SCIENTIA
Series A: *Mathematical Sciences*, Vol. 24 (2013), 33–54
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

An Extension of an Additive Selection Theorem of Z. Gajda and R. Ger to Vector Relator Spaces

Árpád Száz

Dedicated to the memory of Professor Zbigniew Gajda

Abstract. We prove an extension of an additive selection theorem of Z. Gajda and R. Ger to closed-valued, 2–sublinear relations of commutative semigroups to sequentially complete vector relator spaces. Thus, we obtain a further generalization of the classical Hyers–Ulam stability theorem.

1. Introduction

Hyers [**13**] in 1941, giving a partial answer to a general problem formulated by S. M. Ulam, proved a slightly weaker Banach space particular case of the following stability theorem.

Theorem 1.1. *If f is an ε–approximately additive function of a commutative semigroup U to a Banach space X, for some $\varepsilon > 0$, in the sense that*

$$\| f(u+v) - f(u) - f(v) \| \leqslant \varepsilon$$

for all $u, v \in U$, then there exists an additive function g of U to X such that g is ε–near to f in the sense that

$$\| f(u) - g(u) \| \leqslant \varepsilon$$

for all $u \in U$.

Remark 1.1. Hence, by using the $\mathbb{N}$–homogeneity of g, one can infer that

$$g(u) = \lim_{n \to \infty} n^{-1} f(n u)$$

for all $u \in U$. Therefore, the unicity of the additive function g is also true.

2000 *Mathematics Subject Classification.* 23E25, 54C65, 54E15, 46A19, 39B82.

Key words and phrases. Subadditive relations, vector relators, Hyers - Ulam stability.

The work of the author has been supported by the Hungarian Scientific Research Fund (OTKA) Grant NK-81402.

To define g, Hyers originally used the subsequence $\big(2^{-n} f\big(2^n u\big)\big)_{n=1}^{\infty}$ since its convergence can be more easily verified. Moreover, it can also be well used when f is assumed to be only ε–approximately 2–homogeneous in the sense that $\| f(2u) - 2f(u) \| \leqslant \varepsilon$ for all $u \in U$. In this case, it can also be shown that the Hyers sequence is uniformly convergent [42].

By Ger [**11**, p. 4], M. Laczkovich observed that a strict inequality form of the $U = \mathbb{N}$, $X = \mathbb{R}$ and $\varepsilon = 1$ particular case of the above theorem was already proved by Pólya and Szegő [**24**, pp. 17, 171] in 1925. Moreover, this particular case is actually equivalent to the $X = \mathbb{R}$ particular case of the original theorem.

Hyers's stability theorem has later been generalized by several authors by replacing ε by more general quantities and weakening the commutativity property of U. These investigations have led to an enormous theory of the stability of additivity and homogeneity properties.

The interested reader can get a rapid overview on the subject by consulting the surveys of Hyers and Rassias [**15**], Ger [**11**], Forti [**6**], Székelyhidi [**45**], and Sánchez and Castillo [**30**], and the books of Hyers, Isac and Rassias [**14**], Jung [**17**] and Czerwik [**5**].

However, it is now more important to note that Hyers's theorem was also transformed into set-valued settings by W. Smajdor [**32**] and Gajda and Ger [**8**], in 1986 and 1987, respectively, by making use the following observations.

If f and g are as in Theorem 1.1 and $A = \big\{ u \in U : \ \| u \| \leqslant \varepsilon \big\}$, then

$$g(u) - f(u) \in A \qquad \text{and} \qquad f(u+v) - f(u) - f(v) \in A,$$

and hence

$$g(u) \in f(u) + A \qquad \text{and} \qquad f(u+v) \in f(u) + f(y) + A$$

for all $u, v \in U$.

Therefore, by defining

$$F(u) = f(u) + A$$

for all $u \in U$, we can get a set-valued function F of U to X such that g is a selection of F and F is subadditive. That is,

$$g(u) \in F(u) \qquad \text{and} \qquad F(u+v) \subset F(u) + F(v)$$

for all $u, v \in U$. Thus, the essence of Hyers's theorem nothing but the statement of the existence of an additive selection of a certain subadditive set-valued function.

A similar observation, in connection with the Hahn–Banach extension theorems, was already announced by Rodríguez-Salinas and Bou [**28**] in 1974 and Gajda, A. Smajdor and W. Smajdor [**9**] in 1992. (See also [**16**], [**34**] and [**38**].)

Moreover, the existence of additive and linear selection was formerly also investigated by Á. Száz and G. Száz [**44**], Godini [**12**], W. Smajdor [**33**], Nikodem [**21**], A. Smajdor [**31**], Lee and Nashed [**19**], Gajda [**7**], Sablik [**29**], and Á. Száz [**40**].

In particular, Gajda and Ger [**8**] in 1987 proved the following generalization of Theorem 1.1. (See also Gajda [**7**, Theorem 4.2] for a further generalization.)

Theorem 1.2. *If F is a subadditive set-valued function of a commutative semigroup U to a Banach space X such that the values of F are nonempty, closed and convex, and moreover*

$$\sup \left\{ \operatorname{diam}\left(F(u) \right) : \quad u \in U \right\} < +\infty ,$$

then F has an additive selection function f.

Remark 1.2. Hence, by using the $\mathbb{N}$–homogeneity of f and the above boundedness condition on F, one can infer that

$$\{ f(u) \} = \bigcap_{n=1}^{\infty} n^{-1} F(n u)$$

for all $u \in U$. Therefore, the unicity of the additive selection f is also true.

At the same time, Gajda and Ger [**8**] also proved a less attractive extension of this theorem to a separated, sequentially complete topological vector space X. (See also Gajda [**7**, Theorem 4.3] for a further generalization.)

The importance of the observations of W. Smajdor, Gajda and Ger was soon recognized by Hyers and Rassias [**15**], [**26**], Hyers, Isac and Rassias [**14**, pp. 204–231], and Czerwik [**5**, pp. 301–329]. Moreover, the results of Gajda and Ger [**8**] have been generalized and improved by Popa [**25**], Badora, Ger and Páles [**1**], and several other authors.

In the present paper, by using relations and relators instead of multifunction and topologies, we shall proved the following more convenient

Theorem 1.3. *If F is a closed-valued, 2–sublinear relation of a commutative semigroup U to a separated, sequentially complete vector relator space $X(\mathcal{R})$ such that the sequence $\left(2^{-n} F\left(2^{n} x \right) \right)_{n=0}^{\infty}$ is infinitesimal for all $u \in U$, then F has an additive selection f.*

Remark 1.3. Hence, by using the $\mathbb{N}$–homogeneity of f, the above infinitesimality condition on F and the separatedness of X, we can again infer that

$$\{ f(u) \} = \bigcap_{n=1}^{\infty} n^{-1} F(n u)$$

for all $u \in U$. Therefore, the unicity of the additive selection f is also true.

In the above theorem, $\mathcal{R}$ is a nonvoid family of relations on the vector space X which is, to some extent, compatible with the linear operations in X. And the infinitesimality of a sequence $\left(A_n \right)_{n=1}^{\infty}$ of subsets of $X(\mathcal{R})$ means only that for each $R \in \mathcal{R}$ there exist $x \in X$ and $n \in \mathbb{N}$ such that $A_n \subset R(x)$.

The necessary prerequisites concerning relations and relators, which are certainly unfamiliar to the reader, will be briefly laid out in the next preparatory sections. Moreover, for the reader's convenience, we shall also recall some basic facts on additively written groupoids.

2. A few basic facts on relations

A subset F of a product set $U\times X$ is called a relation on U to X. If in particular $F\subset U^2$, then we may simply say that F is a relation on U. Thus, $\Delta_U=\{(u,u):\ u\in U\}$ is a relation on U.

If F is a relation on U to X, then for any $u\in U$ and $V\subset U$ the sets $F(u)=\{x\in X:\ (u,x)\in F\}$ and $F[V]=\bigcup_{v\in V}F(v)$ are called the images of u and V under F, respectively.

Moreover, the sets $D_F=\{u\in U:\ F(u)\neq\emptyset\}$ and $R_F=F[U]=F[D_F]$ are called the domain and range of F, respectively. If in particular $D_F=U$ $(R_F=X)$, then we say that F is a relation of U to X (on U onto X).

If F is a relation on U to X, then $F=\bigcup_{u\in U}\{u\}\times F(u)=\bigcup_{u\in D_F}\{u\}\times F(u)$. Therefore, a relation F on U to X can be naturally defined by specifying $F(u)$ for all $u\in U$, or by specifying D_F and $F(u)$ for all $u\in D_F$.

