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A Uniqueness Theorem for Mellin Transform for Quotient Spaces

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Abstract. In this paper a uniqueness theorem is proved for the Mellin transform for the quotient space (the Boehmians) of analytic functions by using a relation between the Mellin transform and the Fourier transform.

1. Introduction

We know that an analytic function is infinitely differentiable. Let the set of all real analytic functions on a given set p is denoted by $C^w(\ p)$. Then for any open set $U : \Omega \subseteq C$, the set $A(\Omega)$ of all analytic functions $U : \Omega \to C$ is a Fréchet space with respect to the uniform convergence on compact sets.

The construction of Boehmians (or the quotient space) was motivated by the concept of regular operators, see Boehme [**2**]. Nemzer [**5**] constructed a subspace of Boehmians, called Boehmians of analytic type, which is said to possess a uniqueness property. If an analytic function $f(z)$ is bounded in the unit disc $\mathcal{D}$, then it has the uniqueness property that if $\lim_{r\to 1} f(re^{i\theta}) = 0$ on a set of positive measure on the unit circle S', which implies $f(z)$ to be identically zero, where the radial limit $F(re^{i\psi}) = \lim_{h\to 1} f(he^{i\psi})$, almost everywhere on S'. Riesz [**6**] showed that any bounded analytic function in $\mathcal{D}$ has the uniqueness property.

This paper introduces the extended (in other words, modified) Mellin transform and proves the uniqueness theorem for this transform for the quotient space (the Boehmians) of analytic functions.

Let S' denote the unit circle, $C(S')$ is the collection of continuous complex valued function of S'. By $C^N(S')$ we mean collection of sequence of continuous complex valued function on S'. No distinction is made between a function on S' and a 2π-periodic function on the real line R, see [**4, 5**]. For f, $\phi \in C(S')$, the convolution $*$ is defined by

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$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)\phi(t)dt. \tag{0.1}$$

Let G is linear space, i.e., $G = C^{\infty}(R)$, which is also considered as a quasi-normed space that is equipped with the topology of uniform convergence on compact set $S \in \mathcal{D}(R)$, and Δ be the class of sequence from $\mathcal{D}$ which satisfies the following conditions

(i) $\frac{1}{2\pi}\int_{-\pi}^{\pi} \delta_n(x)dx = 1$ for all $n \in N$
(ii) supp $\delta_n \subseteq (-\varepsilon_n, \varepsilon_n)$, where $\varepsilon_n \to 0$ as $n \to \infty$,

where δ_n is a delta sequence (a sequence of continuous non-negative functions).

A pair of sequence (f_n, δ_n), denoted by f_n/δ_n, is called quotient of sequence, if

$$A = \{(\{f_n\}, \{\delta_n\}) : f_k * \delta_n = f_n * \delta_k \text{ for all } n, k \in N\}, \tag{0.2}$$

where $f_n \in C(R)$, $n = 1, 2, ...,$ and $A \subseteq C^N(S') \times \Delta$.

Two quotients of sequence f_n/δ_n and g_m/σ_m are called equivalent if $f_n * \sigma_m = g_m * \delta_n$ for all $m, n \in N$, which is said to be an equivalence relation on A.
The equivalence classes are called periodic Boehmians, defined by

$$\beta = \left\{ \left[\frac{\{f_n\}}{\{\delta_n\}} \right] : (\{f_n\}, \{\delta_n\}] \in A \right\}. \tag{0.3}$$

The natural addition, multiplication and scalar multiplication on β imply

$$f_n/\delta_n + g_n/\sigma_n = (f_n * \sigma_n + g_n * \delta_n)/(\delta_n * \sigma_n) \tag{0.4}$$

$$f_n/\delta_n * g_n/\sigma_n = (f_n * g_n)/(\delta_n * \sigma_n) \tag{0.5}$$

and

$$\alpha(f_n/\delta_n) = \alpha\ f_n/\delta_n, \tag{0.6}$$

where α is a complex number and β becomes a commutative algebra with identity $\delta = \delta_n/\delta_n$.

