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Arithmetic Fuchsian groups of signature $(0; m_1, m_2, m_3, m_4)$

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Abstract. Here we determine the arithmetic data of arithmetic Fuchsian Groups of signature $(0; m_1, m_2, m_3, m_4)$.

1. Introduction

Let us call a Fuchsian group whose signature has the form $(0; m_1, m_2, m_3, m_4)$ a VE group (Vierecksgruppe). We want to determine all conjugacy classes of arithmetic VE groups. There are several classification results in very special cases [1, 2, 9, 12, 19, 20, 24] but these results are far from a complete classification. We took the existence of the particular results as the motivation to determine all conjugacy classes of arithmetic VE groups. We start by determining all commensurability classes (in the wide sense). These commensurability classes are determined by the defining number field k and the quaternion algebra A over k together with the associated arithmetic data.

This paper presents all the necessary theoretical results needed for the complete classification like the bounds of the field degree $[k : \mathbb{Q}]$ and the possible torsion in the arithmetic VE groups. These result are interesting and worth mentioning by their own. Finally, as an application, we determine completely all conjugacy classes of the arithmetic VE groups with $[k : \mathbb{Q}] \geq 4$ which are not subgroups of an arithmetic triangle group. If $[k : \mathbb{Q}] \leq 3$ then the number of conjugacy classes of maximal arithmetic VE groups is enormous (see for instance the lists for the special cases in [1, 2, 9]). For this case $[k : \mathbb{Q}] \leq 3$ we give the complete classification in a forthcoming paper together with the classification of VE subgroups of arithmetic triangle groups.

During the work on our project Colin Maclachlan died on 26 November 2012. The second author lost a wonderful lifelong friend. He is thankful to Dorothy Maclachlan, Rob Archbold and Michael Chung for the permission and the help to continue with the project, especially for putting some material from Colin's office at his disposal.

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2. Maximal arithmetic Fuchsian groups

Let k be a totally real number field and A a quaternion algebra over k. Let k_ν denote the completion of k at the place ν so that $A_\nu = A \otimes_k k_\nu$ is a quaternion algebra over k_ν. The algebra A is said to be ramified at ν if A_ν is a division algebra. The set of ramified places, $\mathrm{Ram}(A)$, is finite of even cardinality and it is the union of $\mathrm{Ram}_\infty(A)$, the set of real ramified places and $\mathrm{Ram}_f(A)$ the set of finite ramified places, each one corresponding to a prime ideal $\mathcal{P}$ of k. For details on material in this and subsequent sections see [**3, 11**].

For Fuchsian groups, we require that $\mathrm{Ram}_\infty(A)$ consist of all real places of k except one so that there exists a k-embedding $\rho : A \to M_2(\mathbb{R})$. An order $\mathcal{O}$ in A is a complete R_k-lattice which is a ring with 1. If

$$\mathcal{O}^1 = \{\alpha \in \mathcal{O} \mid n(\alpha) = 1\}$$

where n is the reduced norm, then $P\rho(\mathcal{O}^1)$ is a Fuchsian group of finite co-area. The class of *arithmetic Fuchsian groups* consists of all those subgroups of $\mathrm{PSL}(2,\mathbb{R})$ which are commensurable with some such $P\rho(\mathcal{O}^1)$.

Since the commensurator of $P\rho(\mathcal{O}^1)$ in $\mathrm{PGL}(2,\mathbb{R})$ is $P\rho(A^*)$, where A^* is the group of invertible elements of A, it is convenient to drop the reference to the embedding ρ so that the commensurability class determined by A will be all groups in $P(A^*)$ commensurable with $P(\mathcal{O}^1)$. We thus consider k as a subfield of $\mathbb{R}$ and A unramified at the identity real place. For $\alpha \in A^*$, by Eichler's norm theorem, $n(\alpha) \in k^*_\infty$, where k^*_∞ consists of all those elements which are positive at all real places in $\mathrm{Ram}_\infty(A)$. Furthermore, if $P(\alpha)$ lies in a Fuchsian group, then $n(\alpha) > 0$ so that $n(\alpha) \in k^*_+$, the group of all totally positive elements.

Let $\mathcal{O}$ denote a maximal order in A and $\mathcal{E} = \mathcal{O} \cap \mathcal{O}'$, the intersection of two maximal orders, an Eichler order. Then $\mathcal{E}$ is said to be of *square-free level* S, where S is a finite set of prime ideals of k, disjoint from $\mathrm{Ram}_f(A)$, such that at each finite place $\mathcal{P} \notin S$, $\mathcal{E}_\mathcal{P} = \mathcal{O}_\mathcal{P} = \mathcal{O}'_\mathcal{P}$, and if $\mathcal{P} \in S$, $\mathcal{E}_\mathcal{P} = \mathcal{O}_\mathcal{P} \cap \mathcal{O}'_\mathcal{P}$ has level $\mathcal{P}$ so that $\mathcal{O}_\mathcal{P}, \mathcal{O}'_\mathcal{P}$ are adjacent orders in the tree of maximal orders in $A_\mathcal{P} \cong M_2(k_\mathcal{P})$. Let $N(\mathcal{O}), N(\mathcal{E})$ denote the normalisers of $\mathcal{O}, \mathcal{E}$ respectively in A^*.

THEOREM 2.1. *(Borel) Every arithmetic Fuchsian group in the commensurability class determined by A is conjugate to a subgroup of some $P(N(\mathcal{O})^+)$ for $\mathcal{O}$ a maximal order or of some $P(N(\mathcal{E})^+)$ for a Eichler order of square-free level.*

The decoration '+' indicates that all elements must have totally positive norm.

3. Co-areas of maximal groups

Let $\mathcal{O}$ be a maximal order in A. Then

$$\text{Co}-\text{area of } P(\mathcal{O}^1) = \frac{8\pi\zeta_k(2)\Delta_k^{3/2}}{(4\pi^2)^{[k:\mathbb{Q}]}} \prod_{\mathcal{P} \in \mathrm{Ram}_f(A)} (N\mathcal{P} - 1) \tag{1}$$

where Δ_k is the discriminant of k, ζ_k the Dedekind zeta function of k and $N\mathcal{P}$ the cardinality of the field $R_k/\mathcal{P}$ (see e.g. [11, §11.1]).

If $\mathcal{E}$ is an Eichler order of square-free level S with $\mathcal{E} \subset \mathcal{O}$, then [**3**],[**11**, §11.5]

$$[P\mathcal{O}^1 : P\mathcal{E}^1] = \prod_{\mathcal{Q} \in S} (N\mathcal{Q} + 1). \tag{2}$$

In the subsequent discussion, we allow S to be empty in which case $\mathcal{E} = \mathcal{O}$ a maximal order. Note also that since we are concerned with VE groups, which are cocompact, $A \neq M_2(\mathbb{Q})$.

Now $P(N(\mathcal{E})^+)/P(\mathcal{E}^1)$ is a finite elementary abelian 2-group and under the norm map is isomorphic to $\overline{H(S)} = H(S)/k^{*2}$ where

$$H(S) = \{x \in k_+^* \mid xR_k \in \mathcal{D}\mathcal{S}I_k^2\}$$

with $\mathcal{D}, \mathcal{S}$ the subgroups of the ideal group I_k generated by prime ideals in $\mathrm{Ram}_f(A)$, S respectively and I_k^2 the subgroup generated by squares of ideals [**3, 7**]. Recall that, if k has class number one, then $|\overline{H(S)}| = [R^*_{f\cup S,+} : {R^*_{f\cup S}}^2]$ where $R_{f\cup S}$ is the ring of elements in k which are integral at all places outside $\mathrm{Ram}_f(A) \cup S$. Now

$$\mathrm{Co-area\ of\ } P(N(\mathcal{E})^+) = |\overline{H(S)}| \times \mathrm{Co-area\ of\ } P(\mathcal{E}^1). \tag{3}$$

Thus from (1), (2), (3) and the determination of $H(S)$, the co-area of maximal groups $P(N(\mathcal{E})^+)$ can be calculated.

4. Torsion in maximal groups

If $m > 2$, then $P(A^*)$ contains an element of order m if and only if $2\cos 2\pi/m \in k$ and $L = k(\mathrm{e}^{2\pi i/m})$ embeds in A. In that case, there is, up to conjugacy, a unique subgroup of order m in $P(A^*)$ generated by $P(1 + \xi_m)$ where ξ_m is the image of $\mathrm{e}^{2\pi i/m}$ in A. Furthermore we can deduce (see [**4, 7**]) that $P(N(\mathcal{E})^+)$ contains an element of order m if and only if all the following conditions hold

- (i) $2\cos 2\pi/m \in k$,
- (ii) no prime $\mathcal{P} \in \mathrm{Ram}_f(A)$ splits in $L = k(\mathrm{e}^{2\pi i/m})$,
- (iii) $(2 + 2\cos 2\pi/m)R_k \in \mathcal{D}\mathcal{S}I_k^2$,
- (iv) for each $\mathcal{Q} \in S$ at least one of the following must hold
  - (a) the highest power of $\mathcal{Q}$ dividing $(2 + 2\cos 2\pi/m)R_k$ is odd,
  - (b) $\mathcal{Q}$ splits in L,
  - (c) $\mathcal{Q}$ divides $(2 - 2\cos 2\pi/m)R_k$.

For future reference, these conditions will be called the *C(m) conditions*.

If $P(N(\mathcal{E})^+)$ contains an element $P(v)$ of order 2, there is an embedding $\sigma : L = k(u) \to A$ where $u^2 = -n = -n(v)$. Now $n \in H(S)$ and since $P(v)$ is only determined up to scalar multiples of v, n determines an element of $\overline{H(S)}$. Then an element $n \in \overline{H(S)}$ gives rise to an element of order 2 in $P(N(\mathcal{E})^+)$ if and only if all the following conditions hold

- (i) no prime $\mathcal{P} \in \mathrm{Ram}_f(A)$ splits in $L = k(\sqrt{-n})$,
- (ii) $nR_k \in \mathcal{D}\mathcal{S}I_k^2$,
- (iii) for each $\mathcal{Q} \in S$ at least one of the following must hold
  - (a) the highest power of $\mathcal{Q}$ dividing nR_k must be odd,
  - (b) $\mathcal{Q}$ splits in L,

(c) $\mathcal{Q}$ divides $2R_k$.

For future reference, these conditions will be call the $C_2(n)$ *conditions*.

Let $\overline{H_2(S)}$ denote the subset of $\overline{H(S)}$ such that conditions $C_2(n)$ are satisfied. Thus if we let $\ell_2(S)$ denote the number of conjugacy classes of elements of order 2 in $P(N(\mathcal{E})^+)$, which is the same as the number of even periods, then

$$\ell_2(S) = \sum_{n \in \overline{H_2(S)}} \ell_2(n) \tag{4}$$

where $\ell_2(n) \geq 1$ is the number of conjugacy classes of elements of order 2 with norm n in $P(N(\mathcal{E})^+)$. Formulas for these numbers $\ell_2(n)$, and for the number of conjugacy classes of elements of orders $m > 2$ are given in [**7**] (see also §9). However, these formulas are quite complicated and require all the arithmetic data to be available for their computation, so we first endeavour to obtain some general results which will reduce the number of cases to be considered.

Let

$$|\overline{H(S)}| = [P(N(\mathcal{E})^+) : P(\mathcal{E}^1)] = 2^{m(S)}.$$

The following result was proved in [**8**].