For instance, if F is a relation on U to X, then the inverse relation F^{-1} of F can be naturally defined such that $F^{-1}(x)=\{u\in U:\ x\in F(u)\}$ for all $x\in X$. Thus, we also have $F^{-1}=\{(x,u):\ (u,x)\in F\}$.

Moreover, if in addition G is a relation on X to Y, then the composition relation $G\circ F$ of G and F can be naturally defined such that $(G\circ F)(u)=G[F(u)]$ for all $u\in U$. Thus, we also have $(G\circ F)[V]=G[F[V]]$ for all $V\subset U$.

A relation R on U is called reflexive, symmetric, and transitive if $\Delta_U\subset R$, $R^{-1}\subset R$, and $R\circ R\subset R$, respectively. Moreover, a reflexive relation is called a tolerance (preorder) relation if it is symmetric (transitive).

In particular, a relation f on U to X is called a function if for each $u\in D_f$ there exists an $x\in X$ such that $f(u)=\{x\}$. In this case, by identifying singletons with their elements, we may write $f(u)=x$ in place of $f(u)=\{x\}$.

If F is a relation on U to X, then a function f of D_F to X is called a selection of F if $f\subset F$, i.e., $f(u)\in F(u)$ for all $u\in D_F$. Thus, the axiom of choice can be briefly expressed by saying that every relation has a selection.

In particular, a function a of the set $\mathbb{N}$ of all natural numbers to X is called a sequence in X. In this case, we usually write a_n, $(a_n)_{n=1}^{\infty}$, and $\{a_n\}_{n=1}^{\infty}$ in place of $a(n)$, a, and R_a, respectively.

If $(a_n)_{n=1}^{\infty}$ is a sequence in the set $\overline{\mathbb{R}}$ of all extended real numbers, then the extended real numbers $\underline{\lim}_{n\to\infty} a_n=\sup_{n\in\mathbb{N}}\inf_{k\geqslant n}a_k$ and $\overline{\lim}_{n\to\infty} a_n=\inf_{n\in\mathbb{N}}\sup_{k\geqslant n}a_k$ are called the lower and upper limits of the sequence $(a_n)_{n=1}^{\infty}$, respectively.

Quite similarly, if $(A_n)_{n=1}^{\infty}$ is a sequence in the family $\mathcal{P}(X)$ of all subsets of X, then the sets $\underline{\lim}_{n\to\infty} A_n=\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}A_k$ and $\overline{\lim}_{n\to\infty} A_n=\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}A_k$ are called the lower and upper limits of the sequence $(A_n)_{n=1}^{\infty}$, respectively.

In particular, a function d of X^2 to $[0,+\infty]$ is called a distance function on X. The distance function d may be called a quasi-pseudo-metric if $d(x,x)=0$, $d(x,y)<+\infty$, and $d(x,z)\leqslant d(x,y)+d(y,z)$ for all $x,y,z\in X$.

Moreover, a function p of a vector space X over the number field $\mathbb{K}=\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ to $[0, +\infty[$ is called a preseminorm on X if $\lim_{\lambda\to 0} p(\lambda x)=0$, $p(\lambda x)\leqslant p(x)$ and $p(x+y)\leqslant p(x)+p(y)$ for all $x, y\in X$ and $\lambda\in\mathbb{K}$ with $|\lambda|\leqslant 1$.

3. A few basic facts on groupoids

Definition 3.1. If U is a nonvoid set, then a function $+$ of U^2 to U is called an operation in U. And the ordered pair $U(+)=(U, +)$ is called a groupoid.

Remark 3.1. In this case, we may simply write $u+v$ in place of $+(u, v)$ for all $u, v\in U$. Moreover, we may also simply write U in place of $U(+)$.

Definition 3.2. If U is a groupoid, then for any $n\in\mathbb{N}$ and $u\in U$ we define

$$n u=u \quad \text{if} \quad n=1 \qquad \text{and} \qquad n u=(n-1)u \quad \text{if} \quad n\neq 1.$$

By induction, we can easily prove the following two theorems.

Theorem 3.1. *If U is a semigroup, then for any $u\in U$ and $n, m\in\mathbb{N}$ we have*

(1) $(n+m)u=nu+mu$, *(2)* $(nm)u=n(mu)$.

Theorem 3.2. *If U is a semigroup, then for any $n, m\in\mathbb{N}$ and $u, v\in U$, with $u+v=v+u$, we have*

(1) $nu+mv=mv+nu$, *(2)* $n(u+v)=nu+nv$.

Definition 3.3. If U is a groupoid, then for any $A, B\subset U$ and $n\in\mathbb{N}$ we define

$$A+B=\{u+v: \quad u\in A, \quad v\in B\} \qquad \text{and} \qquad nA=\{nu: \quad u\in A\}.$$

Remark 3.2. If in particular U is a group, then we may also naturally define $-A=\{-u: u\in A\}$ and $A-B=A+(-B)$ for all $A, B\subset U$.

Moreover, if in particular X is a vector space over $\mathbb{K}$, then we may also naturally define $\lambda A=\{\lambda x: \ x\in A\}$ for all $\lambda\in\mathbb{K}$ and $A\subset X$.

Note that thus only two axioms of vector spaces may fail to hold for the family $\mathcal{P}(X)$. Namely, only the one-point subsets of X can have additive inverses. Moreover, in general we only have $(\lambda+\mu)A\subset\lambda A+\mu A$.

By using the above notations, we can briefly formulate the following

Definition 3.4. A subset A of a vector space X over $\mathbb{K}$ is called

(1) absorbing if $X=\bigcup_{n=1}^{\infty} nA$;

(2) balanced if $\lambda A\subset A$ for all $\lambda\in\mathbb{K}$ with $|\lambda|\leqslant 1$;

(3) convex if $\lambda A+(1-\lambda)A\subset A$ for all $\lambda\in\mathbb{K}$ with $0<\lambda<1$.

Concerning balanced sets, we can easily establish the following

Theorem 3.3. *If A is a balanced subset of a vector space X over $\mathbb{K}$, then for any $\lambda, \mu\in\mathbb{K}$, with $|\lambda|\leqslant|\mu|$, we have $\lambda A\subset\mu A$.*

Hence, it is clear that in particular we also have

COROLLARY 3.1. *If A is a balanced subset of a vector space X over $\mathbb{K}$, then for any $\lambda \in \mathbb{K}$, we have $\lambda A = |\lambda| A$.*

4. A few basic facts on 2–subhomogeneous relations

DEFINITION 4.1. A relation F on one groupoid U to another V is called n–subhomogeneous, for some $n \in \mathbb{N}$, if for any $u \in U$ we have

$$F(n u) \subset n F(u).$$

REMARK 4.1. If the corresponding equality (converse inclusion) holds, then F is called n–homogeneous (n–superhomogeneous).

However, in the sequel we shall actually be interested only in 2–subhomogeneous relations on commutative semigroups to vector spaces.

THEOREM 4.1. *If F is a relation on a groupoid U to a vector space X over $\mathbb{K}$ such that*

$$F(2u) \subset F(u) + F(u) \qquad \text{and} \qquad 2^{-1} F(u) + 2^{-1} F(u) \subset F(u)$$

for all $u \in U$, then F is already 2–subhomogeneous.

PROOF. For any $u \in U$, we have

$$F(2u) \subset F(u) + F(u) = 2\left(2^{-1} F(u) + 2^{-1} F(u)\right) \subset 2 F(u).$$

□

According to Gajda and Ger [**8**], we may naturally introduce the following

DEFINITION 4.2. If F is a relation on a groupoid U to a vector space X over $\mathbb{K}$, then for any $u \in U$ and $n \in \mathbb{N}_0$, with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, we define

$$F_n(u) = 2^{-n} F\left(2^n u\right).$$

Concerning the Hyers sequence $(F_n)_{n=1}^{\infty}$, we can easily prove the following two theorems.

