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A Note On Quasi-Essential Submodule of QTAG-module

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Abstract. The concept of quasi-essential submodules has been studied in [1] and different characterizations were obtained in terms of center of h-purity. In this paper we characterize quasi-essential subsocles which shows that each quasi-essential subscole is, indeed, the center of h-purity (Theorem 2.6).

Introduction: The concept of quasi-essential submodules has been introduced in [1]. A submodule N of a QTAG-module M is called quasi-essential if $M = T + K$ for a complement K of N and T an h-pure submodule of M containing N. The concept of center of h-purity was also introduced as: A submodule N of M is called center of h-purity if every complement of N is h-pure in M. After imposing one more condition on M, many results have been proved to see the relation between center of h-purity and quasi-essential submodules. It has been seen that all subsocles of M^1 are quasi-essential and condition has been obtained under which every quasi-essential subsocle is center of h-purity. In this paper we obtain a similar characterization.

1. Preliminaries: Rings considered here are with unity $(1 \neq 0)$ and modules are unital QTAG-module. A module in which the lattice of its submodules is totally ordered is called a serial module; in addition if it has finite composition length it is called uniserial module. An element $x \in M$ is called uniform if xR is a non zero uniform (hence uniserial) submodule of M. If $x \in M$ is uniform then $e(x) = d(xR)$ (The composition length of xR), $H_M(x) = \sup\{d(yR/xR)/x \in yR$ and $y \in M$ is uniform $\}$ are called exponent of x and height of x in M respectively. For any $n \geqslant 0, H_n(M) = \{x \in M/H_M(x) \geqslant n\}$. A submodule N of M is called h-pure in M if $H_k(N) = N \cap H_k(M)$ for all $k \geqslant 0$, N is h-neat in M if $H_1(N) = N \cap H_1(M)$. The module M is called h-divisible if $H_1(M) = M$. We denote by M^1 as the submodule generated by the uniform elements of infinite height.
Here we impose one more condition on M
$(\mathbf{A})$: For any finitely generated submodule N of $M, R/ann(N)$ is right artinian. For

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other basic concepts of QTAG-module one may see [1,3].

2. Quasi-Essential Submodule

Firstly we state the following lemma, since the proof is of set theoretic nature, therefore it is omitted.

Lemma 2.1: If M is a QTAG-module such that $M = N \oplus K$ such that $N_0 \subseteq N$ and $K_0 \subseteq K$ are submodules, if N' is a complement of N_0 in N and K' is a complement of K_0 in K, then $N' \oplus K'$ is a complement of $K_0 \oplus N_0$ in M.

Lemma 2.2: If S is a quasi-essential subsocle of a QATG-module M and N is an h-pure submodule of M with $Soc(N) = Soc(H_n(M))$. Then $S \cap H_n(M)$ is a quasi-essential subsocle of N.

Proof: Let $N_0 = S \cap H_n(M)$ and $S = N_0 \oplus K_0$, then trivially $K_0 \cap H_n(M) = 0$. Let K be a complement of N in M containing K_0; then since N is h-pure and M/N is bounded, we get $M = K \oplus N$. Now let N' be a complement of N_0 in N and T be an h-pure submodule of N containing N_0. If K' is complement of K_0 in K, then $N' \oplus K'$ is complement of S in M by Lemma 2.1. Now

$$\begin{aligned}(T \oplus K) \cap H_n(M) &= (T \oplus K) \cap (H_n(K) \oplus H_n(N)) \\ &= H_n(K) + (T \oplus K) \cap H_n(N)\end{aligned}$$

Now let $x \in (T \oplus K) \cap H_n(N)$ then $x = a + b, a \in T, b \in K$ and $x \in H_n(N)$, then $x - a = b \in K \cap N = 0$, so $x \in T \cap H_n(N) = H_n(T)$. Hence, we get $(T \oplus K) \cap H_n(M) = H_n(K) \oplus H_n(T) = H_n(K \oplus T)$; so $T \oplus K$ is an h-pure submodule of M. Trivially $S \subseteq T \oplus K$. Since S ie quasi-essential submodule of M, we get $M = T \oplus K + N' \oplus K' = (T + N') \oplus K$. Hence, $N = T + N'$. Therefore, $S \cap H_n(M)$ is quasi-essential in N.

Lemma 2.3: If S be a quasi-essential subsocle of a QTAG-module M satisfying condition (A) and if $Soc(H_n(M)) \neq (S \cap H_n(M)) + Soc(H_{n+1}(M))$ for some $n \in Z^+$, then $S \subset Soc(H_n(M))$.