THEOREM 4.1. *Let $P(N(\mathcal{E})^+)$ be a maximal arithmetic Fuchsian group. If $\overline{H_2(S)} \neq \emptyset$ then*

$$|\overline{H_2(S)}| \geq 2^{m(S)-r-2s_2'}$$

where r is the number of primes in $\mathrm{Ram}_f(A)$ *and s_2' is the number of non-dyadic primes in S.*

COROLLARY 4.2.

$$2^{m(S)} \leq 2^{r+2s_2'}\ell_2(S). \tag{5}$$

If $P(N(\mathcal{E})^+)$ has t periods which are multiples of $2m$ for some fixed $m > 1$ then

$$2^{m(S)} \leq 2^{r+2s_2'}(\ell_2(S) - (t-1)). \tag{6}$$

Proof: The inequality (5) is immediate from the theorem and (4).

For each period which is a multiple of $2m$, there is an element x such that $\langle x \rangle$ represents a conjugacy class of cyclic subgroups of order $2m$ in $P(N(\mathcal{E})^+)$. Now $k(\mathrm{e}^{2\pi i/2m})$ embeds in A and so there exists j with $(j, 2m) = 1$ such that x^j is conjugate to $P(1+\mathrm{e}^{2\pi i/2m})$. Thus as elements of $\overline{H(S)}$, $n(x^j) = n(1+\mathrm{e}^{2\pi i/2m}) = 2+2\cos 2\pi/2m$. Since j is odd, $n(x) = 2 + 2\cos 2\pi/2m$ in $\overline{H(S)}$. But then the element x^m of order 2 defines the element $t_{2m} := (2 + 2\cos 2\pi/2m)^m$ in $\overline{H_2(S)}$. Thus by hypothesis, $\ell_2(t_{2m}) \geq t$. Hence

$$\ell_2(S) \geq \ell_2(t_{2m}) + |\overline{H_2(S)}| - 1 \geq t - 1 + 2^{m(S)-r-2s_2'}.$$

□

5. Bounding the field degree $[k : \mathbb{Q}]$

As with most similar enumeration problems (see [2, 8, 9, 10, 22]), our approach proceeds by first obtaining bounds on $[k : \mathbb{Q}]$, then bounds on Δ_k and finally, by using detailed information on totally real fields k of small degree and discriminants for which tables of data are available.

First note that, for any arithmetic Fuchsian group of genus 0, $[k : \mathbb{Q}] \leq 11$ and if it contains an element of order N, then

$$N \in \{2, 3, 4, \ldots, 15, 16, 18, 20, 22, 24, 26, 28, 30, 36\}$$

(see [8]).

Any VE group is either a subgroup of a triangle group or a subgroup of a VE group [17]. All arithmetic triangle groups have been determined [20] as have all arithmetic VE groups of signatures of the form $(0; N, N, N, N)$, $(0; 2, 2, N, N)$, $(0; 2, 2, 2, N)$ [9, 22]. Note also that any VE group of signature $(0; N_1, N_1, N_2, N_2)$ is of index 2 in a VE group of signature $(0; 2, 2, N_1, N_2)$ [17].

Thus let $\Gamma = P(N(\mathcal{E})^+)$, where $\mathcal{E}$ is an Eichler order of square-free level S or a maximal order, be a maximal arithmetic VE group of signature $(0; m_1, m_2, m_3, m_4)$ not of the form $(0; 2, 2, 2, N)$. Note that the signature of $P(N(\mathcal{E})^+)$ depends only on the level S [7]. [In our notation, we reserve $\mathcal{O}$ for maximal orders, while, as noted, $\mathcal{E}$ can be either an Eichler order or a maximal order.]

$$[\Gamma : P\mathcal{O}^1] = \frac{[P(N(\mathcal{E})^+) : P\mathcal{E}^1]}{[P\mathcal{O}^1 : P\mathcal{E}^1]} = \frac{2^{m(S)}}{\prod_{\mathcal{Q} \in S}(N\mathcal{Q} + 1)}. \tag{7}$$

$$2\pi\Big(2 - \sum_{i=1}^{4} \frac{1}{m_i}\Big)\frac{2^{m(S)}}{\prod_{\mathcal{Q} \in S}(N\mathcal{Q} + 1)} = \frac{2\pi 4\zeta_k(2)\Delta_k^{3/2}}{(4\pi^2)^{[k:\mathbb{Q}]}} \prod_{\mathcal{P} \in \mathrm{Ram}_f(A)} (N\mathcal{P} - 1). \tag{8}$$

We now employ lower bounds for Δ_k where k is a totally real field with $[k : \mathbb{Q}] = n$ due to Odlyzko [13, 14]. These are given in the form

$$\Delta_k > C^n \exp(f - E) \tag{9}$$

where C, E are constants. Furthermore $f = \sum f_q$ where $f_q \geq 0$ is a contribution due to the presence of a prime ideal in k of norm q. These f_q also depend on a constant b related to C, E and an optimal choice from the table of possibilities for C, E, b in [14] is $C = 28.668$, $E = 8.0001$ and $b = 3.0$. We note that, with this choice, $f_q = 0$ for $q > 401$ and f_q can be computed for the other values of q. These calculations yield the following inequalities which will be used subsequently:

$$\frac{q+1}{q^2}\exp\Big(\frac{-3f_q}{2}\Big) < 0.09; \;\; \frac{q-1}{q^2}\exp\Big(\frac{-3f_q}{2}\Big) < 0.06. \tag{10}$$

(see [8, Table 1]). Substituting (9) in (8) and bounding $\zeta_k(2)$ below by the Euler product terms for primes in $\mathrm{Ram}_f(A)$ and S, we obtain, after re-arrangement

$$\left(\frac{C^{3/2}}{4\pi^2}\right)^{[k:\mathbb{Q}]} < \frac{1}{4}\exp(3E/2) \prod_{\mathcal{P} \in \mathrm{Ram}_f(A)} \frac{N\mathcal{P} + 1}{N\mathcal{P}^2} \exp\Big(\frac{-3f_{N\mathcal{P}}}{2}\Big) \tag{11}$$

$$\times \prod_{\mathcal{Q}\in S} \frac{N\mathcal{Q}-1}{N\mathcal{Q}^2} \exp(\frac{-3f_{N\mathcal{Q}}}{2})\,(2-\sum_{i=1}^{4}\frac{1}{m_i})2^{m(S)}.$$

Since $P(N(\mathcal{E})^+)/P\mathcal{E}^1$ is an elementary abelian 2-group

$$2^{m(S)} = |\overline{H(S)}| = [P(N(\mathcal{E})^+) : P\mathcal{E}^1] \leq [\Gamma : \Gamma^{(2)}] = 2^{\ell_2(S)-1}$$

provided $\ell_2(S) \geq 1$ and this index $= 1$ if all periods of Γ are odd.

In the inequality (11), we note that each of the terms involving primes in $\mathrm{Ram}_f(A)$ and S on the r.h.s. of the inequality is clearly less than 1 so we get an upper bound by assuming $\mathrm{Ram}_f(A)$ and S are empty and $(2-\sum 1/m_i)2^{m(S)} < 16$. This yields $[k:\mathbb{Q}] \leq 9$.

$[k:\mathbb{Q}] = 9$. In this case Γ can only have torsion $m_i \in \{2,3,4,6,7,9,14,18\}$. Using the bounds in (10), it follows that (11) cannot hold if either $\mathrm{Ram}_f(A) \neq \emptyset$ or $S \neq \emptyset$. Furthermore inequality (11) fails if $2^{m(\emptyset)} \leq 2$ and this will be true unless all periods are even and no two are multiples of some $2m$ for $m > 1$ by (5) and (6). The r.h.s. of (11) will then be maximal for a signature $(0;2,4,14,18)$ in which case (11) does not hold.

$[k:\mathbb{Q}] = 8$. In this case, Γ can only have torsion $m_i \in \{2,3,4,5,6,8,10,12,16,20,24,30\}$. Since $\mathrm{Ram}_f(A)$ must be non-empty in this case, a simple application of (10) to (11) shows that $\mathrm{Ram}_f(A) = \{\mathcal{P}\}$ and $S = \emptyset$. Employing the same type of argument as in the degree 9 case above but with signature $(0;2,6,10,16)$ shows that (11) must fail.

$[k:\mathbb{Q}] = 7$. Here Γ can only have torsion $m_i \in \{2,3,4,6\}$. By (11), we obtain that $\mathrm{Ram}_f(A)$ is empty and S can consist of at most one prime. For degrees up to 7, tables of fields with the smallest discriminants are available [15]. We thus re-arrange (8) and approximate $\zeta_k(2)$ as before to obtain

$$\Delta_k^{3/2} \leq \frac{(4\pi^2)^{[k:\mathbb{Q}]}}{4} \prod_{\mathcal{P}\in \mathrm{Ram}_f(A)} \frac{N\mathcal{P}+1}{N\mathcal{P}^2} \prod_{\mathcal{Q}\in S} \frac{N\mathcal{Q}-1}{N\mathcal{Q}^2}\,(2-\sum 1/m_i)\,2^{m(S)}. \tag{12}$$

In degree 7, with $\mathrm{Ram}_f(A)$ empty and $S = \{\mathcal{Q}\}$ and signature $(0;4,6,6,6)$, (12) can only hold for the field of smallest discriminant 20,134,393 and $N\mathcal{Q} = 2$. But this field does not have a prime of norm 2 so that $S = \emptyset$ and the order $\mathcal{E}$ must be maximal. But then, if $P(N(\mathcal{O})^+)$ has elements of orders 4 or 6, (iii) of $C(4)$ and $C(6)$ show that $2R_k$ or $3R_k$ is a product of even powers of primes. But this cannot arise when $[k:\mathbb{Q}]$ is odd. The only candidate signature is then $(0;2,3,3,3)$ and for this (12) fails.

6. More on Torsion

In the succeeding sections, we will consider the cases where the fields k are such that $[k:\mathbb{Q}] \leq 6$. The torsion that may appear in the groups $\Gamma = P(N(\mathcal{E})^+)$ imposes certain restrictions on the field k and the prime ideals in these fields via the $C(m)$ conditions.

In addition, we will make extensive use of the inequality (12). As already noted in the arguments for degrees 8 and 9 above, the term $(2 - \sum 1/m_i)2^{m(S)}$ on the r.h.s. of (12) can be bounded by making use of (5) and (6) and we expand on that here.

Furthermore, the torsion in Γ and the term $\displaystyle\prod_{\mathcal{P}\in \mathrm{Ram}_f(A)} \frac{N\mathcal{P}+1}{N\mathcal{P}^2} \prod_{\mathcal{Q}\in S} \frac{N\mathcal{Q}-1}{N\mathcal{Q}^2}$ can be linked using the $C(m)$ conditions.

All these scenarios will be discussed in this section with a view to their application in the subsequent investigations.

TA We make use of the following well-known results on cyclotomic fields $C_N = \mathbb{Q}(\mathrm{e}^{2\pi i/N})$ and $k_N = \mathbb{Q}(\cos 2\pi/N)$ where we assume that if N is even then $4 \mid N$. Let R_N denote the ring of integers in k_N. If $N = p^k$ for some prime p then p is totally ramified in C_N and k_N. Let $\mathbb{Z}_N^*$ denote the group of units in $\mathbb{Z}_N$ and let P_N denote its quotient group where $[i] \equiv [-i]$. If prime $p \nmid N$, then $pR_N = \mathcal{P}_1\mathcal{P}_2 \dots \mathcal{P}_g$ with $N(\mathcal{P}_i) = p^f$ where $[p]$ has order f in P_N. If $N = p^k N'$ with $N' > 1$, then $pR_N = (\mathcal{P}_1\mathcal{P}_2 \dots \mathcal{P}_g)^{\phi(p^k)}$ with $N(\mathcal{P}_i) = p^f$ where $[p]$ has order f in $P_{N'}$. In the cases where $N \neq p^k$, it follows that these ideals $\mathcal{P}_i$ in k_N split in C_N if and only if $[p]$ has the same order in $\mathbb{Z}_N^*$ or $\mathbb{Z}_{N'}^*$ as it has in P_N or $P_{N'}$ respectively.