THEOREM 4.2. *If F is a relation on a semigroup U to a vector space X over $\mathbb{K}$, then for any $u \in U$ and $n \in \mathbb{N}_0$ we have*

$$F_n(2u) = 2 F_{n+1}(u).$$

PROOF. If $u \in U$ and $n \in \mathbb{N}_0$, then by Theorem 3.1 we have

$$F_n(2u) = 2^{-n} F\left(2^n (2u)\right) = 2\, 2^{-(n+1)} F\left(2^{n+1} u\right) = 2\, F_{n+1}(u).$$

□

THEOREM 4.3. *If F is a 2–subhomogeneous relation on a semigroup U to a vector space X over $\mathbb{K}$, then $\left(F_n\right)_{n=1}^{\infty}$ is a decreasing sequence of subsets of F.*

PROOF. If $u \in U$ and $n \in \mathbb{N}$, then by Theorem 3.1 we have

$$F_n(u) = 2^{-n} F\left(2^n u\right) = 2^{-n} F\Big(2\left(2^{n-1} u\right)\Big) \subset$$
$$\subset 2^{-n} 2 F\left(2^{n-1} u\right) = 2^{-(n-1)} F\left(2^{n-1} u\right) = F_{n-1}(u).$$

Therefore, the sequence $\big(F_n(u)\big)_{n=1}^{\infty}$ is decreasing. Moreover, by induction, it is clear that $F_n(u) \subset F_0(u) = F(u)$ also holds. □

REMARK 4.2. If in particular, F is a 2–homogeneous relation on a semigroup U to a vector space X over $\mathbb{K}$, then we can quite similarly see that $F_n = F$ for all $n \in \mathbb{N}$.

According to Gajda and Ger [**8**], we may also naturally introduce the following

DEFINITION 4.3. If F is a relation on a groupoid U to a vector space X over $\mathbb{K}$, then we define

$$F^\star = \bigcap_{n=1}^{\infty} F_n .$$

Concerning the relation $F^\star$, we can easily prove the following two theorems.

THEOREM 4.4. *If F is a 2–subhomogeneous relation on a semigroup U to a vector space X over $\mathbb{K}$, then $F^\star$ is already 2–homogeneous.*

PROOF. If $u \in U$, then by Theorems 4.2 and 4.3 we have

$$F^\star(2u) = \bigcap_{n=1}^{\infty} F_n(2u) = \bigcap_{n=1}^{\infty} 2F_{n+1}(u) =$$
$$= 2\bigcap_{n=1}^{\infty} F_{n+1}(u) = 2\bigcap_{n=2}^{\infty} F_n(u) = 2\bigcap_{n=1}^{\infty} F_n(u) = 2F^\star(u).$$

Namely, the mapping $x \mapsto 2x$, being an injection of X, preserves intersections. Moreover, in particular we have $F_2(u) \subset F_1(u)$, and thus $F_1(u) \cap F_2(u) = F_2(u)$. □

THEOREM 4.5. *If F is a relation of a semigroup U to a vector space X over $\mathbb{K}$ and f is a 2–homogeneous selection of F, then f is also a selection of $F^\star$.*

PROOF. If $u \in U$, then by Remark 4.2 and the selection property of f we have

$$f(u) = f_n(u) = 2^{-n} f\left(2^n u\right) \in 2^{-n} F\left(2^n u\right) = F_n(u)$$

for all $n \in \mathbb{N}$. Therefore, we also have $f(u) \in \bigcap_{n=1}^{\infty} F_n(u) = F^\star(u)$. □

REMARK 4.3. Note that if f is an additive function of one groupoid U to another V, then f is, in particular, n–homogeneous for all $n \in \mathbb{N}$.

5. A few basic facts on subadditive relations

DEFINITION 5.1. A relation F on one groupoid U to another V is called subadditive if for any $u, v \in U$ we have

$$F(u+v) \subset F(u)+F(v).$$

REMARK 5.1. If the corresponding equality (converse inclusion) holds, then F is called additive (superadditive).

THEOREM 5.1. *If F is a subadditive relation on a commutative semigroup U to a vector space X over $\mathbb{K}$, then F_n is also subadditive for all $n \in \mathbb{N}$.*

PROOF. If $u, v \in U$ and $n \in \mathbb{N}$, then by Theorem 3.2 we have

$$F_n(u+v) = 2^{-n} F\big(2^n(u+v)\big) = 2^{-n} F\big(2^n u + 2^n v\big) \subset$$
$$2^{-n}\Big(F\big(2^n u\big) + F\big(2^n v\big)\Big) = 2^{-n} F\big(2^n u\big) + 2^{-n} F\big(2^n v\big) = F_n(u) + F_n(v).$$

□

Extending the terminology of W. Smajdor [**33**, p. 29], we may also naturally introduce the following

DEFINITION 5.2. A relation F on one groupoid U to another V is called M–subadditive, for some $M \subset V$, if for any $u, v \in U$ we have

$$F(u+v) \subset F(u)+F(v)+M.$$

REMARK 5.2. Note that if the groupoid V has a zero element 0 with $0 \in M$, then the subadditivity of F implies the M–subadditivity of F.

DEFINITION 5.3. If F is a relation on one groupoid U to another V and $M \subset V$, then for any $u \in U$ we define

$$(F+M)(u) = F(u)+M.$$

THEOREM 5.2. *If F is an M–subadditive relation on a groupoid U to a commutative semigroup V, then $F+M$ is already subadditive.*

PROOF. If $G = F+M$, then for any $u, v \in U$ we have

$$G(u+v) = F(u+v) + M \subset$$
$$\subset F(u)+F(v)+M+M = F(u)+M+F(v)+M = G(u)+G(v).$$

□

THEOREM 5.3. *If F is a relation on a groupoid U to a commutative monoid V and M is a submonoid of V such that $F+M$ is M–subadditive, then F is also M–subadditive.*

PROOF. If $G = F+M$, then for any $u, v \in U$ we have

$$F(u+v) \subset F(u+v)+M = G(u+v) \subset G(u)+G(v)+M =$$
$$F(u)+M+F(v)+M+M = F(u)+F(v)+M+M+M \subset F(u)+F(v)+M.$$

□

Now, as an immediate consequence of Theorems 5.2 and 5.3 and Remark 5.2, we can also state

COROLLARY 5.1. *If F is a relation on a groupoid U to a commutative monoid V and M is a submonoid of V, then the following assertions are equivalent:*

(1) F is M–subadditive; *(2) $F+M$ is subadditive.*

THEOREM 5.4. *If f is a function of a groupoid U to a commutative group V and M is a subgroup of V such that $f+M$ has an additive selection g, then f is M–subadditive.*

PROOF. For any $u \in U$ we have

$$g(u) \in (f+M)(u) = f(u) + M, \qquad \text{and thus} \qquad f(u) \in g(u) - M \subset g(u) + M.$$

Hence, it is clear that for any $u, v \in U$ we also have

$$f(u+v) \in g(u+v) + M = g(u) + g(v) + M \subset$$
$$\subset f(u) + M + f(v) + M + M = f(u) + f(v) + M + M + M \subset f(u) + f(v) + M.$$

□

To prove a certain converse of this theorem, we shall need the following

LEMMA 5.1. *5.1 If M is a subspace of a vector space X, then there exists a linear selection q of $\Delta_X + M$ such that $q(x) = 0$ for all $x \in M$.*

HINT. By [**4**, p. 15], there exists a subspace N of X such that

$$X = M + N \qquad \text{and} \qquad M \cap N = \{0\}.$$

Therefore, for each $x \in X$, there exists a unique pair $(m, n) \in M \times N$ such that $x = m + n$. Thus, in particular, for each $x \in X$, there exists a unique $q(x) \in N$ such that

$$x - q(x) \in M.$$

Hence, because of $0 \in N$, it is clear that $q(x) = 0$ for all $x \in M$. Moreover, by using the linearity properties of M, we can also easily see that q is linear. Furthermore,

$$q(x) = x - \big(x - q(x)\big) \in x - M \subset x + M = \Delta_X(x) + M = \big(\Delta_X + M\big)(x).$$

□

REMARK 5.3. Note that if q is as in the above lemma, then we also have $M = q^{-1}(0)$ and $q^2 = q$. Moreover, $N = \{x \in X : q(x) = x\}$ is a subspace of X such that $X = M + N$ and $M \cap N = \{0\}$.

Furthermore, from [**40**, Theorem 4.1] we can see that $F = \Delta_X + M$ is a linear equivalence relation on X. Thus, in particular $F^{-1}(0) = F(0) = \big(\Delta_X + M\big)(0) = M$. Therefore, Lemma 4.11 is actually a particular case of [40, Corollary 9.6].

Now, we are ready to prove a useful reformulation of [**3**, Lemma 1] of Z. Boros whose origin goes back to Rätz [**27**, p. 241], Baron [**2**], and Gajda–Smajdor–Smajdor [**9**, p. 249].

