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Some Decomposition Theorems on QTAG-module

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ABSTRACT. It has been observed by different authors that QTAG-modules behave very much like torsion abelian groups. In this paper, in section 3, we characterize quasi-essential submodules (Theorem 3.9) and further find a characterization for an h-pure submodule to be a direct summand (Theorem 3.11). In section 4, we obtained a necessary and sufficient condition for a submodule to be contained in a minimal h-pure submodule (Theorem 4.3).

1. Introduction: Following [9], an unital module M_R is called QTAG-module if it satisfies the following condition:
(I) Any finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.
The structure theory of such modules has been developed by various authors. In section 3, we characterize the quasi-essential submodules and established various conditions under which h-pure submodules are direct summands. In section 4, we obtained necessary and sufficient condition for an h-pure submodules to be a minimal h-pure submodule containing a given submodule.

2. Preliminaries: Rings considered here are with unity $(1 \neq 0)$ and modules are unital QTAG-module. An element $x \in M$ is called uniform if xR is a non zero uniform (hence uniserial) submodule of M. For any module $A_R, d(A)$ denotes the length of the composition series. If $x \in M$ is uniform then $e(x) = d(xR), H_M(x) = \sup\{d(yR/xR)/x \in yR \text{ and } y \in M \text{ is a uniform element in } M\}$ are called exponent of x and height of x in M respectively. For any $n \geqslant 0, H_n(M) = \{x \in M/H_M(x) \geqslant n\}$. A submodule N of M is called h-pure in M if $H_n(N) = N \cap H_n(M)$ for all n and M is called h-divisible if $H_1(M) = M$. A submodule B of M is called a basic submodule if B is h-pure in $M, M/B$ is h-divisible and B is a direct sum of uniserial submodules. We denote by M^1 as the submodule generated by the uniform elements of infinite height. For other basic concepts of QTAG-module one may see [1,3,4,7,8,9].

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3. Quasi-essential Submodules

Firstly we state the following lemmas. Since their proofs are of set theoretic nature, therefore the same is omitted.

Lemma 3.1: If M is QTAG-module and $K \subseteq N \subseteq M$ and T is a complement of K then $T \cap N$ is complement of K in N. Conversely, if L is complement of K in N, then $L = T \cap K$ whenever T is complement of K of M containing L.

Lemma 3.2: If M is QTAG-module and $K \subseteq N \subseteq M$. If T is a complement of K, then every complement of $T \cap N$ in T is a complement of a complement of N in M.

Lemma 3.3: If M is QTAG-module and $K \subseteq N \subseteq M$ and T is a complement of K in N. Then a submodule L containing T is a complement of K in M if and only if L/T is a complement of N/T in M/T.

Lemma 3.4: If M is QTAG-module and N, K are submodules of M such that $N \cap K = 0$, then a submodule T containing K is a complement of N in M if and only if T/K is a complement of $(N \oplus K)/K$ in M/K.

Now we prove few Lemmas which are used later and are of independent interest.

Lemma 3.5: If M is QTAG-module and $K \subseteq N \subseteq T$ are submodules of M and N is h-pure submodules of M. Then T/K is h-pure in M/K if and only if T is h-pure in M.

Proof: If T is h-pure in M then trivially T/K is h-pure in M/K. Conversely, let T/K be h-pure in M/K and let f be the canonical map defined as $f : M/K \longrightarrow M/N$ such that $f(x + K) = x + N$ then $ker\ f \subseteq T/K$ and $f(T/K) = T/N$, therefore T/N is h-pure in M/N. Since N is h-pure in M, so T is h-pure in M.

Lemma 3.6: If M is QTAG-module, N is a submodule of M and B is a h-pure, h-dense submodule of N. Then there exists a h-pure, h-dense submodule K of M such that $K \cap N = B$.

Proof: Since B is h-dense in N, we have $M/B = N/B \oplus K/B$ for some submodule K of M, then by [Proposition 2.5, 6], K is h-pure in M and trivially $K \cap N = B$.

Proposition 3.7: Let M be a QTAG-module and S be a subsocle of $Soc(M)$ such that $S \not\subseteq M^1$. Let K be a maximal h-pure submodule of M such that $Soc(K) \subseteq S$. Then $(S + K)/K$ is contained in the h-reduced part of $(M/K)^1$.