When $k_N \subset k$, these results enable us to estimate the norms of primes in k and the decomposition in the extension $k(\mathrm{e}^{2\pi i/m})$ as required by $C(m)$ (ii).

For later reference, we also note that, if $\hat{R}_N$ is the ring of integers in C_N, then $[\hat{R}_N^* : \langle \mathrm{e}^{2\pi i/N}\rangle R_N^*] = 1$ or 2 according as N is a prime power or not.

TB For other fields k, we may be able to use the following theorem of Hilbert [6] on relative quadratic extensions to determine splitting as required in the $C(m)$ (ii) and $C_2(n)$(i) conditions.

Theorem 6.1. *Let $L = k(\sqrt{\mu})$ be a quadratic extension of k.*

(i) *Let $\mathcal{P}$ be a non-dyadic prime. If $\mathcal{P}$ divides μR_k to an odd power, then $\mathcal{P}$ is ramified in L. Now suppose that $\mathcal{P}$ divides μR_k to an even power. We can adjust μ by a square to assume that $(\mu R_k, \mathcal{P}) = 1$. In that case, $\mathcal{P}$ splits in L if and only if*

$$\mu \equiv x^2 (\mathrm{mod}\ \mathcal{P})$$

has a solution with $x \in R_k$. Otherwise, $\mathcal{P}$ is inert in L.

(ii) *Let $\mathcal{P}$ be a dyadic prime. Let a be the highest power of $\mathcal{P}$ dividing μR_k and let ℓ be the highest power of $\mathcal{P}$ dividing $2R_k$. Then $\mathcal{P}$ is unramified in L if and only if*

$$\mu \equiv x^2 (\mathrm{mod}\ \mathcal{P}^{2\ell+a})$$

has a solution with $x \in R_k$. If $\mathcal{P}$ is not ramified, then a can be taken to be zero. In that case, $\mathcal{P}$ splits in L if and only if

$$\mu \equiv x^2 (\mathrm{mod}\ \mathcal{P}^{2\ell+1})$$

has a solution with $x \in R_k$. Otherwise $\mathcal{P}$ is inert in L.

This theorem can be used in many cases where the specific data is available. We note in particular the following general consequence:

Corollary 6.2. *If $\mathcal{E}$ is an Eichler order of square-free level S and $\mathcal{P}_3 \in S$ (resp. $\mathcal{P}_2 \in S$) with $N\mathcal{P}_3 = 3$ (resp. $N\mathcal{P}_2 = 2$), then $P(N(\mathcal{E})^+)$ has no elements of order 4 (resp. 3).*

Proof: Consider $C(4)(iv)$ with $\mathcal{P}_3 \in S$. Clearly (a) and (c) do not hold. Since $L = k(\sqrt{-1})$ and $-1 \equiv x^2 (\text{mod } \mathcal{P}_3)$ has no solution, condition (b) does not hold and there are no elements of order 4 in Γ. Now suppose $\mathcal{P}_2 \in S$ and $L = k(\sqrt{-3})$. Again (a) and (c) of $C(3)(iv)$ do not hold. Suppose that $\mathcal{P}_2^\ell$ is the highest power of $\mathcal{P}_2$ dividing $2R_k$. Then $x = 1$ satisfies $-3 \equiv x^2 (\text{mod } \mathcal{P}_2^{2\ell})$. If $-3 \equiv x^2 (\text{mod } \mathcal{P}_2^{2\ell+1})$, then $x = 1 + y$ with $y \in \mathcal{P}_2^\ell$. So $4 + 2y + y^2 \in \mathcal{P}_2^{2\ell+1}$. Now $4, 2y, y^2 \in \mathcal{P}_2^{2\ell}$ and $\mathcal{P}_2^{2\ell}/\mathcal{P}_2^{2\ell+1}$ is a one-dimensional vector space over $R_k/\mathcal{P}_2$ which is the field of two elements. But $4 \notin \mathcal{P}_2^{2\ell+1}$ and $2y \in \mathcal{P}_2^{2\ell+1}$ if and only if $y^2 \in \mathcal{P}_2^{2\ell+1}$. This contradiction shows that $C(3)(iv)$ does not hold and Γ has no elements of order 3.

TC In applying the inequality (12), one of the ingredients is to obtain as tight an upper bound as possible for the product term

$$T = (2 - \sum_{i=1}^{4} \frac{1}{m_i})\, 2^{m(S)}.$$

To do this we make use of the bounds (5) and (6) and now introduce some terminology for this. These inequalities are generally applicable when $r + 2s_2' = 0$ or 1. We say that the signature $(0; m_1, m_2, m_3, m_4)$ satisfies the *even multiple condition 1* (EMC1) if all periods m_i are even and no two are multiples of $2m$ for some $m > 1$. Notice that $2^{m(S)}$ is always ≤ 8 but if the signature fails EMC1 then $2^{m(S)} \leq 2^{r+2s_2'} \times 2$. A signature for which EMC1 holds and gives a maximal value of $2 - \sum 1/m_i$ will be called an extremal EMC1 signature. When EMC1 fails, we say that the signature satisfies the *even multiple condition 2* (EMC2) if one the following conditions hold:

- $\ell_2(S) = 2$ and both periods are not multiples of $2m$ for some $m > 1$,
- $\ell_2(S) = 3$ and no three periods are multiples of $2m$ for some $m > 1$,
- $\ell_2(S) = 4$ and not all periods are multiples of $2m$ for some $m > 1$.

A signature which fails EMC2 yields $2^{m(S)} \leq 2^{r+2s_2'}$. Again a signature for which EMC2 holds and gives a maximal value of $2 - \sum 1/m_i$ will be called an extremal EMC2 signature.

TD One of the arguments used in degree 7 above will apply for any odd degree. Thus, if $[k : \mathbb{Q}]$ is odd, $\text{Ram}_f(A) = \emptyset$ and $\mathcal{O}$ is a maximal order, then $P(N(\mathcal{O})^+)$ has no elements of orders 4 or 6. For, by $C(4)$(iii), $2R_k \in I_k^2$. Thus if $2R_k = \mathcal{P}_1^{e_1}\mathcal{P}_2^{e_2} \ldots \mathcal{P}_g^{e_g}$, all e_i are even and $[k : \mathbb{Q}] = \sum_{i=1}^{g} e_i f_i$ would be even. Similarily for $C(6)$.

TE To obtain good bounds for Δ_k using (12), we would like to force the product term

$$P = \prod_{\mathcal{P} \in \mathrm{Ram}_f(A)} \frac{N\mathcal{P}+1}{N\mathcal{P}^2} \prod_{\mathcal{Q} \in S} \frac{N\mathcal{Q}-1}{N\mathcal{Q}^2}$$

to be as small as possible. The existence of certain torsion in Γ forces either primes of large norm or several primes to lie in $\mathrm{Ram}_f(A) \cup S$ and hence decrease the size of this term.

Suppose 2 is not ramified in k. Thus $2R_k = \mathcal{P}_1\mathcal{P}_2 \ldots \mathcal{P}_g$. If $P(N(\mathcal{E})^+)$ has torsion of order 4 then $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_g\} \in \mathrm{Ram}_f(A) \cup S$ by $C(4)$(iii).

We emphasise the simple case where $\mathrm{Ram}_f(A) = \{\mathcal{P}\}$, $\mathcal{O}$ is a maximal order and $P(N(\mathcal{O})^+)$ contains an element of order 4. This means that either 2 is ramified in k or $N\mathcal{P} = 2^n$ where $n = [k : \mathbb{Q}]$.

Essentially the same statement can be made with 2 replaced by 3 and 4-torsion replaced by 6-torsion.

It may be possible to make similar statements about elements of order $2p^t$ where $(p, \Delta_k/\Delta_{\mathbb{Q}(\cos 2\pi/p^t)}^{[k:\mathbb{Q}(\cos 2\pi/p^t)]}) = 1$.

7. Degree 6

We continue the assumption that Γ is a maximal arithmetic VE group which is not of the form $(0; 2, 2, 2, N)$. Where the defining field k has degree 6 over $\mathbb{Q}$, Γ can have torsion $m_i \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 18, 26, 28, 36\}$. In addition to the obvious connections to the work in [9, 22], our investigations here occasionally overlap with other earlier work [8, 10], and we take advantage of this. Postulating the existence of certain torsion will determine particular degree 6 fields and we consider these first of all. Recall that, in all cases, $\mathrm{Ram}_f(A) \neq \emptyset$.

$k = \mathbb{Q}(\cos 2\pi/36)$. The possible torsion in Γ is then $\{2, 3, 4, 6, 9, 12, 18, 36\}$. Now $\Delta_k = 2^6.3^9$ and k has a prime of norm 3, one of norm 8 and all others have norms ≥ 37 (see **TA**). By (12) we must have $\mathrm{Ram}_f(A) = \{\mathcal{P}\}$ and $S = \emptyset$ with $N\mathcal{P} = 3$ or 8. If $N\mathcal{P} = 8$ then (12) fails unless the signature satisfies EMC1. But with this choice of $\mathrm{Ram}_f(A)$, condition $C(36)$ holds since $\mathcal{P}$ does not split in C_{36} as described in **TA** and $2 + 2\cos 2\pi/36$ is a unit. So there must be 36 torsion in Γ. But then the only signature satifying EMC1 is $(0; 2, 2, 2, 36)$. If $N\mathcal{P} = 3$, the signature of this group was determined in [8] as $(0; 2^{(4)}, 4, 36)$.

$k = \mathbb{Q}(\cos 2\pi/28)$. The possible torsion is $\{2, 3, 4, 6, 7, 14, 28\}$. Here $\Delta_k = 2^6.7^5$ and k has a prime of norm 7, of norm 8 and the rest of norms ≥ 27. By (12), $\mathrm{Ram}_f(A) = \{\mathcal{P}\}$ and $S = \emptyset$ with $N\mathcal{P} = 7$ or 8. If $N\mathcal{P} = 8$, then $\mathcal{P}$ splits in $k(\mathrm{e}^{2\pi i/7})$ and there are no elements of orders 7,14 or 28 in Γ. There is also no element of order 6 by **TE**. Using the extremal signature $(0; 2, 4, 4, 4)$ (12) fails. If $N\mathcal{P} = 7$ then again this group has been determined in [8] to have signature $(0; 2^{(8)}, 4, 28)$.

$k = \mathbb{Q}(\cos 2\pi/13)$. The possible torsion is $\{2, 3, 4, 6, 13, 26\}$. Here $\Delta_k = 13^5$ and k has primes of norms 13,25,27,53,64 and the rest ≥ 79. By (12) $\mathrm{Ram}_f(A) = \{\mathcal{P}\}$ and $S = \emptyset$ with $N\mathcal{P} \leq 64$. But then EMC yields $N\mathcal{P} \leq 27$.