Theorem 5.5. *If f is an M–subadditive function of a groupoid U to a vector space X, for some subspace M of X, then $f+M$ has an additive selection.*

Proof. Let q be as in Lemma 5.1 and define $g = q \circ f$. Then, for any $u \in U$, we evidently have

$$g(u) = q\big(f(u)\big) \in \big(\Delta_X + M\big)\big(f(u)\big) = f(u) + M = \big(f + M\big)(u).$$

Moreover, if $u, v \in U$, then by using the M–subadditivity of f we can easily see that

$$g(u+v) - g(u) - g(v) = q\big(f(u+v)\big) - q\big(f(u)\big) - q\big(f(v)\big) = \\ = q\big(f(u+v) - f(u) - f(v)\big) \in q[M] = \{0\}$$

and thus $g(u+v) = g(u) + g(v)$ is also true. □

Now, as an immediate consequence of Theorems 5.4 and 5.5, we can also state

Corollary 5.2. *If f is a function of a groupoid U to a vector space X and M is a subspace of X, then the following assertions are equivalent:*

(1) f is M–subadditive; *(3) $f+M$ has an additive selection.*

Remark 5.4. Definition 5.3 can also be well-motivated by [**40**, Theorem 8.6] which shows that a relation F of one vector space X to another Y is linear if and only if there exist a linear function f of X to Y and a subspace M of Y such that $F = f + M$.

6. A few basic facts on relators

Definition 6.1. If $\mathcal{R}$ is a family of relations on a set X, then we say that the family $\mathcal{R}$ is a relator on X and the ordered pair $X(\mathcal{R}) = (X, \mathcal{R})$ is a relator space.

Remark 6.1. Thus, relator spaces are natural generalizations of ordered sets and uniform spaces [**35**]. Moreover, all reasonable generalizations of the usual topological structures can be easily derived from relators [**36**].

However, to include the theory of Galois connections and formal contexts [**10**, p. 17] and to briefly express continuity properties of relations [**39**], relators on one set to another have also to be considered.

Example 6.1. If $\mathcal{A}$ is a family of subsets of X, then the family of $\mathcal{R}_{\mathcal{A}}$ of all Pervin relations [**23**, p. 177]

$$R_A = A^2 \cup A^c \times X,$$

where $A \in \mathcal{A}$ and $A^c = X \setminus A$, is an important relator on X.

Namely, all minimal structures, generalized topologies and ascending systems on X can be easily derived from $\mathcal{R}_{\mathcal{A}}$ according to [**41**].

Example 6.2. If $\mathcal{D}$ is a family of distance functions on X, then the family $\mathcal{R}_{\mathcal{D}}$ of all surroundings

$$B_r^d = \big\{(x, y) \in X^2 : \quad d(x, y) < r\big\},$$

where $d \in \mathcal{D}$ and $r > 0$, is an important relator on X.

Namely, each topology can be derived from a family of quasi-pseudo-metrics according to [**23**, Theorem 11.1.2 and an analogue of Theorem 11.3.4]. Moreover, the relator $\mathcal{R}_{\mathcal{D}}$ is usually a more convenient tool than the family $\mathcal{D}$.

Remark 6.2. Note that R_A is always a preorder relation on X. While, B_r^d is, in general, only a tolerance relation on X even if d is a metric on X.

Therefore, besides preorder relators, tolerance relators are also important particular cases of reflexive relators. Note that a relator may, for instance, be naturally called reflexive if each of its members is reflexive.

Among the several basic algebraic and topological structures derivable from relators, we shall actually need here only the induced closures and sequential limits.

Definition 6.2. If $\mathcal{R}$ is a relator on X, then for any $x \in X$ and $A \subset X$ we write

$$x \in \mathrm{cl}_{\mathcal{R}}(A) \quad \Longleftrightarrow \quad \forall\ R \in \mathcal{R}: \ \exists\ a \in A: \quad a \in R(x).$$

A simple application of the corresponding definitions immediately yields

Example 6.3. If $\mathcal{D}$ is family of distance functions on X, then for any $x \in X$ and $A \subset X$, we have

$$x \in \mathrm{cl}_{\mathcal{R}_{\mathcal{D}}}(A) \quad \Longleftrightarrow \quad \forall\ d \in \mathcal{D}: \quad d(x, A) = 0.$$

Moreover, by using the corresponding definitions, we can also easily establish the following

Theorem 6.1. *If $\mathcal{R}$ is a relator on X, then for any $A \subset X$ we have*

$$\mathrm{cl}_{\mathcal{R}}(A) = \bigcap_{R \in \mathcal{R}} R^{-1}[A].$$

Hence, it is clear that in particular we also have

Corollary 6.1. *If $\mathcal{R}$ is a relator on X and $\delta_{\mathcal{R}} = \bigcap \mathcal{R}$, then for any $x \in X$ we have*

$$\mathrm{cl}_{\mathcal{R}}(x) = \delta_{\mathcal{R}^{-1}}(x) = \delta_{\mathcal{R}}^{-1}(x).$$

Moreover, from Theorem 6.1, it is quite obvious that the following two theorems are also true.

Theorem 6.2. *If $\mathcal{R}$ is a relator on X, then*

(1) $\mathrm{cl}_{\mathcal{R}}(\emptyset) = \emptyset$ *if* $\mathcal{R} \neq \emptyset$,

(2) $\mathrm{cl}_{\mathcal{R}}(A) \subset \mathrm{cl}_{\mathcal{R}}(B)$ *if* $A \subset B \subset X$.

Theorem 6.3. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is reflexive;* *(2)* $A \subset \mathrm{cl}_{\mathcal{R}}(A)$ *for all* $A \subset X$.

Definition 6.3. If $\mathcal{R}$ is a relator on X, then the members of the family

$$\mathcal{F}_{\mathcal{R}} = \big\{ A \subset X : \quad \mathrm{cl}_{\mathcal{R}}(A) \subset A \big\}$$

are called the closed subsets of the relator space $X(\mathcal{R})$.

Because of Theorem 6.2, we evidently have the following

Theorem 6.4. *If $\mathcal{R}$ is a relator on X, then*

(1) $\emptyset \in \mathcal{F}_{\mathcal{R}}$ *if* $\mathcal{R} \neq \emptyset$, *(2)* $\bigcap \mathcal{A} \in \mathcal{F}_{\mathcal{R}}$ *if* $\mathcal{A} \subset \mathcal{P}(X)$.

Remark 6.3. From (2), we can also at once see that $X \in \mathcal{F}_{\mathcal{R}}$ for any relator $\mathcal{R}$ on X.

Thus, in general, $\mathcal{F}_{\mathcal{R}}$ is only a closure (or convexity) structure, and its dual $\mathcal{T}_{\mathcal{R}} = \{ A^c : \ A \in \mathcal{F}_{\mathcal{R}} \}$ is only a generalized topology on X.

7. Some further results on relators

Definition 7.1. If $\mathcal{R}$ is a relator on X, then for any $x \in X$ and sequence a in X we write

(1) $x \in \lim_{\mathcal{R}}(a) \iff \forall\ R \in \mathcal{R} : \ \exists\ n \in \mathbb{N} : \ \forall\ k \geqslant n : \ \ a_k \in R(x)$;

(2) $x \in \mathrm{adh}_{\mathcal{R}}(a) \iff \forall\ R \in \mathcal{R} : \ \forall\ n \in \mathbb{N} : \ \exists\ k \geqslant n : \ \ a_k \in R(x)$.

A simple application of the corresponding definitions immediately yields

Example 7.1. If $\mathcal{D}$ is family of distance functions on X, then for any $x \in X$ and sequence a in X we have

(1) $x \in \lim_{\mathcal{R}_{\mathcal{D}}}(a) \iff \forall\ d \in \mathcal{D} : \ \ \overline{\lim}_{n \to \infty} d(x, a_n) = 0$;

(2) $x \in \mathrm{adh}_{\mathcal{R}_{\mathcal{D}}}(a) \iff \forall\ d \in \mathcal{D} : \ \ \underline{\lim}_{n \to \infty} d(x, a_n) = 0$.

Moreover, by using the corresponding definitions, we can also easily establish the following two theorems.