Proof: Trivially S has at least one element of finite height, therefore, there exists at least one h-pure submodule T of M such that $Soc(T) \subseteq S$. Using Zorn's Lemma we get a maximal h-pure submodule K of M such that $Soc(K) \subseteq S$. Trivially $(S + K)/K \subseteq Soc(M/K)$. If $(S + K)/K$ has an element of finite height then $M/K = K'/K \oplus L/K$ such that $Soc(K'/K) \subseteq (S + K)/K$, hence $Soc(K') \subseteq S$ and since K' is h-pure in M, we get a contradiction to the maximality of K. Therefore, $(S+K)/K \subseteq (M/K)^1$. Since h-divisible submodules are absolute summands, therefore, we ultimately get $(S+K)/K$ contained in the h-reduced part of $(M/K)^1$.

As defined in [5], A submodule N of a QTAG-module M is called quasi-essential of M if $M = T + K$, where T is a complement of N and K is h-pure submodule of M containing N.

Proposition 3.8: If M is a QTAG-module such that $M = B \oplus D$ where B is bounded and D is h-divisible, then every h-pure submodule K of M is the direct sum of bounded and h-divisible submodule.

Proof: Let $M = B \oplus D$ where B is bounded and D is h-divisible. Let K be an h-pure submodule of M, then $K \cap D = K^1$. Let T be a complement of K^1 in K, then $T \cap D = 0$ and T is therefore bounded. Hence, $K = T \oplus (K \cap D)$ where $(K \cap D) \cong K/T$ is h-divisible.

Proposition 3.9: If M be a QTAG-module and $N \subseteq M$, then N is quasi-essential submodule of M if and only if K/T is an absolute summand of M/T whenever K is a h-pure submodule of M containing N and T is a complement of K.

Proof: Let A/T be a complement of K/T in M/T, then by Lemma 3.4, A is a complement of N and if N is quasi-essential, then we get $M = A + K$. Therefore, $M/T = A/T \oplus K/T$. Conversely, let A be a complement of N in M, then by Lemma 3.2, $A \cap K$ is a complement of N in K. Hence, $K/(A \cap K)$ is an absolute summand of $M/(A \cap K)$ and by Lemma 3.4, $A/(A \cap K)$ is a complement of $K/(A \cap K)$ in $M/(A \cap K)$. Therefore, $M/(A \cap K) = A/(A \cap K) \oplus K/(A \cap K)$ and we get $M = A + K$. Therefore, N is quasi-essential submodule of M.

Theorem 3.10: If M is a QTAG-module and S is a subsocle of M^1. Then every h-pure submodule of M containing S is summand of M if and only if M is a direct sum of a bounded submodule and h-divisible submodule.

Proof: Let K be a complement of M^1, then K is h-pure and M/K is h-divisible [Theorem 7 and Proposition 13, 1]. If K is unbounded then K contains a proper basic submodule B of K and hence, $M/B = K/B \oplus T/B$ where T can be chosen to contain M^1 as $K \cap M^1 = 0$. Appealing to [Proposition 2.5, 6], T is h-pure submodule of M and $S \subseteq T$. Therefore, $M = T \oplus A$ and A is h-divisible, which is a contradiction. Hence, K is bounded and therefore, K is a summand of M i.e. $M = K \oplus D$ where D is h-divisible. For the converse we refer to Proposition 3.8.

Theorem 3.11: If M is a QTAG-module and S is a subsocle of M. Then the following are equivalent:

(i) $S \supseteq Soc(M^1)$ and every h-pure submodule of M containing S is a summand of M.
(ii) Every h-pure submodule of M containing S is a cobounded summand of M.
(iii) $S \supseteq Soc(H_n(M))$, for some positive integer n.

Proof: We establish (ii)$\rightarrow$ (i)$\rightarrow$ (iii)$\rightarrow$ (ii)
(ii)$\rightarrow$ (i) Let x be a uniform element in $Soc(M^1)$ and $x \notin S$, then $xR \cap S = 0$. Embedding S into a complement K of xR. Then K is h-pure submodule of M and M/K is h-divisible, which is a contradiction. Therefore, $x \in S$ and we get $Soc(M^1) \subseteq S$.

(i)$\rightarrow$ (iii) Let $S = M^1$, then by Theorem 3.10, $M = B \oplus D$ where B is bounded and D is h-divisible. Let $H_n(B) = 0$, then clearly $Soc(H_n(M)) \subseteq S$. Let $S \neq M^1$ and K be a maximal h-pure submodule of M such that $Soc(K) \subseteq S$, then by Proposition 3.7, $(K + S)/K \subseteq (M/K)^1$. Now every h-pure submodule A/K of M/K containing $(K + S)/K$ is a summand of M/K as A is h-pure submodule of M containing S. Hence, M/K is a direct sum of a bounded submodule and a h-divisible submodule. Thus M/K is h-pure complete, which is a contradiction. Therefore, $Soc(K) = S$ and M/K is bounded. Hence, for some n, $H_n(M/K) = 0$ and we get $Soc(H_n(M)) \subseteq S$.