- $N\mathcal{P} = 13$. This group has signature $(0; 2^{(5)}, 26)$ by [8].
- $N\mathcal{P} = 25$. Condition $C(13)$ holds but $C(26)$ does not. There are also no elements of orders 4 or 6 by **TE**. So EMC1 cannot hold, an extremal signature for EMC2 is $(0; 2, 2, 3, 13)$ and (12) fails.
- $N\mathcal{P} = 27$. Condition $C(13)$ does not hold so there are no periods of orders 13, 26 and also none of order 4. An extremal EMC2 signature is then given by $(0; 2, 6, 6, 6)$ and (12) fails.

Thus we have deduced that there are no VE groups with 36,28,26 or 13 torsion.

$k = \mathbb{Q}(\cos 2\pi/9, \cos 2\pi/5)$ so that Γ can have torsion $\{2, 3, 4, 5, 6, 9, 10, 18\}$. Now $\Delta_k = 81^2.5^3$ and k has primes of norm 9, 19 and ≥ 64. Thus $\mathrm{Ram}_f(A) = \{\mathcal{P}\}$ and $S = \emptyset$ with $N\mathcal{P} = 9$. With $N\mathcal{P} = 9$, there are no periods of orders 4 or 10 so that EMC1 cannot hold. An extremal signature for EMC2 is $(0; 2, 9, 18, 18)$ and with that (12) fails.

$k = \mathbb{Q}(\cos 2\pi/7, \cos 2\pi/5)$ giving torsion $\{2, 3, 4, 5, 6, 7, 10, 14\}$. Here $\Delta_k = 49^2.5^3$ and k has primes of norms 29,41,49,64,71 and ≥ 97. Thus $\mathrm{Ram}_f(A) = \{\mathcal{P}\}$ and $S = \emptyset$ with $N\mathcal{P} \leq 64$.

- $N\mathcal{P} = 29$. There are no elements of orders 4,6,7,10 or 14 and extremal signature $(0; 3, 5, 5, 5)$ shows that (12) fails.
- $N\mathcal{P} = 41$. This time there are no elements of orders 4,6,10,14 and extremal signature $(0; 5, 7, 7, 7)$ gives a contradiction.
- $N\mathcal{P} = 49$. There is no torsion of orders 4,6 or 10 so that EMC1 cannot hold. For EMC2 the extremal signature $(0; 2, 7, 14, 14)$ shows that (12) fails.
- $N\mathcal{P} = 64$. Only possible torsion is 2,3,4,5 and contradiction follows.

$k = \mathbb{Q}(\cos 2\pi/9, \cos 2\pi/8)$. With the smallest prime in this field having norm 8, (12) fails.

$k = \mathbb{Q}(\cos 2\pi/7, \cos 2\pi/8)$. Thus $\Delta_k = 49^2.8^3$ and k has primes of norm 7 and 8 and ≥ 41. Only possibilities are $\mathrm{Ram}_f(A) = \{\mathcal{P}\}$ and $S = \emptyset$ with $N\mathcal{P} = 7$. In this case, there is no 6 torsion and (12) fails unless EMC1 holds. An extremal signature is $(0; 2, 2, 8, 14)$ and for that (12) fails.

$k = \mathbb{Q}(\cos 2\pi/7, \cos 2\pi/12)$. Again with the smallest prime having norm 8, it follows that (12) fails.

We now consider other possible fields of degree 6 over $\mathbb{Q}$ and for this we make use of the PARI archives of such fields of small discriminant [15] and also, where appropriate, the PARI-gp calculator.

Suppose that $\mathbb{Q}(\cos 2\pi/7) \subset k$ and is not one of the fields considered above. Then Γ may have torsion of orders $\{2, 3, 4, 6, 7, 14\}$.Thus $49^2 \mid \Delta_k$ and the smallest discriminant of a field (not so far considered) with this property is 434,581. Furthermore, since $\mathbb{Q}(\cos 2\pi/7)$ has primes of norm 7,8,13,27,29 and larger, k cannot have primes of any smaller norms. Thus (12) shows that $\mathrm{Ram}_f(A) = \{\mathcal{P}\}$ and $S = \emptyset$ and using the extremal signature $(0; 2, 4, 6, 14)$ shows that $N\mathcal{P} \leq 27$. This implies that $\Delta_k < 1160930$ and there are 4 fields within this bound for which $49^2 \mid \Delta_k$. By **TE**, we note that either $2 \mid \Delta_k$ or $P(N(\mathcal{O})^+)$ has no torsion of order 4, either $3 \mid \Delta_k$ or $P(N(\mathcal{O})^+)$ has no 6-torsion and either $7 \mid \Delta_k/49^2$ or $P(N(\mathcal{O})^+)$ has no 14-torsion. Examining Δ_k in each of the four cases shows that for all but one, $P(N(\mathcal{O})^+)$ has no 4,6 or 14 torsion. But then using signature $(0; 3, 7, 7, 7)$ shows that (12) fails.

We are left to consider the field with defining polynomial $x^6 - x^5 - 6x^4 + 6x^3 + 8x^2 - 8x + 1$ which has discriminant 453789. In this case there is no 4 torsion and (12) shows that $N\mathcal{P} = 27$ is not possible. Furthermore, there are no primes of norms 8 or 13 in this field. Thus $N\mathcal{P} = 7$. By **TB**, $\mathcal{P}$ splits in $k(\sqrt{-3})$ and so there is no 3-torsion. Using the PARI-gp calculator, we obtain that the rational $\dfrac{4\,\zeta_k(2)\,\Delta_k^{3/2}}{(4\pi^2)^6} = \dfrac{1}{3}$ and $2^{m(\emptyset)} = [R^*_{f,+} : {R^*_f}^2] = 4$ (see §3). But then, see (8), there are no solutions to

$$(2 - \sum_{i=1}^{4} \frac{1}{m_i})4 = 2$$

with $m_i \in \{2, 7, 14\}$.

Now suppose that $\mathbb{Q}(\cos 2\pi/9) \subset k$. Thus $81^2 \mid \Delta_k$ and Γ can have torsion $\{2, 3, 4, 6, 9, 18\}$. Proceeding as in the case above, the smallest discriminant of a candidate field is 1292517 and this shows that $\mathrm{Ram}_f(A) = \{\mathcal{P}\}$ and $S = \emptyset$ with $N\mathcal{P} = 3$. This gives an upper bound of 1876538 for Δ_k and there are three candidate fields. But two of the fields have no prime of norm 3. Thus we are left to consider the field with defining polynomial $x^6 - 3x^5 - 3x^4 + 10x^3 + 3x^2 - 6x + 1$ and discriminant 1397493. But using **TB**, we see that $\mathcal{P}$ splits in $k(\sqrt{-3})$ so there are no elements of order 3. With only possible torsion of orders 2 and 4 we get a contradiction.

If $\mathbb{Q}(\cos 2\pi/12) \subset k$, then $12^3 \mid \Delta_k$ and Γ can have torsion of orders $\{2, 3, 4, 6, 12\}$. Assuming $\mathrm{Ram}_f(A) = \{\mathcal{P}\}$ and $N\mathcal{P} = 2$, the r.h.s. of (12) gives a bound of 4323956 and the only fields with discriminant such that $12^3 \mid \Delta_k$ have already been considered.

If $\mathbb{Q}(\cos 2\pi/8) \subset k$, then $8^3 \mid \Delta_k$ and Γ may have torsion of orders $\{2, 3, 4, 6, 8\}$. The r.h.s. of (12) gives a bound of 4093275 assuming k has a prime of norm 2 and a bound of 2887853 if not. There are five fields with $8^3 \mid \Delta_k$ satisfying the first bound and two satisfying the second. Of the four largest ones, two do not have a prime of norm 2 and the other two must have $\mathrm{Ram}_f(A) = \{\mathcal{P}\}$, $S = \emptyset$ with $N\mathcal{P} = 2$. In both cases, $(3, \Delta_k) = 1$ so there is no 6 torsion and using extremal EMC2 signature $(0; 2, 8, 8, 8)$, inequality (12) fails.

It remains to consider the field k where $\Delta_k = 1081856$, $k = \mathbb{Q}(x)$ where $x^6 - 6x^4 - 2x^3 + 7x^2 + 2x - 1 = 0$. This field has primes of norms 7 and 8 so that $\mathrm{Ram}_f(A) = \{\mathcal{P}\}$, $S = \emptyset$ with $N\mathcal{P} = 7$ or 8. Now Γ has no 6 torsion and so again using $(0; 2, 8, 8, 8)$ (12) fails.

If $\mathbb{Q}(\cos 2\pi/5) \subset k$, then $5^3 \mid \Delta_k$ and Γ can have $\{2, 3, 4, 5, 6, 10\}$ torsion. Since k cannot have primes of norms ≤ 4, we obtain $\Delta_k < 2361112$. There are 11 such fields. For all of them $(3, \Delta_k) = 1$, $(5, \Delta_k/5^3) = 1$ and for all but two $(2, \Delta_k) = 1$. Thus for 9 of these fields, we have either no 4,6 or 10 torsion or $\mathrm{Ram}_f(A) \cup S$ contains all primes in at least one of the unramified products $2R_k$, $3R_k$ or $\sqrt{5}R_k$ as discussed in **TE**. In any of the cases where $\mathrm{Ram}_f(A) \cup S$ contains these primes, this forces P to be small and hence (12) fails. If there is no 4,6 or 10 torsion, then the signature $(0; 3, 5, 5, 5)$ with $P \leq 5/16$ gives $\Delta_k \leq 463430$ which is smaller than the smallest discriminant of the 11 fields. Of the remaining two fields, one has discriminant 1922000 and with no 6 or 10 torsion, (12) fails.

We are left to consider the field with $\Delta_k = 722000$ and defining polynomial $x^6 - x^5 - 6x^4 + 7x^3 + 4x^2 - 5x + 1$. This field does not have a prime of norm 5 but $2R_k = \mathcal{P}_4^3$. So we must have $\mathrm{Ram}_f(A) = \{\mathcal{P}_4\}$ and $S = \emptyset$. Using PARI we get that $\dfrac{4\zeta_k(2)\Delta_k^{3/2}}{(4\pi^2)^6} = \dfrac{7}{10}$, $[R^*_{k,+} : R^{*2}_k] = 1$ and that $2^{m(\emptyset)} = [R^*_{f,+} : R^{*2}_f] = 2$. Using (8) there could be a solution for a VE group with $(0; 4, 4, 4, 5)$. So we need to look more closely at the formula for the number of classes of elements of order 4 in $P(N(\mathcal{O})^+)$ given in [7] (see also §9). Again using PARI we obtain that for $L = k(\sqrt{-1})$, $h(L) = 1$, $\delta_{L|k}$ is trivial and $\mathcal{P}_4$ is ramified in L. The formula then yields only one conjugacy class of elements of order 4 and we do not get a VE group in this case.

We now consider fields k which do not contain any of the fields $\mathbb{Q}(\cos 2\pi/N)$ different from $\mathbb{Q}$ in which case Γ can only have torsion of orders $\{2, 3, 4, 6\}$. First we expand slightly on arguments used above resulting from **TE** which reduce the sizes of P and T.