Theorem 7.1. *If $\mathcal{R}$ is a relator on X, then for any sequence a in X we have*

$$\lim_{\mathcal{R}}(a) = \bigcap_{R \in \mathcal{R}} \underline{\lim}_{n \to \infty} R^{-1}(a_n) \qquad \textit{and} \qquad \mathrm{adh}_{\mathcal{R}}(a) = \bigcap_{R \in \mathcal{R}} \overline{\lim}_{n \to \infty} R^{-1}(a_n).$$

Theorem 7.2. *If $\mathcal{R}$ is a relator on X, then for any sequence a in X we have*

$$\mathrm{adh}_{\mathcal{R}}(a) = \bigcap_{n=1}^{\infty} \mathrm{cl}_{\mathcal{R}}\Big(\{ a_k \}_{k=n}^{\infty} \Big).$$

Definition 7.2. A sequence a in a relator space $X(\mathcal{R})$ is called convergent (adherent) if $\lim_{\mathcal{R}}(a) \neq \emptyset$ ($\mathrm{adh}_{\mathcal{R}}(a) \neq \emptyset$).

Moreover, the sequence a is called convergence (adherence) Cauchy if it is convergent (adherent) in each of the simple relator spaces $X(R)$, where $R \in \mathcal{R}$.

Remark 7.1. By the corresponding definitions, it is clear that

$$\lim_{\mathcal{R}}(a) = \bigcap_{R \in \mathcal{R}} \lim_{R}(a) \qquad \text{and} \qquad \operatorname{adh}_{\mathcal{R}}(a) = \bigcap_{R \in \mathcal{R}} \operatorname{adh}_{R}(a) .$$

Therefore, a convergent (adherent) sequence is, in particular, convergence (adherence) Cauchy, but the converse need not be true.

Definition 7.3. A relator $\mathcal{R}$ on X, or the relator spaces $X(\mathcal{R})$, is called sequentially complete if each convergence Cauchy sequence in it is adherent.

Remark 7.2. It is a remarkable fact that, by [**37**, Theorem 2.1], the relator

$$\mathcal{R}^{\wedge} = \big\{ S \subset X^2 : \quad \forall\ x \in X : \quad \exists\ R \in \mathcal{R} : \quad R(x) \subset S(x) \big\}$$

is always complete. It is actually the largest relator on X such that $\operatorname{cl}_{\mathcal{R}^{\wedge}} = \operatorname{cl}_{\mathcal{R}}$. Moreover, we also have $\lim_{\mathcal{R}^{\wedge}} = \lim_{\mathcal{R}}$.

Definition 7.4. A subset A of a relator space $X(\mathcal{R})$ is called infinitesimal if for each $R \in \mathcal{R}$ there exists $x \in X$ such that $A \subset R(x)$.

Moreover, a sequence $(A_n)_{n=1}^{\infty}$ of subsets of $X(\mathcal{R})$ is called infinitesimal if for each $R \in \mathcal{R}$ there exist $x \in X$ and $n \in \mathbb{N}$ such that $A_n \subset R(x)$.

Remark 7.3. Thus, in particular $A = \bigcap_{n=1}^{\infty} A_n$ is an infinitesimal subset of $X(\mathcal{R})$.

Moreover, as a straightforward extension of Cantor's intersection theorem [**18**, p. 186], we can prove the following

Theorem 7.3. *If $(A_n)_{n=1}^{\infty}$ is a decreasing infinitesimal sequence of nonvoid closed subsets of a sequentially complete relator space $X(\mathcal{R})$ and $A = \bigcap_{n=1}^{\infty} A_n$, then A is a nonvoid closed infinitesimal subset of $X(\mathcal{R})$.*

Proof. By the countable axiom of choice, there exists a sequence a in X such that $a_n \in A_n$ for all $n \in \mathbb{N}$. Moreover, by Definition 7.4, for each $R \in \mathcal{R}$ there exist $x \in X$ and $n \in \mathbb{N}$ such that $A_n \subset R(x)$. Hence, it is clear that for any $k \geqslant n$, we have

$$a_k \in A_k \subset A_n \subset R(x) .$$

Therefore, $x \in \lim_{R}(a)$. This shows that a is a convergence Cauchy sequence in $X(\mathcal{R})$. Hence, by the assumed completeness of $\mathcal{R}$, it follows that $\operatorname{adh}_{\mathcal{R}}(a) \neq \emptyset$.

On the other hand, by using Theorem 7.2 and the corresponding properties of the sets A_n, we can also easily see that

$$\operatorname{adh}_{\mathcal{R}}(a) = \bigcap_{n=1}^{\infty} \operatorname{cl}_{\mathcal{R}}\Big(\{ a_k \}_{k=n}^{\infty} \Big) \subset \bigcap_{n=1}^{\infty} \operatorname{cl}_{\mathcal{R}}(A_n) \subset \bigcap_{n=1}^{\infty} A_n = A .$$

Therefore, $A \neq \emptyset$ is also true. Now, since the remaining assertions are quite obvious by Theorem 6.4 and Remark 7.3, the proof is complete. □

Remark 7.4. By the results of [**37**], it is clear that for some particular relator spaces a certain converse of the above theorem is also true.

Definition 7.5. A relator $\mathcal{R}$ on X is called strictly T_2–separating, or the relator space $X(\mathcal{R})$ is called strictly T_2–separated, if for any $x, y \in X$, with $x \neq y$, there exists $R \in \mathcal{R}$ such that $R(x) \cap R(y) = \emptyset$.

Theorem 7.4. *If A is an infinitesimal subset of a relator space $X(\mathcal{R})$ such that the the relator $\mathcal{R}^{-1}$ is strictly T_2–separating, then A is at most a singleton.*

Proof. Assume, on the contrary, that there exist $a, b \in A$ such that $a \neq b$. Then, since $\mathcal{R}^{-1}$ is strictly T_2–separating, there exists $R \in \mathcal{R}$ such that $R^{-1}(a) \cap R^{-1}(b) = \emptyset$. Moreover, since A is infinitesimal, there exists $x \in X$ such that $A \subset R(x)$. Hence, in particular it follows that $a, b \in R(x)$, and thus $x \in R^{-1}(a) \cap R^{-1}(b) = \emptyset$. This contradiction proves the theorem. □

8. A few basic facts on vector relators

Definition 8.1. A nonvoid relator $\mathcal{R}$ on vector space X over $\mathbb{K}$ is called a vector relator on X if

(1) $R(x) = x + R(0)$ for all $x \in X$ and $R \in \mathcal{R}$;

(2) $R(0)$ is an absorbing balanced subset of X for all $R \in \mathcal{R}$;

(3) for each $R \in \mathcal{R}$ there exists $S \in \mathcal{R}$ such that $S(0) + S(0) \subset R(0)$.

The appropriateness of this definition is apparent from the following

Example 8.1. If $\mathcal{P}$ is a nonvoid family of preseminorms on X, then it can be easily seen that the family $\mathcal{R}_{\mathcal{P}}$ of all surroundings

$$B_r^p = \left\{ (x, y) \in X^2 : \quad p(x-y) < r \right\},$$

where $p \in \mathcal{P}$ and $r > 0$, is a vector relator on X. Note that if in particular p is a seminorm on X, then $B_r^p(0)$ is, in addition, convex for all $r > 0$.

Remark 8.1. It is well-known that each vector topology $\mathcal{T}$ on X can be derived from a nonvoid directed family $\mathcal{P}$ of preseminorms on X. (See [**20**] for a rather thorough treatment.)

Therefore, vector relators are somewhat more general objects than vector topologies. Namely, if $\mathcal{R}$ is a vector relator on X, then $\mathcal{T}_{\mathcal{R}}$ is a vector topology on X if and only if for any $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T(0) \subset R(0) \cap S(0)$.

Concerning vector relators, we shall only quote here the following three basic theorems of [**43**].

Theorem 8.1. *If $\mathcal{R}$ is a vector relator on X, then*

(1) for each $R \in \mathcal{R}$ there exists $S \in \mathcal{R}$ such that $S(x) + S(y) \subset R(x+y)$ for all $x, y \in X$;

(2) for any $R \in \mathcal{R}$ and $n \in \mathbb{N}$ there exists $S \in \mathcal{R}$ such that $\lambda S(x) \subset R(\lambda x)$ for all $x \in X$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leqslant n$.

Remark 8.2. Thus, we may actually write any subsets A and B of X in place of the points x and y in the above assertions.

Theorem 8.2. *If $\mathcal{R}$ is a vector relator on X, then $\mathcal{R}$ is a tolerance relator on X such that for each $R \in \mathcal{R}$ there exists $S \in \mathcal{R}$ such that $S \circ S \subset R$.*

Remark 8.3. The reflexivity and the above strict uniform transitivity of $\mathcal{R}$ imply, in particular, that $\mathcal{R}$ is a topological relator on X in the sense that for each $x \in X$ and $R \in \mathcal{R}$ there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $x \in V \subset R(x)$.