2. Mellin Transform for Boehmians of Analytic Functions

Let $M_{a,b}$ is the space of all smooth complex valued functions $\theta\ (x)$ on $I = (0,\infty)$ such that for each non-negative integer k,

$$\eta_k(\theta) = \eta_{a,b,k}(\theta) \triangleq \sup_{0<x<\infty} |r_{a,b}(x) x^{k-1} D_x^k \theta(x)| < \infty$$

$$= \sup\{r_{a,b}(x)x^{k+1}|\theta^k(x)| : x \in I\} < \infty, k = 0, 1, 2... \tag{0.7}$$

where

$$r_{a,b}(x) = \begin{cases} x^{-a} & 0 < x \leqslant 1 \\ b^{-a} & 0 < x < 1 \end{cases}$$

and a and b are fixed numbers, $0 < a < b < \infty$. Therefore,

$$\eta_{a,b}(x) = K_{a,b}(t), \tag{0.8}$$

where

$$K_{a,b}(t) = \begin{cases} e^{at} & 0 \leqslant t < \infty \\ e^{bt} & -\infty < t < 0; a, b, t \in R'. \end{cases}$$

η_k are seminorms on $M_{a,b}$ and η_0 is a norm. $M'_{a,b}$ is the dual of the space $M_{a,b}$, which is equipped with a weak topology. By the convergence property, if $f_n \to f$ as $n \to \infty$ in $M'_{a,b}$, then $\int f_n(x)\theta(x)dx - \int f(x)\theta(x)dx \to 0$ as $n \to \infty$ for each $\theta \in M_{a,b}$.

The classical Mellin transform of $f \in {M'}_{a,b}$ and its inverse are, respectively, defined by

$$M\{f(x); s\} = \int_0^\infty f(x)x^{s-1}dx \tag{0.9}$$

and

$$f(x) = \frac{1}{2\pi\, i} \int_{c-i\infty}^{c+i\infty} F(s)x^{-s}ds. \tag{0.10}$$

Mf is an analytic function with a polynomial growth [**7**]. The relation between the Mellin transform and the Fourier transform, see [**1**], is given by,

$$M\{f(x); s\} = F\{f(e^x); is\}, \tag{0.11}$$

which may also be written as

$$\tilde{f}(is) = \int f(e^x)e^{sx}dx, \tag{0.12}$$

whereas the inverse is given by

$$f(e^x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(is)e^{-sx}ds. \tag{0.13}$$

The Mellin transform $\tilde{f}(is)$ of slowly increasing function f is the distribution, given by

$$\left\langle \tilde{f}(is), \overline{\tilde{\varphi}(is)} \right\rangle = 2\pi \left\langle f(e^x), \overline{\varphi(e^x)} \right\rangle. \tag{0.14}$$

The k th Mellin coefficients for a function $f \in C(S')$ is given by

$$c_k(f(ik)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^x)e^{kx}dx, \quad k \in Z. \tag{0.15}$$

LEMMA 1: *Let* $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta$. *Then for each* k, *the sequence* $\{c_k(f_n(ik))\}_{n=1}^{\infty}$ *converges.*

Proof: Let $k \in$ Z. Since $\{\varphi_w(ik)\}_{w=1}^{\infty}$ is a delta sequence, there exists a $w \in N$ such that $\tilde{\varphi}_w(ik) \neq 0$. Now,

$$c_k(f_n(ik)) = c_k(f_n(ik)\frac{\tilde{\varphi}_w(ik)}{\tilde{\varphi}_w(ik)}$$
$$= \frac{c_k(f_n * \varphi_w)(ik)}{\tilde{\varphi}_w(ik)}$$
$$= \frac{c_k(f_w * \varphi_n)(ik)}{\tilde{\varphi}_w(ik)}$$
$$= \frac{c_k(f_w(ik)}{\tilde{\varphi}_w(ik)} \cdot \tilde{\varphi}_n(ik) \to \frac{c_k(f_w(ik))}{\tilde{\varphi}_w(ik)}, as\ n \to \infty.$$

Thus the lemma is proved.

DEFINITION 1: *Let* $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta$. *Then the k-th Mellin coefficient, see also* (0.15), *of* F *is*

$$c_k F(ik) = \lim_{n\to\infty} c_k(f_n(ik)). \tag{0.16}$$

DEFINITION 2: [**5**] *A Boehmian* F *is said to be zero on an open set* Ω, *denote by* $F = 0$ *on* Ω, *if there exists a delta sequence* $\{\delta_n\}$ *such that* $F * \delta_n \in C(S')$ *for all* $n \in N$ *and* $F * \delta_n \to 0$ *uniformly on compact subset of* Ω *as* $n \to \infty$.