Thus suppose that 2 is unramified in k. Then either Γ has no 4 torsion or $2R_k = \mathcal{P}_1\mathcal{P}_2\ldots\mathcal{P}_g$ and $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_g\} \subset \mathrm{Ram}_f(A) \cup S$. If $\mathcal{P}_i$ has norm 2^{f_i} then $\sum f_i = 6$. Considering the possible products for $2R_k$ we obtain that $P \leq (3/4)^4.(5/16) \approx 0.0989$ which is the maximum value achieved with 4 primes of norm 2 and one of norm 4 all lying in $\mathrm{Ram}_f(A)$. Note that in all cases, the signature $(0; 4, 6, 6, 6)$ yields $T \leq (2 - (3/6) - (1/4))8 = 10$. Thus if Γ has 4 torsion then $PT < 10 \times 0.0989$. On the other hand, if Γ has no 4 torsion, the signature $(0; 2, 6, 6, 6)$ gives $T \leq 8$ and $PT \leq 3/4 \times 8 = 6$ if k has a prime of norm 2 and $PT \leq 32/9$ if not. Thus, from (12), if 2 is unramified in k we have $\Delta_k \leq 3182974(2245625)$.

Now suppose that 3 is unramified in k. If Γ has 6 torsion, we get $P \leq (4/9)^6.(3/4) \approx 0.00578$ this maximum occuring when $3R_k$ decomposes completely and $\mathrm{Ram}_f(A)$ consists of these six primes of norm 3 together with one additional one of norm 2. On the other hand, if there is no 6 torsion, the signature $(0; 2, 4, 4, 4)$ yields $T \leq 6$. Thus if 3 is unramified we have $PT \leq \max(3/4 \times 6, 0.00578 \times 10)$ if k has a prime of norm 2. As above this yields $\Delta_k \leq 2627487(1853722)$ with the second figure the bound if k does not contain a prime of norm 2.

Now suppose that both 2 and 3 are unramified in k. Then from the four possible cases, the largest value of PT arises if Γ has no 6 torsion but may have 4 torsion. Note that with no 4 or 6 torsion, the only possible signature is $(0; 2, 3, 3, 3)$ which gives $PT \leq 3/4 \times 1/2$. Thus in this case we have $\Delta_k \leq 824625(376407)$.

With no restrictions, we note that $PT \leq 3/4 \times 10$ which gives the overall bound of $\Delta_k \leq 3693514(2605817)$. We include the bounds in the cases where k does not have a prime of norm 2 as, it is generally quite straightforward to decide if a field has a prime of norm 2.

A. Suppose that $2 \nmid \Delta_k$ and $3 \nmid \Delta_k$. There is one field within the bound given above. It does not have a prime of norm 2 and its discriminant exceeds the bound in that case.
B. Suppose that $2 \mid \Delta_k$ but $3 \nmid \Delta_k$. There are three fields to consider. One field has discriminant 1997632 and no prime of norm 2 so can be discarded. The second has discriminant 2540864 and 2 is totally ramified in k. Since 3 is inert in k, there are no elements of order 6 and we must have $\mathrm{Ram}_f(A) = \{\mathcal{P}_2\}$ and $S = \emptyset$. In this case we get that $\dfrac{4\zeta_k(2)\Delta_k^{3/2}}{(4\pi^2)^6} = \dfrac{35}{6}$ and $2^{m(\emptyset)} = 2$ so that (8) has no solution for a VE group. In the third case $\Delta_k = 810448$, $2R_k = \mathcal{P}_4^3$ and the decomposition of primes shows that $\mathrm{Ram}_f(A) = \{\mathcal{P}_4\}$ and $S = \emptyset$. Further, we compute that $\dfrac{4\zeta_k(2)\Delta_k^{3/2}}{(4\pi^2)^6} = \dfrac{5}{6}$ and $[R^*_{f,+} : R^{*\,2}_f] = 2$. There is then no 6 torsion and so no solution to (8) with $m_i \in \{2, 3, 4\}$.
C. Suppose that $2 \nmid \Delta_k$ and $3 \mid \Delta_k$. There are two candidate fields. One has discriminant 2565429 and no prime of norm 2. The other also has no prime of norm 2 but discriminant 1387029. In this case, 2 is inert in k and $3R_k = \mathcal{P}_3^2\mathcal{P}_{3^4}$. So there are no elements of orders 4 or 6 and we get a contradiction.
D. Finally suppose that $6 \mid \Delta_k$. There is one candidate field of discriminant 2847312. Here $2R_k = \mathcal{P}_4^3$ and $3R_k = \mathcal{P}_3^3\mathcal{P}_{3^3}$ and (12) is violated.

Thus there are no arithmetic VE groups with a defining field of degree 6 over $\mathbb{Q}$.

8. Degree 5

We proceed as in degree 6. In this case the possible torsion is $\{2, 3, 4, 6, 11, 22\}$. Recall that (see (8) from the equality

$$(13)\qquad (2 - \sum_{i=1}^{4} \frac{1}{m_i})2^{m(S)} = \frac{4\zeta_k(2)\,\Delta_k^{3/2}}{(4\pi^2)^5} \prod_{\mathcal{P}\in \mathrm{Ram}_f(A)} (N\mathcal{P} - 1) \prod_{\mathcal{Q}\in S} (N\mathcal{Q} + 1)$$

we deduced, in the notation of §7,

$$(14)\qquad \Delta_k^{3/2} \leq \frac{(4\pi^2)^5}{4} \times P \times T$$

where $P = \displaystyle\prod_{\mathcal{P}\in \mathrm{Ram}_f(A)} \frac{N\mathcal{P} + 1}{N\mathcal{P}^2} \prod_{\mathcal{Q}\in S} \frac{N\mathcal{Q} - 1}{N\mathcal{Q}^2}$ and $T = (2 - \sum 1/m_i)2^{m(S)}$.

If Γ has 11 or 22 torsion, then $k = \mathbb{Q}(\cos 2\pi/11)$. Then $\Delta_k = 11^4$ and, from **TA**, k has primes of norms 11,23,32,43. Since $\mathrm{Ram}_f(A)$ has an even number of primes, (14) shows that $\mathrm{Ram}_f(A) = \emptyset$. But then $P(N(\mathcal{O})^+)$ has signature $(0; 2, 3, 11)$ [16] and (13) yields

$$(15)\qquad T = (2 - \sum_{i=1}^{4} \frac{1}{m_i})2^{m(S)} = \frac{5}{66} \prod_{\mathcal{Q}\in S} N\mathcal{Q} + 1.$$

Thus $S = \{\mathcal{Q}\}$ with $N\mathcal{Q} < 179$. If Γ has an element of order 22, then $N\mathcal{Q} = 11$. But then $P(N(\mathcal{E})^+)$ has signature $(0; 2, 2, 2, 22)$ [22] and there is only one conjugacy class of such groups (see also §9). If $N\mathcal{Q} \neq 2^5$, Γ has no 4 or 6 torsion. With $m_i \in \{2, 3, 11\}$, $T < 1.4$ and so $N\mathcal{Q} < 19$. If $N\mathcal{Q} = 32$, Γ has no 6 or 22 torsion and from (15), $T = 5/2$. But there is no solution to $(2 - \sum 1/m_i)2^{m(S)} = 5/2$ with $m_i \in \{2, 3, 11\}$.

Now consider other possible fields k of degree 5 and note that Γ can only have torsion of orders $\{2, 3, 4, 6\}$. Furthermore, by **TD**, if $\mathrm{Ram}_f(A) \cup S = \emptyset$, then Γ has no 4 or 6 torsion. If Γ has no 4 or 6 torsion, Γ must have signature $(0; 2, 3, 3, 3)$ for which $T = 1/2$. But then (14) gives $\Delta_k < 52377$. There are four such fields, one of which is $\mathbb{Q}(\cos 2\pi/11)$, fully discussed above. For the others, (14) shows that $\mathrm{Ram}_f(A) \cup S = \emptyset$. But then for each of these three fields $P(N(\mathcal{O})^+)$ has signature $(0; 2, 2, 2, 3)$ [1, 9] and the number of conjugacy classes of these groups is also given there (see discussion in §9).

Thus for other fields we can assume that $\mathrm{Ram}_f(A) \cup S \neq \emptyset$ and that Γ has either 4 or 6 torsion. For the three cases where, respectively, Γ has 4 and 6 torsion, only 6 torsion and only 4 torsion, we obtain $T \leq 10, 8, 6$ respectively using the signatures $(0; 4, 6, 6, 6)$, $(0; 2, 6, 6, 6)$, and $(0; 2, 4, 4, 4)$. We consider 4 cases depending on the ramification of 2 and 3 in k.

(1) $2 \mid \Delta_k, 3 \mid \Delta_k$. An upper bound for PT is then $10/3$ occuring when Γ has 4 and 6 torsion and $\mathrm{Ram}_f(A)$ consists of primes of norm 2 and 3. Thus $\Delta_k \leq 185527$.
(2) $2 \nmid \Delta_k, 3 \mid \Delta_k$. Then $PT \leq 8/3$ occuring when Γ has 6 torsion and $\mathrm{Ram}_f(A)$ has primes of norms 2 and 3. Thus $\Delta_k \leq 159883$.
(3) $2 \mid \Delta_k, 3 \nmid \Delta_k$. Then $PT \leq 54/16$ with Γ having 4 torsion and $\mathrm{Ram}_f(A)$ two primes of norm 2. Thus $\Delta_k \leq 187071$.
(4) $2 \nmid \Delta_k, 3 \nmid \Delta_k$. Then $PT \leq 6.(3/4)^5.(4/9)$ with Γ having 4 torsion and $\mathrm{Ram}_f(A)$ consisting of 5 primes of norm 2 and one of norm 3. Then $\Delta_k \leq 61283$.

We now use the Pari archive [15] to determine fields which satisfy these bounds and, for those that do, we employ the PARI-gp calculator to find primes of small norm and, where appropriate, to determine the rational

$$R = \frac{4\,\zeta_k(2)\,\Delta_k^{3/2}}{(4\pi^2)^5}$$

appearing in (13).

(1) There are no fields satifying these bounds.
(2) There is one field with $\Delta_k = 149169$, $2R_k = \mathcal{P}_{2^5}$, $3R_k = \mathcal{P}_3^2\mathcal{P}_{3^3}$. But having 4 or 6 torsion, violates (14).
(3) There are 7 fields satisfying this bound. The bound was obtained assuming that k had two primes of norm 2 but none of the 7 fields have such primes. We obtain a bound of 131981 assuming a prime of norm 2 and one of norm 3. There are two fields with discriminant less than this. One has $\Delta_k = 126032$, $R = 17/6$ and has one prime of norm 2 and one of norm 3 and all others have norms > 5. If $\mathrm{Ram}_f(A) = \emptyset$ and $S = \{\mathcal{P}_2\}$, then Γ has no elements

of order 3 by **TB** and (13) has no solution. The only other possibility is $\mathrm{Ram}_f(A) = \{\mathcal{P}_2, \mathcal{P}_3\}$, $S = \emptyset$. But with no 6 torsion (13) has no solution with $m_i \in \{2, 3, 4\}$. The other field has $\Delta_k = 117688$, a prime of norm 2, norm 8 and norm 3^5. With $R = 7/3$ and either 4 or 6 torsion, (13) has no solution.

(4) Only the four fields with smallest discriminant, discussed above, can arise. Examining the primes in these fields we obtain that, as Γ has either 4 or 6 torsion, $\mathrm{Ram}_f(A) \cup S$ contains a prime of norm 2^5, or one of norm 3^5 or one of norm 3 together with one of norm 3^4. Also $R = 1/6.1/3$ or $11/3$. Only possibility for (13) is $R = 1/6$, $\mathrm{Ram}_f(A) = \emptyset$ and $S = \{\mathcal{P}_{2^5}\}$. But then (13) has no solution with $m_i \in \{2, 3, 4\}$.