Therefore, as a useful consequence of the above theorem, we can also state

Corollary 8.1. *If $\mathcal{R}$ is a vector relator on X, then for any $A \subset X$ we have*

$$\mathrm{cl}_{\mathcal{R}}(A) = \bigcap \{ W \in \mathcal{F}_{\mathcal{R}} : \ A \subset W \} .$$

Remark 8.4. Moreover, it can also be easily shown that $\mathcal{R}$ is well-chained in the sense that $X^2 = R^{\infty} = \bigcup_{n=0}^{\infty} R^n$ for all $R \in \mathcal{R}$, where $R^0 = \Delta_X$ and $R^n = R^{n-1} \circ R$ for all $n \in \mathbb{N}$. Thus, $R[A] \subset A$ implies $A \in \{ \emptyset , X \}$ for all $A \subset X$ and $R \in \mathcal{R}$. (See [**22**, Theorem 12.8].)

Theorem 8.3. *If $\mathcal{R}$ is a vector relator on X, then*

(1) $\mathrm{cl}_{\mathcal{R}}(\lambda A) = \lambda \, \mathrm{cl}_{\mathcal{R}}(A)$ *for all* $0 \neq \lambda \in \mathbb{K}$ *and* $A \subset X$*;*

(2) $\mathrm{cl}_{\mathcal{R}}(x + A) = x + \mathrm{cl}_{\mathcal{R}}(A)$ *for all* $x \in X$ *and* $A \subset X$*;*

(3) $\mathrm{cl}_{\mathcal{R}}(A) + \mathrm{cl}_{\mathcal{R}}(B) \subset \mathrm{cl}_{\mathcal{R}}(A + B)$ *for all* $A, B \subset X$*.*

Remark 8.5. Hence, we can also see that $\mathrm{cl}_{\mathcal{R}}(A+B) = \mathrm{cl}_{\mathcal{R}}\big(\mathrm{cl}_{\mathcal{R}}(A)+\mathrm{cl}_{\mathcal{R}}(B)\big)$ for all $A, B \subset X$.

Therefore, we can also state that $\mathrm{cl}_{\mathcal{R}}(A + B) = \mathrm{cl}_{\mathcal{R}}(A) + \mathrm{cl}_{\mathcal{R}}(B)$ if and only if $\mathrm{cl}_{\mathcal{R}}(A) + \mathrm{cl}_{\mathcal{R}}(B) \in \mathcal{F}_{\mathcal{R}}$.

However, it is now more important to note that, as an immediate consequence of the above theorem, we also have

Corollary 8.2. *If $\mathcal{R}$ is a vector relator on X, then*

(1) $\lambda A \in \mathcal{F}_{\mathcal{R}}$ *for all* $0 \neq \lambda \in \mathbb{K}$ *and* $A \in \mathcal{F}_{\mathcal{R}}$*;*

(2) $x + A \in \mathcal{F}_{\mathcal{R}}$ *for all* $x \in X$ *and* $A \in \mathcal{F}_{\mathcal{R}}$*.*

Moreover, by using Theorems 8.3 and 6.4 and Corollary 8.1, we can also easily establish the following

Theorem 8.4. *If $\mathcal{R}$ is a vector relator on X and A is a linear subspace of X, then $\mathrm{cl}_{\mathcal{R}}(A)$ is a closed linear subspace of the vector relator space $X(\mathcal{R})$.*

Hence, it is clear that in particular we also have

Corollary 8.3. *If $\mathcal{R}$ is a vector relator on X, then $\mathrm{cl}_{\mathcal{R}}(0)$ is a closed linear subspace of the vector relator space $X(\mathcal{R})$.*

Definition 8.2. A vector relator $\mathcal{R}$ on X is called separating, or the vector relator space $X(\mathcal{R})$ is called separated, if for each $x \in X$, with $x \neq 0$, there exists $R \in \mathcal{R}$ such that $x \notin R(0)$.

The importance of this definition lies mainly in the following simple consequence of the corresponding definitions and Theorems 6.1 and 8.3.

Theorem 8.5. *If $\mathcal{R}$ is a vector relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is separating;

(2) $\{x\} \in \mathcal{F}_{\mathcal{R}}$ for all $x \in X$;

(3) $\mathcal{R}$ is strictly T_2–separating.

Remark 8.6. Actually, if either $\{0\} \in \mathcal{F}_{\mathcal{R}}$ or $\mathcal{R}$ is T_0–separating, then $\mathcal{R}$ is already separating.

A relator $\mathcal{R}$ on X is called T_0–separating if for any $x, y \in X$, with $x \neq y$, there exists $R \in \mathcal{R}$ such that either $y \notin R(x)$ or $x \notin R(y)$. This is equivalent to the requirement that the relator $\mathcal{R}$ be weakly antisymmetric in the sense that the relation $\delta_{\mathcal{R}} = \bigcap \mathcal{R}$ is antisymmetric.

In this respect, it is also worth mentioning that a relator is T_1–separating if and only if it is both weakly symmetric and weakly antisymmetric. Moreover, weak symmetry corresponds to the famous weak regularity introduced by N. A. Shanin in 1943 which was called R_0-regularity by A. S. Davis in 1961.

9. Infinitesimal sequences in vector relator spaces

Definition 9.1. If $X(\mathcal{R})$ is a vector relator space over $\mathbb{K}$, then for any $A \subset X$ and $R \in \mathcal{R}$ we define

$$\rho_R(A) = \inf \{ t \in \mathbb{K}_+ : \quad \exists\ x \in X : \quad A \subset x + t\,R(0) \},$$

where $\mathbb{K}_+ = \mathbb{K} \cap [\,0\,, +\infty\,[$. (Recall that $\mathbb{K} = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$.)

Remark 9.1. By Gajda and Ger [**8**], we may also naturally define

$$d_R(A) = \inf \big\{ t \in \mathbb{K}_+ : \quad A - A \subset t\,R(0) \big\}.$$

However, the radius function ρ_R seems now to be more appropriate than the diameter function d_R.

Namely, we evidently have $\rho_R(A) \leqslant d_R(A)$. Since, if $A - A \subset t\,R(0)$ for some $t \in \mathbb{K}_+$, then we also have $A \subset a + t\,R(0)$ for any $a \in A$. Therefore, $\rho_R(A) \leqslant t$, and thus the stated inequality is true.

Theorem 9.1. *9.1 If $X(\mathcal{R})$ is a vector relator space over $\mathbb{K}$, and moreover $A \subset X$ and $R \in \mathcal{R}$ such that $\rho_R(A) < +\infty$, then for any $\lambda \in \mathbb{K}$ we have*

$$\rho_R(\lambda A) = |\lambda|\,\rho_R(A).$$

Proof. By Definition 9.1, for each $\varepsilon > 0$, there exists $t \in \mathbb{K}_+$, with $t < \rho_R(A) + \varepsilon$, such that $A \subset x + t\,R(0)$ for some $x \in X$. Hence, since $R(0)$ is balanced, it follows that

$$\lambda A \subset \lambda x + \lambda t\,R(0) = \lambda x + |\lambda|\,t\,R(0).$$

Therefore, by Definition 9.1, we have $\rho_R(\lambda A) \leqslant |\lambda|\,t \leqslant |\lambda|\big(\rho_R(A) + \varepsilon\big)$. Hence, by letting $\varepsilon \to 0$, we can infer that

$$\rho_R(\lambda A) \leqslant |\lambda|\,\rho_R(A).$$

Hence, it is clear that $\rho_R(0\,A) = 0 = 0\,\rho_R(A)$. Moreover, if $\lambda \neq 0$, then by putting λ^{-1} in place of λ and λA in place A, we can easily see that

$$\rho_R(A) = \rho_R\big(\lambda^{-1}\lambda A\big) \leqslant |\lambda|^{-1}\rho_R(\lambda A),$$

and thus $|\lambda|\,\rho_R(A) \leqslant \rho_R(\lambda A)$ is also true. □

Remark 9.2. If in particular $R(0)$ is convex, then we can also easily prove that $\rho_R(A + B) \leqslant \rho_R(A) + \rho_R(B)$ for all $A, B \subset X$.