Proof: Let $A_0 = S \cap H_{n+1}(M)$ and $S = A_0 \oplus B_0$. Let $Soc(H_{n+1}(M))$ support a h-pure submodule A of M. Let B be a complement of A in M such that $B_0 \subset B$. Then as done in Lemma 2.2, $M = A \oplus B$. Let K be a h-pure submodule of B such that $Soc(K) = B_0$ and B' be a complement of K in B. Then B' is also a complement of B_0. Let A' be a complement of A_0 in A, then $A' \oplus B'$ is complement of S in M. Since S is quasi-essential in M and as done in Lemma 2.2, $A \oplus K$ is an h-pure submodule of M containing S. Therefore, $M = A \oplus K + A' \oplus B' = A \oplus (K \oplus B')$. Thus, we get $B = K \oplus B'$, so K is an absolute direct summand of B. Now appealing to [Theorem 12, 1], we get $Soc(H_{k+1}(B)) \subseteq B_0 \subseteq Soc(H_k(B))$ for some $k \in Z^+$. Since $Soc(H_n(M)) = Soc(A) \oplus Soc(H_n(B))$ and $Soc(H_n(M)) \neq (S \cap H_n(M)) + Soc(H_{n+1}(M))$, we get $Soc(H_n(B)) \subset B_0$. Thus $n \leqslant k$, so $B_0 \subseteq Soc(H_n(B))$. Now

$S = A_0 + B_0 \subseteq Soc(H_{n+1}(M)) \oplus Soc(H_n(B)) = Soc(H_n(M))$.

Lemma 2.4: If S is quasi-essential subsocle of a QTAG-module M satisfying condition (A) and is h-dense in M. Then either $S \subseteq M^1$ or $S = Soc(M)$.

Proof: Appealing to [Theorem 2.10, 2] we see that S supports an h-pure submodule and is quasi-essential. Now if $S \not\subset M^1$, then by [Theorem 12, 1], $Soc(H_{k+1}(M)) \subseteq S \supset Soc(H_k(M))$ for some $k \in Z^+$. Since $Soc(M) = S + Soc(H_{k+1}(M))$ and as $Soc(H_{k+1}(M)) \subseteq S$, we get $S = Soc(M)$.

Lemma 2.5: If S be a quasi essential subsocle of a QTAG-module M satisfying condition (A) and if $Soc(H_k(M)) = (S \cap H_k(M)) + Soc(H_{k-1}(M))$ for every $k > n$, then either $H_{n+1}(M)$ is h-divisible or $Soc(H_{n+1}(M)) \subset S$.

Proof: Let K be an h-pure submodule supported by $Soc(H_{n+1}(M))$, then $Soc(H_k(M)) = Soc(H_k(K))$ and $S \cap H_k(M) = S \cap H_k(K)$ for $k > n$, consequently $Soc(H_k(K)) = (S \cap H_k(K)) + Soc(H_{k+1}(K))$ for every $k > n$. Since K is h-pure and $Soc(H_{n+1}(M)) = Soc(K)$, we get $Soc(K) = Soc(H_{n+1}(K))$. Using induction it is easy to see that $Soc(H_{n+1}(K)) = (S \cap H_{n+1}(K)) + Soc(H_{n+m}(K))$ for all $m \geqslant 1$. Thus $S \cap H_{n+1}(K)$ is h-dense in $Soc(K)$ and is quasi-essential in $Soc(K)$ (see Lemma 2.2). Now by Lemma 2.4, either $S \cap H_{n+1}(K) \subseteq K^1$ or $S \cap H_{n+1}(K) = Soc(K)$. If $S \cap H_{n+1}(K) \subseteq K^1$, then as $S \cap H_{n+1}(K)$ is h-dense in K, therefore K is h-divisible; consequently $H_{n+1}(M)$ is h-divisible. If $S \cap Soc(H_{n+1}(K)) = Soc(K)$ then $S \cap Soc(H_{n+1}(M)) = Soc(H_{n+1}(M))$and we get $Soc(H_{n+1}(M)) \subset S$.

Now we state and prove the main result.

Theorem 2.6: If M is a QTAG-module satisfying condition (A) and S is a subsocle of M, then S is quasi-essential if and only if one of the following conditions holds:

(i) $S \subset M^1$.
(ii) $Soc(H_{n+1}(M)) \subseteq S \subseteq Soc(H_n(M))$ for some $n \geqslant 0$.

Proof: The sufficiency follows from [Corollary 2, Corollary 8, 1]. Conversely, suppose S is quasi-essential. Now if $Soc(H_n(M)) \neq (S \cap H_n(M)) + Soc(H_{n+1}(M))$ for arbitrarily large n, then by Lemma 2.3, $S \subset M^1$. If not so, then there exists $n \in Z^+$ such that $Soc(H_n(M)) \neq (S \cap H_n(M)) + Soc(H_{n+1}(M))$ and equality holds for every $k > n$. Thus $S \subseteq Soc(H_n(M))$ by Lemma 2.3 and either $Soc(H_{n+1}(M)) \subseteq S$ or $H_{n+1}(M)$ is h-divisible by Lemma 2.5. If $Soc(H_{n+1}(M)) \subseteq S$, then the condition (ii) is satisfied. If $H_{n+1}(M)$ is h-divisible then every subsocles of M will support an h-pure submodule. Thus S supports an absolute direct summand. Therefore appealing to [Theorem 12, 1], we see that either (i) or (ii) is satisfied.

Appealing to above theorem, the following immediately follows:

Corollary 2.7: If M is a QTAG-module satisfying condition (A) then a subsocle S of M supports an absolute direct summand if and only if S is quasi-essential and $S \subset M^1$ implies $S \subset D$, where D is the maximal h-divisible submodule of M.

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