Thus there are no new arithmetic VE groups with a defining field k with $[k : \mathbb{Q}] = 5$ unless they occur as subgroups of the $(0; 2, 3, 11)$ triangle group. Numerically there are just two possibilities for this and both arise [2]. The groups have signatures $(0; 3, 3, 3, 11)$, $(0; 2, 3, 11, 11)$ of indices 12 and 13 respectively. These and other examples arising from triangle groups will be discussed in a forthcoming paper.

9. Signatures and conjugacy classes of maximal arithmetic Fuchsian groups

So far we have been engaged in eliminating many possible candidates as arithmetic VE groups. For this, it has, in general, sufficed to use the results on the existence of torsion in maximal arithmetic Fuchsian groups (see §4) and the allied results and consequences which flow from that as given in **TA** to **TE** in §6. However, in the cases of fields k where $[k : \mathbb{Q}] \leq 4$, we not only rule out cases but rule several in by showing the existence of new maximal arithmetic VE groups. For this, we use the formula by which the number of conjugacy classes of finite cyclic subgroups in maximal arithmetic Fuchsian groups are enumerated [7]. Furthermore, for each new arithmetic VE group which is established we must determine the number of conjugacy classes of such groups in $\mathrm{PGL}(2, \mathbb{R})$ [9, 24]. These formulae are discussed in this section.

As noted in §4, finite cyclic subgroups of order $m > 2$ in maximal arithmetic Fuchsian groups are generated by conjugates of images of $u = 1 + e^{2\pi i/m}$ and those of order 2 by conjugates of images of $u = \sqrt{-n}$ where n represents an element of $\overline{H_2(S)}$. In all cases, $L = k(u)$ is a totally imaginary quadratic extension of k. The number of conjugacy classes of these finite cyclic subgroups in $P(N(\mathcal{E})^+)$, where $\mathcal{E}$ is a maximal order or Eichler order of square-free level S is then a sum $\sum \ell(B)$of integers $\ell(B) \geq 1$ over suitable commutative orders B in L, called *Candidate orders* which contain u.

Describing the set of orders in such a quadratic extension which contain a fixed element u is given in [16] and is as follows: Let

$$\mathrm{disc}(u) R_k = \mathcal{A}_0(u)^2 \delta_{L|k}$$

where $\delta_{L|k}$ is the relative discriminant and $\mathcal{A}_0(u)$ is an integral ideal in R_k. Then there exists $a_0 \in R_k$ such that

$$R_L = R_k \oplus \mathcal{A}_0(u)^{-1}(u - a_0)$$

and all orders B of L which contain u are of the form

$$B = B(\mathcal{A}) := R_k + \mathcal{A}^{-1}(u - a_0)$$

where $\mathcal{A}$ is an ideal of R_k dividing $\mathcal{A}_0(u)$. The conductor $f(B)$ in these cases is $\mathcal{A}_0(u)\mathcal{A}^{-1}$ and. following Eichler [5], we extend the Legendre symbol as follows for an order B in L and a prime ideal $\mathcal{P}$ in k

$$\left\{\frac{B}{\mathcal{P}}\right\} = \left(\frac{L}{\mathcal{P}}\right) \text{ if } \mathcal{P} \nmid f(B) \text{ and } = 1 \text{ otherwise.}$$

We also define $\mathcal{V}_u = \{t \in k \mid tu \in R_L\}$ and J_u the integral ideal such that $J_u^{-1} = \mathcal{V}_u$.

Definition 9.1. *A Candidate order for u is a commutative order $B = B(\mathcal{A})$ in $L = k(u)$ as defined above which additionally satisfies the conditions that $\{\frac{B}{\mathcal{P}}\} \neq 1$ for all $\mathcal{P} \in \mathrm{Ram}_f(A)$, $\{\frac{B}{\mathcal{Q}}\} \neq -1$ for all $\mathcal{Q} \in S$ and $J_u \mid \mathcal{A} \mid \mathcal{A}_0(u)$.*

The term $\ell(B)$ then gives the number of conjugacy classes of embeddings of L in A which give the number of conjugacy classes of finite cyclic subgroups in $P(N(\mathcal{E})^+)$.

$$\ell(B) = \frac{h(L)}{h}\frac{1}{[R_{k,+}^* : R_k^{*2}]}\prod_{\mathcal{P}\in\mathrm{Ram}_f(A)}\left(1 - \left\{\frac{B}{\mathcal{P}}\right\}\right)\prod_{\mathcal{Q}\in S}\left(1 + \left\{\frac{B}{\mathcal{Q}}\right\}\right)$$

$$\times\frac{t(\mathcal{A})}{[R_L^* : B^*][H(S) : H(S)\cap N_{L|k}(G(B,S))R_{k,+}^*{k^*}^2]}. \tag{16}$$

$$t(\mathcal{A}) = N_{k|\mathbb{Q}}(\mathcal{A}_0(u)\mathcal{A}^{-1})\prod_{\mathcal{P}\mid\mathcal{A}_0(u)\mathcal{A}^{-1}}(1 - (\frac{L}{\mathcal{P}})N\mathcal{P}^{-1})$$

and with $H(S)$ as defined in §3,

$$G(B,S) = \{\alpha \in B \mid N_{L|k}(\alpha)R_k \in \mathcal{DS}I_k^2, \mathrm{disc}(\alpha)/N_{L|k}(\alpha) \in R_k, \mathcal{V}_\alpha\alpha \subset B\}.$$

The following observations ease the computation of this rather complicated formula in commonly occuring circumstances. Generally, if there are Candidate orders for u, R_L, the full ring of integers in L is a Candidiate order for u. When that occurs, i.e. $B = R_L$, then $\mathcal{A} = \mathcal{A}_0(u)$ and $t(\mathcal{A}) = [R_L^* : B^*] = 1$.

a) In particular, if $\mathcal{V}_u u$ contains an element η such that $\{1, \eta\}$ is a relative integral basis of L over k, then $J_u = \mathcal{A}_0(u)$ and R_L is the only Candidate order.
b) If each prime ideal $\mathcal{P}$ dividing $\mathcal{A}_0(u)$ lies in $\mathrm{Ram}_f(A)$, then R_L is the only Candidate order.
c) The ring R_L is the only Candidate order in the following commonly occuring situation.

Lemma 9.2. *Let k be the cyclotomic field $k_N = \mathbb{Q}(\cos 2\pi/N)$ where, if N is even, then $4 \mid N$. Let p be an odd prime such that $p^\alpha \mid N$. Suppose A is defined over k and $\Gamma = P(N(\mathcal{E})^+)$, where $\mathcal{E}$ is a maximal order or Eichler order of square-free level, has an element of order $2p^\alpha$. Then for* *each* $u = 1 + \mathrm{e}^{2\pi i/2p^t}$, $1 \le t \le \alpha$, $k(u) = L = C_N = \mathbb{Q}(\mathrm{e}^{2\pi i/N})$ *and* R_L *is the only Candidate order for* u.

Proof: Clearly $k_N(u)$ is a totally imaginary quadratic extension of k_N contained in C_N so that $k_N(u) = C_N$. Now

$$\mathrm{disc}(u)R_k = (4\sin^2 2\pi/2p^t)R_k = \mathcal{A}_0(u)^2\delta_{L|k}. \tag{17}$$

If N is composite, $\delta_{L|k} = R_k$. In that case, $\dfrac{(1+\mathrm{e}^{2\pi i/2p^t})^2}{(4\sin^2 2\pi/2p^t)}$ is a unit in R_L. Thus $\mathcal{V}_u = \mathcal{A}_0(u)^{-1}$ and R_L is the only Candidate order.

If $N = p^\alpha$, $\mathcal{A}_0(u)^2 = (4\sin^2 2\pi/2p^t)/(4\sin^2 2\pi/p^\alpha)R_k$. Now

$$\frac{4\sin^2 2\pi/p^\alpha}{4\sin^2 2\pi/2p^t}(1+\mathrm{e}^{2\pi i/2p^t})^2 = (1-\mathrm{e}^{2\pi i/p^\alpha})^2\tau$$

where $\tau \in R_L^*$. Note that $\mathcal{Q}_p = (1-\mathrm{e}^{2\pi i/p^\alpha})R_L$ is a prime ideal in L. Thus $\mathcal{A}_0(u)^{-1} \subset \mathcal{V}_u$ and $\mathcal{V}_u u \mid \mathcal{Q}_p$. So $\mathcal{V}_u u = R_L$ or $\mathcal{Q}_p$. But $\mathrm{e}^{2\pi i/p^\alpha} \notin \mathcal{V}_u u$ and so $\mathcal{V}_u = \mathcal{A}_0(u)^{-1}$ and R_L is the only candidate order. $\square$

For brevity, we refer to the term $[H(S) : H(S) \cap N(G(B,S))R_{k,+}^* k^{*2}]$ appearing in (16) as the H-index. Recall from §3, if k has class number one, $H(S)/k^{*2} \cong R_{f\cup S,+}^* k^{*2}/k^{*2}$. Clearly, if $R_{f\cup S,+}^* k^{*2} = R_{k,+}^* k^{*2}$ then the H-index is 1. Less obviously, we have

Lemma 9.3. *Suppose that* k *has class number one, and that* $\eta \in \mathcal{V}_u u$ *where* $\{1,\eta\}$ *is a relative integral basis with, additionally,* $\mathrm{disc}(\eta) \in R_k^*$. *If* $[R_{f\cup S,+}^* k^{*2} : R_{k,+}k^{*2}] = 2$, *then the H-index is 2.*

Proof: Under these circumstances $B = R_L$. Let $\alpha = a + b\eta$, $a, b \in R_k$ so that $N_{L|k}(\alpha) = a^2 + b^2|\eta|^2 + 2abRe(\eta)$ and $\mathrm{disc}(\alpha) = -b^2\mathrm{disc}(\eta)$. Now $R_{f\cup S,+}^*$ can be generated by $z \in R_k$ where zR_k is a square-free ideal. If $\alpha \in G(R_L, S)$ has $N_{L|k}(\alpha) \in zR_{k,+}^* k^{*2}$, then $zR_k \mid bR_k$ and so $zR_k \mid aR_k$. But that is a contradiction since then $z^2R_k \mid N_{L|k}(\alpha)R_k$. $\square$

Now let us consider the number of conjugacy classes of these maximal groups. Clearly if two groups are conjugate in $\mathrm{PGL}(2,\mathbb{R})$, then they are commensurable in the wide sense. But then the quaternion algebras are isomorphic and conjugacy is within $P(A^*)$. More precisely, we have (see [11, Theorem 8.4.7], [24, Theorem 3.5], [9]):

Lemma 9.4. *The groups* $P(N(\mathcal{E}_1)^+)$ *and* $P(N(\mathcal{E}_2)^+)$ *are conjugate in* $\mathrm{PGL}(2,\mathbb{R})$ *if and only if they are defined by quaternion algebras* A_1, A_2 *over the field* k *which admits an automorphism* τ *such that* $\tau(\mathrm{Ram}(A_1)) = \mathrm{Ram}(A_2)$ *and the induced isomorphism* $\tau^* : A_1 \to A_2$ *is such that* $\tau^*(\mathcal{E}_1) = c\mathcal{E}_2c^{-1}$ *for some* $c \in A_2^*$.