However, it is now more important to note that we also have the following

Theorem 9.2. *If $(\lambda_n)_{n=1}^{\infty}$ is a null sequence in $\mathbb{K}$ and $(A_n)_{n=1}^{\infty}$ is a sequence of subsets of vector relator space $X(\mathcal{R})$ over $\mathbb{K}$ such that the sequence $\big(\rho_R(A_n)\big)_{n=1}^{\infty}$ is bounded for all $R \in \mathcal{R}$, then the sequence $(\lambda_n A_n)_{n=1}^{\infty}$ is infinitesimal.*

Proof. If $R \in \mathcal{R}$, then by the assumption there exists a number c such that $\rho_R(A_n) \leqslant c$ for all $n \in \mathbb{N}$. Hence, by Theorem 9.1, it follows that

$$\rho_R(\lambda_n A_n) = |\lambda_n|\,\rho_R(A_n) \leqslant |\lambda_n|\,c$$

for all $n \in \mathbb{N}$. Therefore, $\rho_R(\lambda_n A_n) < 1$ for some $n \in \mathbb{N}$. Hence, by using Definition 9.1, we can see that there exist $t \in \mathbb{K}_+$, with $t < 1$, and $x \in X$ such that

$$\lambda_n A_n \subset x + t\,R(0) \subset x + R(0) = R(x).$$

Therefore, by Definition 7.4, the required assertion is true. □

Theorem 9.3. *If $(A_n)_{n=1}^{\infty}$ and $(B_n)_{n=1}^{\infty}$ are decreasing infinitesimal sequences of subsets of a vector relator space $X(\mathcal{R})$ over $\mathbb{K}$, then*

(1) $(A_n + B_n)_{n=1}^{\infty}$ is also a decreasing infinitesimal sequence;

(2) $(\lambda A_n)_{n=1}^{\infty}$ is also a decreasing infinitesimal sequence for all $\lambda \in \mathbb{K}$.

Proof. If $R \in \mathcal{R}$, then by Theorem 8.1 there exists $S \in \mathcal{R}$ such that

$$S(x) + S(y) \subset R(x + y)$$

for all $x, y \in X$. Moreover, by Definition 7.4, there exist $x, y \in X$ and $k, l \in \mathbb{N}$ such that

$$A_k \subset S(x) \qquad \text{and} \qquad B_k \subset S(y).$$

Hence, by taking $n = \max\{k, l\}$, we can already see that

$$A_n + B_n \subset A_k + B_l \subset S(x) + S(y) \subset R(x + y).$$

Therefore, the sequence $(A_n+B_n)_{n=1}^{\infty}$ is also infinitesimal.

On the other hand, if $\lambda\in\mathbb{K}$ and $R\in\mathcal{R}$, then by Theorem 8.1 there exists $S\in\mathcal{R}$ such that $\lambda S(x)\subset R(\lambda x)$ for all $x\in X$. Moreover, by Definition 7.4, there exist $x\in X$ and $n\in\mathbb{N}$ such that $A_n\subset S(x)$. Hence, we can already see that $\lambda A_n\subset\lambda S(x)\subset R(\lambda x)$. Therefore, the sequence $(\lambda A_n)_{n=1}^{\infty}$ is infinitesimal even if $(A_n)_{n=1}^{\infty}$ is not supposed to be decreasing. □

In the sequel, we shall also need the following

THEOREM 9.4. *If A is an infinitesimal subset of a vector relator space $X(\mathcal{R})$, then*

$$A-A\subset \mathrm{cl}_{\mathcal{R}}(0).$$

PROOF. If $R\in\mathcal{R}$, then by Theorem 8.1 there exists $S\in\mathcal{R}$ such that

$$S(x)+S(y)\subset R(x+y)$$

for all $x, y\in X$. Moreover, by Definition 7.4, there exists $x\in X$ such that $A\subset S(x)$. Now, by using Definition 8.1, Corollary 3.1 and the corresponding properties of S and R, we can already see that

$$-S(x)=-\big(x+S(0)\big)=-x-S(0)=-x+S(0)=S(-x),$$

and thus

$$A-A\subset S(x)-S(x)=S(x)+S(-x)\subset R(0)=R^{-1}(0).$$

Hence, by Theorem 6.1, it is clear that

$$A-A\subset\bigcap_{R\in\mathcal{R}}R^{-1}(0)=\mathrm{cl}_{\mathcal{R}}(0).$$

□

COROLLARY 9.1. *If A is an infinitesimal subset of a separated vector relator space $X(\mathcal{R})$, then A is at most a singleton.*

PROOF. By Theorems 9.4 and 8.5, we now have $A-A\subset\mathrm{cl}_{\mathcal{R}}(0)=\{0\}$. Therefore, either $A=\emptyset$ or $A=\{x\}$ for some $x\in X$. □

REMARK 9.3. Note that this corollary is also an immediate consequence of Theorems 7.4, 8.2 and 8.5.

10. An additive selection theorem for 2–sublinear relations

DEFINITION 10.1. A relation F on one groupoid U to another a groupoid X will be called n-sublinear, for some $n\in\mathbb{N}$, if it is subadditive and n–subhomogeneous.

THEOREM 10.1. *If F is a 2–sublinear relation on a commutative semigroup U to a vector relator space $X(\mathcal{R})$ over $\mathbb{K}$ such that the sequence $\big(F_n(u)\big)_{n=1}^{\infty}$ is infinitesimal for all $u\in U$, then for any $u, v\in U$ we have*

$$F^{\star}(u+v)-F^{\star}(u)-F^{\star}(v)\subset\mathrm{cl}_{\mathcal{R}}(0).$$

Proof. If $u, v \in U$, then by Definition 4.3 and Theorem 5.1 we have

$$F^\star(u) + F^\star(v) \subset F_n(u) + F_n(v)$$

and

$$F^\star(u+v) \subset F_n(u+v) \subset F_n(u) + F_n(v)$$

for all $n \in \mathbb{N}$. Hence, it is clear that, under the notation

$$A = \bigcap_{n=1}^{\infty} \big(F_n(u) + F_n(v) \big),$$

we have

$$F^\star(u+v) - F^\star(u) - F^\star(v) = F^\star(u+v) - \big(F^\star(u) + F^\star(v) \big) \subset A - A.$$

Moreover, from Theorem 4.3, we know that the sequences $\big(F_n(u)\big)_{n=1}^{\infty}$ and $\big(F_n(v)\big)_{n=1}^{\infty}$ are decreasing. Hence, by using Theorem 9.3 we can infer that the sequence $\big(F_n(u) + F_n(v)\big)_{n=1}^{\infty}$ is also infinitesimal. Therefore, by Remark 7.3 and Theorems 9.4, we necessarily have

$$A - A \subset \mathrm{cl}_{\mathcal{R}}(0),$$

and thus the required inclusion is also true. □

Definition 10.2. A relation F on a set U to a relator space $X(\mathcal{R})$ is called closed-valued if $F(u)$ is a closed subset of $X(\mathcal{R})$ for all $u \in U$.

Now, we are ready to prove the following generalization of the existence part of [**8**, Theorem 3] of Z. Gajda and R. Ger.

Theorem 10.2. *If F is a closed-valued, 2–sublinear relation of a commutative semigroup U to a sequentially complete vector relator space $X(\mathcal{R})$ over $\mathbb{K}$ such that the sequence $\big(F_n(u)\big)_{n=1}^{\infty}$ is infinitesimal for all $u \in U$, then F has an additive selection.*

Proof. If $u \in U$, then by Theorem 4.3 the sequence $\big(F_n(u)\big)_{n=1}^{\infty}$ is decreasing. Moreover, by Definition 4.2 and Corollary 8.2, it is clear that $F_n(u)$ is a nonvoid closed subset of $X(\mathcal{R})$ for all $n \in N$. Hence, by using Theorem 7.3, we can already infer that

$$F^\star(u) = \bigcap_{n=1}^{\infty} F_n(u) \neq \emptyset,$$

and thus U is the domain of $F^\star$. Moreover, by the axiom of choice, the relation $F^\star$ has a selection g.

Now, by Theorem 10.1, it is clear that for any $u, v \in U$ we have

$$g(u+v) - g(u) - g(v) \in F^\star(u+v) - F^\star(u) - F^\star(v) \subset \mathrm{cl}_{\mathcal{R}}(0).$$

Hence, by taking $M = \mathrm{cl}_{\mathcal{R}}(0)$, we can see that g is an M–subadditive function of U to X. Moreover, from Corollary 8.3 we can see that M is a subspace of X. Therefore, by Theorem 5.5, the relation $g + M$ has an additive selection f. Hence, by Theorem 8.3 and the inclusions $g \subset F^\star \subset F$, it is clear that for any $u \in U$ we have

$$f(u) \in (g + M)(u) = g(u) + M = g(u) + \mathrm{cl}_{\mathcal{R}}(0) = \\ = \mathrm{cl}_{\mathcal{R}}\big(g(u)\big) \subset \mathrm{cl}_{\mathcal{R}}\big(F^\star(u)\big) \subset \mathrm{cl}_{\mathcal{R}}\big(F(u)\big) = F(u).$$

□

To prove an extension of the unicity part of [**8**, Theorem 3], we can easily establish the following

THEOREM 10.3. *If F is a relation of a commutative semigroup U to a separated vector relator space $X(\mathcal{R})$ over $\mathbb{K}$ such that the sequence $\big(F_n(u)\big)_{n=1}^{\infty}$ is infinitesimal for all $u \in U$, then F can have at most one 2–homogeneous selection.*

PROOF. If f is a 2–homogeneous selection of F, then by Theorem 4.5, for any $u \in U$, we also have $f(u) \in F^\star(u)$.