(iii)$\rightarrow$ (ii) Let K be a h-pure submodule of M such that $S \subseteq K$, then $H_n(M) \subseteq K$ and hence, K is a cobounded summand of M.

Corollary 3.12: If M is a h-reduced QTAG-module and S is a subsocle of M, then every h-pure submodule K of M containing S is summand of M if and only if $S \supseteq Soc(H_n(M))$ for some n.

Proof: Due to above Theorem it is sufficient to show that $Soc(M^1) \subseteq S$. Let x be a uniform element in $Soc(M^1)$ and let $x \notin S$. Let K be a complement of xR and $S \subseteq K$ then by [Theorem 7 and Proposition 13, 1], K is h-pure submodule of M and $M = K \oplus D$ where $M/K \cong D$ is h-divisible, which is a contradiction as M is h-reduced. Therefore, $x \in S$ and we get $Soc(M^1) \subseteq S$.

Proposition 3.13: If M is QTAG-module and N is a submodule of M such that no proper h-pure submodule contains N. Then every h-pure submodule containing $Soc(N)$ is a cobounded summand of M.

Proof: Let T be a submodule of M such that $T \cap N = 0$, then T is bounded, since otherwise T will contain a proper basic submodule B and we will have $M/B = T/B \oplus K/B$. Appealing to [Proposition 2.5, 6], we get K to be h-pure submodule containing N, which is a contradiction. Now let A be a h-pure submodule of M such that $Soc(N) \subset A$, then M/A has a bounded basic submodule. Otherwise, if B/A is unbounded basic submodule of M/A, then $B = A \oplus L$ where $L \cong B/A$ and $A \cap N = 0$, which is a contradiction as L is unbounded. Therefore, $M/A = B/A \oplus D/A$ where B/A is bounded and D/A is h-divisible. Now we show that $D/A = 0$. Let $D/A \neq 0$, then M/B is h-divisible and B is h-pure submodule of M. This implies that $Soc(B)$

is proper dense in $Soc(M)$ and $Soc(N) \subseteq Soc(B)$, which is a contradiction. Hence, M/A is bounded. As A is h-pure in M, A is a summand of M.

Corollary 3.14: If M is QTAG-module and N is a submodule of M and T is a minimal h-pure submodule of M containing N. Then $T = B \oplus K$ where B is bounded and $Soc(K) = Soc(N)$.

Proof: Appealing to Proposition 3.13 and Theorem 3.11, we see that $Soc(N)$ supports an h-pure submodule K of T and T/K is bounded. Therefore, $T = B \oplus K$.

Let M be a QTAG-module satisfying
($\star$) $M/K = B/K \oplus D/K$ where B/K is bounded and D/K is h-divisible, whenever K is h-pure submodule of M containing M^1.

Definition 3.15: A QTAG-module M is called essentially finitely indecomposable (e.f.i) if it has no unbounded direct sum of uniserial submodules summand.

Theorem 3.16: If M is a QTAG-module and if M satisfies ($\star$), then every h-pure submodule of M containing M^1 is e.f.i.

Proof: Let A be h-pure submodule of M containing M^1, then A satisfies ($\star$), because if K is h-pure submodule of A containing $A^1 = M^1$, then A/K is h-pure submodule of M/K and the assertion follows from Proposition 3.8. Therefore, A satisfies ($\star$). Now let A be not e.f.i, then $A = S \oplus T$ where S is unbounded direct sum of uniserial submodules. Therefore, T is h-pure submodule of A containing A^1 and A/T is unbounded, a contradiction. Hence, A is e.f.i..

In the last of this section we prove the following result which is of independent interest.

Let us consider one more condition as mentioned below
(A) For any finitely generated submodule N of M, $R/ann(N)$ is right artinian.

Theorem 3.17: If M is a QTAG-module satisfying condition (A) and N is a quasi-essential submodule of M such that $Soc(N) \nsubseteq M^1$. Then every h-pure submodule K of M containing N is a cobounded summand of M.

Proof: Let K be h-pure submodule of M with $N \subseteq K$, then by Proposition 3.9, K/T is an absolute summand of M/T where T is any complement of N in K. Since $Soc(N) \nsubseteq M^1$, then [Corollary 10, 8] implies that K/T is not h-divisible for some complement T of N in K, as $K^1 \subseteq M^1$. Now appealing to [Theorem 12, 5], there exists a positive integer n such that

$$Soc(H_{n+1}(M/T)) \subseteq Soc(K/T) \subseteq Soc(H_n(M/T)$$

Therefore, $Soc(H_{n+1}(M)) \subseteq K$ and as K is h-pure we get $H_{n+1}(M) \subseteq K$ [Lemma 2, 3]. Hence, K is cobounded summand of M.