To apply this we need to know about the Galois automorphisms of the field k and also the number of conjugacy classes within a quaternion algebra of maximal or Eichler orders of square-free level. These are referred to as *type numbers*. In these cases where the quaternion algebra satisfies the Eichler condition, i.e. there is an infinite unramified place, the type number is the order of the group

$$T_S(A) = \frac{I_k}{P_{k,\infty}\mathcal{D}\mathcal{S}I_k^2}$$

where $P_{k,\infty}$ is the subgroup of principal ideals which have a generator in k_∞^*. If the order is maximal $S = \emptyset$ and $\mathcal{S}$ is trivial. It is useful to note that the order of the ray class group $I_k/P_{k,\infty}$ is $\dfrac{2^{[k:\mathbb{Q}]-1}h}{[R_k^* : R_{k,\infty}^*]}$.

10. Degree 4

We continue with the assumption that $\Gamma = P(N(\mathcal{E})^+)$ is a maximal arithmetic VE group whose signature is not of the form $(0; 2, 2, 2, N)$. In the cases where $[k : \mathbb{Q}] = 4$, we can have torsion of orders $\{2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24, 30\}$, and we here consider in detail the fields which give rise to the higher orders of torsion. Note that several examples of arithmetic triangle and VE groups are known to be defined over fields of degree 4 [9, 20, 21, 22]. We make liberal use of these results and also of the PARI-gp calculator to obtain number theoretic data on the fields involved. With notation as before, for a maximal arithmetic VE group of signature $(0; m_1, m_2, m_3, m_4)$ we have

$$(18) \qquad (2 - \sum_{i=1}^{4} \frac{1}{m_i})2^{m(S)} = R \prod_{\mathcal{P}\in \mathrm{Ram}_f(A)} (N\mathcal{P} - 1) \prod_{\mathcal{Q}\in S} (N\mathcal{Q} + 1).$$

Notationally, we will use $\mathcal{P}$ for primes in $\mathrm{Ram}_f(A)$ and $\mathcal{Q}$ for primes in S and their suffixes will give their norm.

A. $k = \mathbb{Q}(\cos 2\pi/15)$ so that torsion of orders 2,3,4,5,6,10,15,30 is possible. Note $\Delta_k = 1125$, $R = 1/15$ and there are primes of norms 5,9,16,29,31,59,61,etc. From (18), we must have $\mathrm{Ram}_f(A) = \{\mathcal{P}\}$ and $S = \emptyset$ or $\{\mathcal{Q}\}$.

a) $\mathrm{Ram}_f(A) = \{\mathcal{P}_5\}$ in which case $P(N(\mathcal{O})^+)$ has signature $(0; 2, 3, 30)$ with $2^{m(\emptyset)} = 2 = [R_{f,+}^* : {R_f^*}^2] = [R_{k,+}^* : {R_k^*}^2]$. So for $S = \{\mathcal{Q}\}$, $2^{m(S)} = [R_{f\cup S,+}^* : {R_{f\cup S}^*}^2] \leq 4$ and (18) yields $N\mathcal{Q} = 9$ or 16.

(1) $S = \{\mathcal{Q}_9\}$. Since $\mathcal{Q}_9$ does not split in C_{15}, there is no 30,15,10,6 or 5 torsion in Γ and there is also no 4 torsion. Thus (18) has no solution.

(2) $S = \{\mathcal{Q}_{16}\}$. Now $\mathcal{Q}_{16}$ splits in $L = C_{15}$ so that Γ has torsion of all orders dividing 30. Also $[R_{f\cup S,+}^* : {R_{f\cup S}^*}^2] = 4$. Note that, for $u = 1 + \mathrm{e}^{2\pi i/30}$, $\{1, u\}$ is a relative integral basis and $\mathrm{disc}(u) \in R_k^*$. Thus there is just one Candidate order R_L for u and $\ell(R_L) = 1$ since $h(L) = 1$, $(L/\mathcal{P}_5) = -1$, $(L/\mathcal{Q}_{16}) = 1$ and the H-index is 2 by Lemma 9.3. Likewise, there is only one Candidate order R_L for $u = 1 + \mathrm{e}^{2\pi i/15}$, and, as above, $\ell(R_L) = 1$.This is also true for $u = 1 + \mathrm{e}^{2\pi i/10}$ and $u = 1 + \mathrm{e}^{2\pi i/6}$ by Lemma 9.2. For $u = 1 + \mathrm{e}^{2\pi i/5}$, $\mathcal{A}_0(u) = \mathcal{P}_5$ and, using remark b) in §9 there is again only one Candidate order for this u. Note that $\mathcal{P}_5$ splits in $k(\sqrt{-1})$ and so there are no elements of order 4. For $u = 1 + \mathrm{e}^{2\pi i/3}$, $\mathcal{A}_0(u) = \mathcal{P}_9$ and $\mathcal{V}_u = R_k$. So there are two Candidate orders and at

least two classes of order 3. Thus if Γ ha signature $(g; m_1, m_2, \ldots, m_r)$, we have $m_1 = 30, m_2 = 3$, and the remaining $m_i \in \{2, 3\}$ with $2(g-1) + \sum_{i=1}^{r}(1 - 1/m_i) = 17/15$. Thus Γ must have signature $(0; 2, 2, 2, 3, 30)$ and is not a VE group.

b) $\mathrm{Ram}_f(A) = \{\mathcal{P}_9\}$ in which case $P(N(\mathcal{O})^+)$ has signature $(0; 2, 5, 30)$, $2^{m(\emptyset)} = 2$. So for $S = \{\mathcal{Q}\}$ then $N\mathcal{Q} = 5$. But in that case, Γ has no torsion of orders 3,4,6,10,15 or 30 and $2^{m(S)} = 4$. But then (18) has no solution.

c) $\mathrm{Ram}_f(A) = \{\mathcal{P}_{16}\}$. Here $2^{m(\emptyset)} = 4$ so $S = \emptyset$ or $\{\mathcal{Q}_5\}$ or $\{\mathcal{Q}_9\}$. But $\mathcal{P}_{16}$ splits in $L = C_{15}$ and so none of these groups have torsion of orders 30,15,10,6,5 or 3. For $\Gamma = P(N(\mathcal{O})^+)$, this group has signature $(0; 2, 2, 2, 4)$ and there is one conjugacy class of such groups.

d) $\mathrm{Ram}_f(A) = \{\mathcal{P}_{29}\}$ where this is one of the four primes of norm 29.We must have $S = \emptyset$ and we compute that $2^{m(\emptyset)} = 2$. Since $\mathcal{P}_{29}$ does not split in C_{15}, $P(N(\mathcal{O})^+)$ has torsion of all orders dividing 30. For $u = 1 + \mathrm{e}^{2\pi i/30}$, there is only one Candidate order R_L and easily $\ell(R_L) = 1$. For $u = 1 + \mathrm{e}^{2\pi i/15}$ there is again only one Candidate order and by Lemma 9.2 this is also true for $u = 1 + \mathrm{e}^{2\pi i/10}$ and $u = 1 + \mathrm{e}^{2\pi i/6}$. For $u = 1 + \mathrm{e}^{2\pi i/5}$, $\mathcal{A}_0(u) = \mathcal{P}_5$ and $J_u = R_k$ so there are two Candidate orders and at least two classes of order 5. The same holds for $u = 1+\mathrm{e}^{2\pi i/3}$. Thus if Γ has signature $(g; m_1, m_2 \ldots, m_r)$, we have $m_1 = 30$, $m_2 = 5$, $m_3 = 3$ and the remaining $m_i \in \{2, 3, 4, 5\}$. Thus Γ must have signature $(0; 2, 3, 5, 30)$ and is a maximal VE group. [This can be consolidated by showing that $\ell(B) = 1$ for the non-maximal Candidate orders B arising from $u = 1 + \mathrm{e}^{2\pi i/5}$ and $u = 1 + \mathrm{e}^{2\pi i/3}$ and a similar calculation for elements of order 2, so giving one additional class of groups of orders 2,3 and 5. For $u = 1 + \mathrm{e}^{2\pi i/5}$, we have $B = R_k + R_k\mathrm{e}^{2\pi i/5}$ and so $[R_L^* : B^*] = 6$ (see **TA**) thus giving $\ell(B) = 1$. A similar argument holds for $u = 1 + \mathrm{e}^{2\pi i/3}$. For elements of order 2, note that $\mathcal{P}_{29}$ splits in $k(\sqrt{-1})$ and so there are no elements of order 4 or of order 2 corresponding to $1 \in \overline{H(\emptyset)}$. Thus all elements of order 2 correspond to $(2 + 2\cos 2\pi/30)$ in $\overline{H(\emptyset)}$ giving $u = 2i\sin 2\pi/30$. Then $R_L = R_k \oplus (2i\sin 2\pi/30 + 2\cos 2\pi/30)/2R_k$ so that $B = R_k + 2\mathrm{e}^{2\pi i/30}R_k$. Note that $R_L^*/\langle \mathrm{e}^{2\pi i/15}\rangle R_k^*$ has order 2 generated by $1 - \mathrm{e}^{2\pi i/15} = -\mathrm{e}^{2\pi i/30}2i\sin 2\pi/30$. Thus $[R_L^* : B^*] = 15$ and formula (16) yields $\ell(B) = 1$].

In this case there are four commensurability classes of such groups and the type number is one for each. Thus there are 4 conjugacy classes of such groups.

e) The only remaining possibilities will be $\mathrm{Ram}_f(A) = \{\mathcal{P}\}$, $S = \emptyset$ and $N\mathcal{P} = 31, 59, 61$ or 89. When $N\mathcal{P} = 31, 61$, $\mathcal{P}$ splits in C_{15} and Γ can only have 2 and 4 torsion so (18) has no solution. For $N\mathcal{P} = 59, 89$ we compute that $2^{m(\emptyset)} = 2$ and again (18) has no solution.

B. $k = \mathbb{Q}(\cos 2\pi/24)$ so possible torsion is $\{2, 3, 4, 6, 8, 12, 24\}$. Here $\Delta_k = 2304, R = 1/4$ and k has primes of norms $2, 9, 23, 25, 47, 71, 73$,etc. From (18), we have $\mathrm{Ram}_f(A) = \{\mathcal{P}\}$ and $S = \emptyset$ or $\{\mathcal{Q}\}$.

a) $\mathrm{Ram}_f(A) = \{\mathcal{P}_2\}$ in which case $P(N(\mathcal{O})^+)$ has signature $(0; 2, 3, 24)$, $[R^*_{f,+} : {R^*_f}^2] = [R^*_{k,+} : {R^*_k}^2] = 2$. Thus when $S = \{\mathcal{Q}\}$, $2^{m(S)} \leq 4$ and so $N\mathcal{Q} \leq 27$.