LEMMA 2: *Let* $f(z), g(z) \in \mathcal{D}$ *and* $f * \varphi_n = g * \varphi_n$ *for all* $n \in N$. *Then* $f(z) = g(z)$ *is in* $\mathcal{D}$.

Proof **:** Since the Fourier transform is an isomorphism from $\mathcal{D}$ into itself, it is enough to prove that $\tilde{f}(i\psi) = \tilde{g}(i\psi)$, $\forall\psi\in Q$, where the space $Q(R)$ is dense in $\mathcal{D}(R)$. By virtue of (0.11), for the Mellin transform, we write

$$\tilde{f}(i\psi) = \tilde{f}\left(\frac{\tilde{\varphi}_n(i\psi)}{\tilde{\varphi}_n(i\psi)}\right)$$
$$= \tilde{\varphi}_n\tilde{f}\left(\frac{(i\psi)}{\tilde{\varphi}_n(i\psi)}\right) = \tilde{\varphi}_n\tilde{g}\left(\frac{(i\psi)}{\tilde{\varphi}_n(i\psi)}\right)$$
$$= \tilde{g}\left(\frac{\tilde{\varphi}_n(i\psi)}{\tilde{\varphi}_n(i\psi)}\right) = \tilde{g}(i\psi), \forall n \in N.$$

This completes the proof.

DEFINITION 3: *A Boehmian* $F = \left[\frac{f_n}{\delta_n}\right] \in \beta$ *is said to be of analytic type if* $\tilde{F}(ik) = 0$ *for* k = -1,-2...

THEOREM 1: *If F is a Boehmian of analytic functions such that F= 0 on some open arc* Ω*, then F*$\equiv$ 0.

Proof: Let $F = \left[\frac{f_n}{\delta_n}\right] \in \beta$ be a Boehmian of analytic functions such that F=0 on Ω. If $MF = \tilde{F}(ik)$, then by Definition 3, $\tilde{F}(ik) = 0$ for k = -1,-2,..; while for each k,

$$\tilde{f}_n(ik) = \tilde{F}(ik)\widetilde{\delta}_n(ik) = 0 \quad \text{for} \quad k = -1, -2, ... \tag{0.17}$$

Invoking the Definition of Boehmians, $f_n * \varphi_w = \varphi_w * f_n$ for all $n \in N$, we have

$$f_n = f_n - (f_n * \delta_w) + (f_n * \delta_w), \text{ for all } \quad n,\ w \in N\ . \tag{0.18}$$

Since $\{\delta_w\}$ is a delta sequence, for each w, $f_n * \delta_w \to f_n$ uniformly on T as $w \to \infty$. Let J be any closed subinterval on Ω. Then there exists a closed interval I, for an α >0, $J \subset I \subset \Omega$, and (-$\alpha$,$\alpha$) +$J \subseteq I$. Also there exists an $n_0 \in N$ such that supp $\delta_n \subseteq$ (-α,α), for all $n \geqslant n_0$. Let n_0 be any fixed integer, $n > n_0$. Then for all $w \geqslant w_0$, let $\varepsilon > 0$. Since $f_w \to 0$ uniformly on I as $w \to \infty$, there exists a $w_0 \in N$ such that for all $w \geqslant w_0$, $|f_w(x)| < \varepsilon$ for all $x \in I$. Then

$$\begin{aligned} |(f_n * \delta_w)(ix)| &= |(f_w * \delta_n)(ix)| \\ &\leqslant \frac{1}{2\pi}\int_{-\alpha}^{\alpha} |f_w(ix - t)|\delta_n(t)dt \\ &\leqslant \frac{\varepsilon}{2\pi}\int_{-\alpha}^{\alpha} \delta_n(t)dt = \varepsilon, \quad \text{for all} \quad x \in J. \end{aligned}$$

Therefore $f_w * \delta_w \to 0$ uniformly on J as $w \to \infty$, for each $n \geqslant n_0$. By combining (17), (18) and (20), we see that for each $n \geqslant n_0$, f_n vanishes on J. This completes the proof of theorem.

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