Moreover, from Definition 4.3, by Remark 7.3, and Corollary 9.1, we can see that $F^\star(u)$ is at most a singleton. Therefore, we actually have $\{f(u)\} = F^\star(u)$. □

Now, an immediate consequence of Theorems 10.2, 10.3 and 9.2, we can also state

COROLLARY 10.1. *If F is a closed-valued, 2–sublinear relation of a commutative semigroup U to a separated, sequentially complete vector relator space $X(\mathcal{R})$ over $\mathbb{K}$ such that the sequence $\big(\rho_R\big(F(2^n u)\big)_{n=1}^{\infty}$ is bounded for all $u \in U$ and $R \in \mathcal{R}$, then F has a unique additive selection.*

NOTE. This paper is a somewhat shortened and updated version of a Technical Report of the author, sent to several mathematicians in 2006.

References

[1] R. Badora, R. Ger and Zs. Páles, *Additive selections and the stability of the Cauchy functional equation*, ANZIAM J. **44** (2003), 323–337.

[2] K. Baron, *Functions with differences in subspaces*, Proceedings of the 18th International Symposium on Functional Equations, Faculty of Mathematics, University of Waterloo, 1980, 8–9.

[3] Z. Boros, *Stability of the Cauchy equation in ordered fields*, Math. Pannon. **11** (2000), 191–197.

[4] M. Cotlar and R. Cignoli, *An Introduction to Functional Analysis*, North-Holland, Amsterdam, 1974.

[5] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, London, 2002.

[6] G. L. Forti, *Hyers–Ulam stability of functional equations in several variables*, Aequationes Math. **50** (1995), 143–190.

[7] Z. Gajda, *Invariant means and representations of semigroups in the theory of functional equations*, Prace Nauk. Uniw. Śląsk. Katowic. **1273** (1992), 1–81.

[8] Z. Gajda and R. Ger, *Subadditive multifunctions and Hyers–Ulam stability*, In: W. Walter (Ed.), General Inequalities 5, Internat. Ser. Numer. Math. (Birkhäuser, Basel) **80** (1987), 281–291.

[9] Z. Gajda, A. Smajdor and W. Smajdor, *A theorem of the Hahn–Banach type and its applications*, Ann. Polon. Math. **57** (1992), 243–252.

[10] B. Ganter and R. Wille, *Formal Concept Analysis*, Springer-Verlag, Berlin, 1999.

[11] R. Ger, *A survey of recent results on stability of functional equations*, Proceedings of the 4th International Conference on Functional Equations and Inequalities Pedagogical University of Cracow, 1994, 5–36.

[12] G. Godini, *Set-valued Cauchy functional equation*, Rev. Roumaine Math. Pures Appl. **20** (1975), 1113–1121.

[13] D. H. Hyers *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A **27** (1941), 222–224.

[14] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, 1998.

[15] D. H. Hyers and Th. M. Rassias, *Approximate homomorphisms*, Aequationes Math. **44** (1992), 125–153.

[16] A. D. Ioffe, *A new proof of the equivalence of the Hahn–Banach extension and the least upper bound properties*, Proc. Amer. Math. Soc. **82** (1981), 385–389.

[17] S.-M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.

[18] K. Kuratowski, *Introduction to Set Theory and Topology*, Pergamon Press, Oxford, 1972.

[19] S. J. Lee and M. Z. Nashed, *Algebraic and topological selections of multi-valued linear relations*, Ann. Scuola Norm. Sup. Pisa. **17** (1990), 111–126.

[20] R. Metzler and H. Nakano, *Quasi-norm spaces*, Trans. Amar. Math. Soc. **123** (1966), 1–31.

[21] K. Nikodem, *Additive selections of additive set-valued functions*, Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak., Ser. Mat. **18** (1988), 143–148.

[22] G. Pataki and Á. Száz, *A unified treatment of well-chainedness and connectedness properties*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **19** (2003), 101–165 (electronic).

[23] W. J. Pervin, *Foundations of General Topology*, Academic Press, New York, 1964.

[24] Gy. Pólya and G. Szegő, *Aufgaben und Lehrsätze aus der Analysis I*, Verlag von Julius Springer, Berlin, 1925.

[25] D. Popa, *Additive selections of* (α, β)*–subadditive set valued maps*, Glasnik Mat. **36** (2001), 11–16.

[26] Th. M. Rassias, *Stability and set-valued functions*, In: Analysis and Topology, World Sci. Publ., River Edge, NJ, 1998, 585–614.

[27] J. Rätz, *On approximately additive mappings*, In: E. F. Beckenbach (Ed.), General Inequalities 2 (Oberwolfach, 1978), Int. Ser. Num. Math. (Birkhäuser, Basel-Boston) **47** (1980), 233–251.

[28] B. Rodríguez-Salinas and L. Bou, *A Hahn-Banach theorem for arbitrary vector spaces*, Boll. Un. Mat. Ital. **10** (1974), 390–393.

[29] M. Sablik, *A functional congruence revisited*, Grazer Math. Ber. **316** (1992), 181–200.

[30] F. C. Sánchez and J. M. F. Castillo, *Banach space techniques underpinning a theory for nearly additive mappings*, Dissertationes Math. **404** (2002), 1–73.

[31] A. Smajdor, *Additive selections of superadditive set-valued functios*, Aequationes Math. **39** (1990), 121–128.

[32] W. Smajdor, *Subadditive set-valued functions*, Glasnik Mat. **21** (1986), 343–348.

[33] W. Smajdor, *Subadditive and subquadratic set-valued functions*, Prace Nauk. Univ. Śląsk. Katowic. **889** (1987), 1–73.

[34] W. Smajdor and J. Szczawińska, *A theorem of the Hahn–Banach type*, Demonstratio Math. **28** (1995), 155–160.

[35] Á. Száz *Basic tools and mild continuities in relator spaces*, Acta Math. Hungar. **50** (1987), 177–201.

[36] Á. Száz *Structures derivable from relators*, Singularité **3** (1992), 14–30.

[37] A. Száz, *Cauchy nets and completeness in relator spaces*, Colloq. Math. Soc. János Bolyai **55** (1993), 479–489.

[38] Á. Száz, *The intersection convolution of relations and the Hahn–Banach type theorems*, Ann. Polon. Math. **69** (1998), 235–249.

[39] Á. Száz, *Somewhat continuity in a unified framework for continuities of relations*, Tatra Mt. Math. Publ. **24** (2002), 41–56.

[40] Á. Száz *Linear extensions of relations between vector spaces*, Comment. Math. Univ. Carolin. **44** (2003), 367–385.

[41] Á. Száz *Minimal structures, generalized topologies, and ascending systems should not be studied without generalized uniformities*, Filomat (Niš) **21** (2007), 87–97.

[42] Á. Száz, *An instructive treatment of a generalization of Hyers's stability theorem*, In: Th. M. Rassias and D. Andrica (Eds.), Inequalities and Applications, Cluj University Press, Romania, 2008, 245–271.

[43] Á. Száz, *Foundations of the theory of vector relators*, Adv. Stud. Contemp. Math. (Kyungshang) **20** (2010), 139–195.

[44] Á. Száz and G. Száz, *Additive relations*, Publ. Math. Debrecen **20** (1973), 259–172.

[45] L. Székelyhidi, *Ulam's problem, Hyers's solution – and to where they led*, In: Th. M. Rassias (Ed.), Functional Equations and Inequalities, Math. Appl. 518, Kluwer Acad. Publ., Dordrecht, 2000, 259–285.

Received 23 11 2012, revised 17 06 2013

Institute of Mathematics,
University of Debrecen,
H-4010 Debrecen, Pf. 12,
Hungary
E-mail address: szaz@science.unideb.hu