v = 1 + \cos\theta, \qquad e^u \sin v = \sin\theta, \qquad -\pi < \theta \leqslant \pi \tag{4.1}$$

(θ is the parameter). Dividing the second equation above by the first and observing that the first equation forces $\cos v \geqslant 0$ one gets

$$v = \theta/2 + 2n\pi, \qquad n \in \mathbb{Z}.$$

Using this in, say, the second parametric equation of (4.1) yields

$$u = \ln(2\cos v), \qquad v \in (2n\pi - \pi/2 < v \leqslant 2n\pi + \pi/2), \quad n \in \mathbb{Z},$$

which is the Cartesian description of $\partial\tilde{\Omega}$. From that (2.9) follows □

Next, a property of nilpotent operators on finite-dimensional spaces.

Lemma A1. Let V be a finite-dimensional vector space over a field $\mathbb{F}$ of scalars with $\dim V < \infty$ and A, B two (linear) nilpotent operators on V. If there exists a polynomial $p(x) \in \mathbb{F}[x]$, with $p(0) \neq 0$ such that

$$A\,p(A) = B\,p(B), \tag{4.2}$$

then $A = B$.

Proof. Let us assume, without loss of generality, that $p(0) = 1$. Observe that $p(A)$ (and hence $P(B)$) is invertible, since $p(A) = I + A_1$, where A_1 is nilpotent.

For any integer $k \geqslant 1$ (4.2) implies that

$$A^k p(A)^k = B^k p(B)^k.$$

Thus, due to the invertibility of $p(A)$ and $P(B)$ we must have that A and B have the same nilpotency index, say $n \geqslant 1$.

If $n = 1$, then $A = B = O$. Suppose $n \geqslant 2$. Then, from (4.2) we get

$$A^{n-1}p(A)^{n-1} = B^{n-1}p(B)^{n-1}. \tag{4.3}$$

Since $A^n = B^n = O$ and $p(x)^{n-1} = 1 + (n-1)c_1 x +$ (higher powers of x), where c_1 is the coefficient of x in $p(x)$, (4.3) implies that

$$A^{n-1} = B^{n-1}. \tag{4.4}$$

If $n = 2$ we are done. Assuming $n \geqslant 3$ and using (4.2) again we get

$$A^{n-2}p(A)^{n-2} = B^{n-2}p(B)^{n-2}$$

and hence, by (4.4) and the fact that $p(x)^{n-2} = 1 + (n-2)c_1 x +$ (higher powers of x) we obtain

$$A^{n-2} = B^{n-2}.$$

If $n \geqslant 4$ we use (4.2) again to get $A^{n-3} = B^{n-3}$. Hence, whenever we have $A^{n-k} = B^{n-k}$, for some $k \leqslant n-2$, by using the above procedure we also have $A^{n-(k+1)} = B^{n-(k+1)}$. Therefore $A = B$. □

Corollary A1. Let V be a vector space over $\mathbb{C}$ (or $\mathbb{R}$) with $\dim V < \infty$. If A and B are two nilpotent operators on V such that

$$e^A = e^B, \tag{4.5}$$

then $A = B$.

Proof. Let $n := \max\{n_A, n_B\}$, where n_A and n_B are the nilpotency indices of A and B respectively. Then (4.6) can be written as

$$I + \sum_{k=1}^{n} \frac{A^k}{k!} = I + \sum_{k=1}^{n} \frac{B^k}{k!}. \tag{4.6}$$

In other words we have

$$A\,p(A) = B\,p(B),$$

where $p(x) := \sum_{k=1}^{n} x^{k-1}/k!$. □

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