(1) $S = \{\mathcal{Q}_9\}$. With $L = C_{24}$, $\mathcal{Q}_9$ splits in L and Γ has torsion of all orders dividing 24. We compute that $[R^*_{f\cup S,+} : {R^*_{f\cup S}}^2] = 4$ and so $2 - \sum_{i=1}^4 1/m_i = 5/8$. With $u = 1 + \mathrm{e}^{2\pi i/24}$, R_L is the only Candidate order and by Lemma 9.3, the H-index is 2 so that $\ell(R_L) = 1$. With $u = 1 + \mathrm{e}^{2\pi i/12}$, R_L is again the only Candidate order. With $u = 1 + \mathrm{e}^{2\pi i/8}$ and $u = 1 + \mathrm{e}^{2\pi i/4}$, $\mathcal{A}_0(u)$ is a power of $\mathcal{P}_2$ and since $\mathcal{P}_2 \in \mathrm{Ram}_f(A)$ R_L is the only Candidate order in these cases also. With $u = 1 + \mathrm{e}^{2\pi i/6}$, R_L is the only Candidate order by Lemma 9.2. With $u = 1 + \mathrm{e}^{2\pi i/3}$, $\mathcal{A}_0(u) = \mathcal{P}_9$, $\mathcal{V}_u = R_k$ so there are two Candidate orders. If Γ has signature $(g; m_1, m_2, \ldots, m_r)$ then $2(g-1) + \sum(1 - 1/m_i) = 5/8$ with $m_1 = 24, m_2 = 3$ and the remaining $m_i \in \{2, 3\}$. Thus γ must have signature $(0; 2, 2, 3, 24)$. Furthermore, since there is just one conjugacy class of $P(N(\mathcal{O})^+)$, the triangle group $(0; 2, 3, 24)$, there is also just one conjugacy class of this arithmetic VE group of signature $(0; 2, 2, 3, 24)$.

(2) $S = \{\mathcal{Q}_{23}\}$ where $\mathcal{Q}_{23}$ is one of the primes of norm 23. But $\mathcal{Q}_{23}$ does not split in C_{24} so there is no torsion of order > 2 and so Γ cannot be a VE group.

(3) $S = \{\mathcal{Q}_{25}\}$ where $\mathcal{Q}_{25}$ is one of the two primes of norm 25. In this case Γ has torsion of all orders dividing 24 and we compute that $[R^*_{f\cup S,+} : {R^*_{f\cup S}}^2] = 4$. For $u = 1 + \mathrm{e}^{2\pi i/24}$ there is just one Candidate order R_L and by Lemma 9.3, the H-index is 2 and $\ell(R_L) = 1$. As above there is only one class of subgroups of each of the orders 12,8,6,4 and at least two of order 3. Solving $2(g-1) + \sum(1 - 1/m_i) = 13/8$ does not yield a VE group.

b) $\mathrm{Ram}_f(A) = \{\mathcal{P}_9\}$. But $\mathcal{P}_9$ splits in C_{24} so that there is only 2 torsion at most and Γ is not a VE group.

c) $\mathrm{Ram}_f(A) = \{\mathcal{P}_{23}\}$. In this case we must have $S = \emptyset$ and $P(N(\mathcal{O})^+)$ will have torsion of all orders dividing 24. Now $2^{m(\emptyset)} = 2$ (This follows from the formula for $\ell(R_L)$. It shows that the H-index must be one. If $2^{m(\emptyset)} = 4$, then Lemma 9.3 would imply that the H-index was 2). But then $(2 - \sum_{i=1}^4 1/m_i)2 = 11/2$ has no solutions.

d) The remaining possibility is $\mathrm{Ram}_f(A) = \{\mathcal{P}_{25}\}$ which, as at b) above, does not give a VE group.

C. $k = \mathbb{Q}(\cos 2\pi/20)$. Thus $\Delta_k = 2000$, $R = 1/6$ and k has primes of norms 4,5,19,41,59,61,79,81. Γ will have possible torsion of orders $\{2, 3, 4, 5, 6, 10, 20\}$. As before we have $\mathrm{Ram}_f(A) = \{\mathcal{P}\}$ and $S = \emptyset$ or $\{\mathcal{Q}\}$.

a) $\mathrm{Ram}_f(A) = \{\mathcal{P}_4\}$. Then $P(N(\mathcal{O})^+)$ has signature $(0; 2, 5, 20)$ and $[R^*_{f,+} : {R^*_f}^2] = [R^*_{k,+} : {R^*_k}^2] = 2$. Thus the only possibility is $S = \{\mathcal{Q}_5\}$. Now $\mathcal{Q}_5$ splits in C_{20} and $P(N(\mathcal{E})^+)$ has torsion of all orders dividing 20 and furthermore $[R^*_{f\cup S,+} : {R^*_{f\cup S}}^2] = 4$. Arguing as before, for $u = 1 + \mathrm{e}^{2\pi i/20}$, there is just one Candidate order R_L and $\ell(R_L) = 1$ using Lemma 9.3.

For $u = 1 + e^{2\pi i/10}$ there is only one Candidate order by Lemma 9.2. For $u = 1 + e^{2\pi i/5}$, $\mathcal{A}_0(u) = \mathcal{P}_5$ and since u is a unit, there are two Candidate orders and at least two classes of groups of order 5. Since $\mathcal{P}_4$ splits in $k(\sqrt{-3})$ there are no elements of orders 3 or 6. For $u = 1 + e^{2\pi i/4}$, $\mathcal{A}_0(u) = \mathcal{P}_4^2$ and with $\mathcal{P}_4 \in \mathrm{Ram}_f(A)$, there will only be one Candidate order. In solving $2(g-1) + \sum_{i=1}^{r}(1 - 1/m_i) = 3/4$ with $m_1 = 20$, $m_2 = 5$ and the remaining $m_i \in \{2, 5\}$ we obtain that $P(N(\mathcal{E})^+)$ must have signature $(0; 2, 2, 5, 20)$. There will be just one conjugacy class of such arithmetic VE groups. For there is just one commensurability class and the type number of this Eichler order will be one as $T_S(A)$ is a factor group of $T_\emptyset(A)$ which must be one since it gives the triangle group $(0; 2, 5, 20)$.

b) $\mathrm{Ram}_f(A) = \{\mathcal{P}_5\}$. Since $\mathcal{P}_5$ splits in C_{20} no group in the commensurability class has elements of orders 20,10,5 or 4. Now $P(N(\mathcal{O})^+)$ has signature $(0; 2, 2, 2, 3)$. Here $[R^*_{k,+} : {R^*_k}^2] = 2$ and $[R^*_{f,+} : {R^*_f}^2] = 4$. Thuis if $S = \{\mathcal{Q}\}$ then $\mathcal{Q} = \mathcal{Q}_4$. But then Γ has no 6 torsion and cannot be a VE group.

c) $\mathrm{Ram}_f(A) = \{\mathcal{P}_{19}\}$ where $\mathcal{P}_{19}$ is one of the prime ideals of norm 19. We must have $S = \emptyset$ and Γ has torsion of all orders dividing 20. We calculate that $[R^*_{f,+} : {R^*_f}^2] = 2$ and observe that Γ has no 6-torsion. With $u = 1 + e^{2\pi i/20}$, there is just one Candidate order R_L and $\ell(R_L) = 1$. By Lemma 9.2, there is just one Candidate order for $u = 1 + e^{2\pi i/10}$. But with $m_1 = 20$ and the remaining $m_i \in \{2, 3, 4, 5\}$ Γ cannot be a VE group.

D. $k = \mathbb{Q}(\cos 2\pi/16)$. In this case $\Delta_k = 2048$, $R = 5/24$ and k has primes of norms 2,17,31,47,49. Γ may have torsion of orders 2,3,4,6,8,16. Now $[R^*_{k,+} : {R^*_k}^2] = 1$ and so we must have $\mathrm{Ram}_f(A) = \{\mathcal{P}_2\}$ and $S = \emptyset$ or $\{\mathcal{Q}\}$. Here $P(N(\mathcal{O})^+)$ has signature $(0; 2, 3, 16)$ and $[R^*_{f,+} {R^*_f}^2] = 2$. The only possibility is that $S = \{\mathcal{Q}_{17}\}$. Now $\mathcal{Q}_{17}$ splits in C_{16} and so there is torsion of all orders dividing 16. We calculate that $[R^*_{f\cup S,+} : {R^*_{f\cup S}}^2] = 4$. For $u = 1 + e^{2\pi i/16}$, $L = C_{16}$, $\delta_{L|k} = \mathcal{P}_2$ and $\mathcal{A}_0(u) = R_k$. So there is just one Candidiate order R_L. Taking $\mathcal{Q}_{17} = (2x - 1)R_k$ where $x = 2\cos 2\pi/16$, then $R^*_{f\cup S,+}$ contains $(2x-1)u$ where $u \in R^*_k$. But then a similar argument as that employed in Lemma 9.3 shows that this element does not belong to $N_{L|k}(G(R_L, S))R^*_{k,+}{k^*}^2$ so that the H-index is 2 and $\ell(R_L) = 1$. Since $\mathrm{Ram}_f(A) = \{\mathcal{P}_2\}$, there is only one Candidate order for each of $u = 1 + e^{2\pi i/8}$ and $u = 1 + e^{2\pi i/4}$. Also $\mathcal{Q}_{17}$ does not split in $k(\sqrt{-3})$ so there are no elements of orders 3 or 6. But then with only possibly 2-torsion remaining (18) has no solution.

For the remaining fields of degreee 4, taking the above arguments together with the arithmetic data and the results in [1, 2, 9, 18, 19, 20, 21, 22] we easily obtain all conjugacy classes of the arithmetic VE groups with $[k : \mathbb{Q}] = 4$ which are not subgroups of an arithmetic triangle group. The list will appear in a forthcoming paper.

11. Concluding remarks

Finally, for maximal arithmetic VE groups, we are left with the field degrees $[k : \mathbb{Q}] \leq 3$. Here the computational calculations are time consuming and the number of conjugucy classes of maximal arithmetic VE groups is enormous.

As already mentioned any VE group is either a subgroup of a triangle group or a subgroup of a VE group. Hence, for the final classification, first we have to determine the conjugacy classes which belong to maximal arithmetic VE groups with $[k:\mathbb{Q}] \leq 3$ and, hence, to arithmetic VE groups with $[k:\mathbb{Q}] \leq 3$ which are not subgroups of an arithmetic triangle group (see $[1,2,9]$ for a list of such VE groups). Second, we have to determine the conjugacy classes which belong to subgroups of some arithmetic triangle groups. The arithmetic triangle groups can be found in [21]. Again, this number is also enormous. We give here two examples.

Example 1 In the triangle groups $(0;2,3,24)$ we have -up to conjugacy- subgroups of the following signatures:

Signature	Index	Number of conjugacy classes
(0;2,2,2,8)	3	1
(0;2,2,3,6)	4	1
(0;2,2,8,8)	6	1
(0;3,3,3,4)	6	1
(0;2,2,6,12)	6	1
(0;2,2,6,8)	7	1
(0;2,3,4,24)	7	2
(0;3,3,6,6)	8	1
(0;3,3,4,12)	8	1
(0;2,4,12,24)	9	1
(0;2,6,8,12)	9	1
(0;3,3,24,24)	10	1
(0;3,12,24,24)	12	1
(0;4,8,12,24)	12	1
(0;6,6,12,12)	12	1
(0;8,8,8,8)	12	1

Example 2 In the triangle group $(0;2,3,30)$ we have -up to conjugacy- subgroups of the following signatures:

Signature	Index	Number of conjugacy classes
(0;2,2,2,10)	3	1
(0;2,3,3,6)	5	1
(0;3,3,3,5)	6	1
(0;2,2,10,10)	6	1
(0;2,2,6,30)	6	1
(0;2,3,6,15)	7	1
(0;2,3,5,30)	7	2
(0;3,3,5,15)	8	1
(0;2,6,10,30)	9	1
(0;2,5,15,30)	9	1
(0;3,6,10,15)	10	1
(0;6,6,30,30)	12	1
(0;5,10,15,30)	12	1
(0;10,10,10,10)	12	1

We will give the complete classification for $[k : \mathbb{Q}] \leq 3$ in a forthcoming paper, together with the VE groups which are subgroups of arithmetic triangle groups.

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