4. Minimal *h*-pure Submodule

Definition 4.1: A submodule N of a QTAG-module M is called almost dense in M if for every h-pure submodule K of M containing N, M/K is h-divisible.

Theorem 4.2: Let N be a submodule of a QTAG-module M. Then there is no proper h-pure submodule of M containing N if and only if N is almost dense in M and $Soc(H_n(M)) \subseteq N$ for some n.

Proof: Let N be almost dense in M and $Soc(H_n(M)) \subseteq N$. Let K be a h-pure submodule of M such that $N \subseteq K$, then $Soc(H_n(M)) \subseteq K$ and hence by [Lemma 2, 3], $H_n(M) \subseteq K$, consequently M/K is bounded but it is also h-divisible which is not possible and we get $M/K = 0$ i.e. $M = K$. Conversely, if no proper h-pure submodule of M contains N, clearly N is almost h-dense in M and by Theorem 3.11 and Proposition 3.13, we get $Soc(H_n(M)) \subseteq N$ for some positive integer n.

Now we prove the following useful criterion:

Theorem 4.3: Let N be a submodule of a QTAG-module M. Then N is contained in a minimal h-pure submodule of M if and only if there exists a h-pure submodule K of M such that $Soc(H_n(M)) \subseteq N \subseteq K$ for some $n \in Z^+$.

Proof: If N is contained in a minimal h-pure submodule of M then the result follows from [Theorem 6, 7]. Conversely, suppose that there exists an h-pure submodule K of M such that $Soc(H_n(M)) \subseteq N \subseteq K$ for some $n \in Z^+$. If $n = 0$, then trivially K itself is an h-pure submodule containing N. If $n \geqslant 1$, then for every h-pure submodule T of K containing N, we define $E(T) = \{l \geqslant 1/Soc(T_{l-1}) \nsubseteq N + H_l(T)\}$ and set $m(T) = 0$ if $E(T) = \phi$ and $m(T) = max\{m \in E(T)\}$ if $E(T) \neq \phi$. Trivially, $m(T) \leqslant n$ and therefore, there exists an h-pure submodule A of M containing N for which $m(A)$ is minimal. Now by [Lemma 4, 7], we see that $m(A) = 0$ i.e. $A \supseteq N \supseteq Soc(H_n(A))$ and $Soc(H_{l-1}(A)) \subseteq N + H_l(A)$ for all $l \geqslant 1$. Hence, by [Theorem 6, 7], A is a minimal h-pure submodule of M containing N.

Theorem 4.4: If N is a submodule of a QTAG-module such that M/N is a direct sum of uniserial submodules. If K is minimal h-pure submodule of M containing N then M/K is also a direct sum of uniserial submodules.

Proof: By Theorem 4.3, there exists $n \in Z^+$ such that $Soc(H_n(K)) \subseteq N$. Since K is h-pure in M, therefore by [Lemma 2.7, 6], $Soc(H_n(M/K)) = (Soc(H_n(M)) + K)/K$. It is trivial to see that the natural homomorphism $f : M/N \longrightarrow M/K$ defined by $f(x+N) = x+K$ is onto and maps $(Soc(H_n(M))+N)/N$ onto $(Soc(H_n(M))+K)/K$. Since we know that homomorphism never decreases heights. We show that f is height preserving. Let x be a uniform element in $Soc(H_n(M))$ and $x+K \in (Soc(H_n(M))+K)/K$, then we can find a uniform element $y \in Soc(H_n(M))$ such that $x+K = y+K$, then trivially $x-y \in Soc(K)$ and as K is h-pure, $x-y \in Soc(H_n(K)) \subseteq N$. Hence, $x+N = y+N \in (Soc(H_n(M))+N)/N$ and we get $H_{M/K}(x+K) \leqslant H_{M/N}(x+N)$. Since $(Soc(H_n(M))+N)/N$ is the union of the ascending chain of submodules of bounded height in $M/N, (Soc(H_n(M))+K)/K$ is also the union of an ascending chain of submodules of bounded height in M/K. Thus, $H_n(M/K)$ is a direct sum of uniserial submodules and M/K is direct sum of uniserial submodules.

Finally we prove the following:

Theorem 4.5: If N is a submodule of a basic submodule B of a QTAG-module M. If N is contained in a minimal h-pure submodule K of M, then K is a direct sum of uniserial submodules.

Proof: Since $N \subseteq B$ and K is h-pure submodule of M, then using [Theorem 4, 2], N can be extended to a basic submodule A of K. Since K is minimal h-pure containing N, $A = K$ and therefore, K is direct sum of uniserial submodules.

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