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Small Homomorphisms and Large Submodules of $QTAG$-Modules

Alveera Mehdi, Sabah A R K Naji and Ayazul Hasan

Abstract. A module M over an associative ring R with unity is a $QTAG$-module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.Over the past several years $QTAG$-modules have been the subject of intense investigation yet there is much to explore.The impetus for these efforts stems from the fact that the rings considered here are almost restriction free.This factor motivates us to continue. A fully invariant submodule L of M is large in M if $L + B = M$, for every basic submodule B of M. We define Ulm sequences of the elements of M to study the structure of large submodules and essentially isomorphic $QTAG$-modules. Closed $QTAG$-modules are also investigated and the relation between large submodules and Ulm invariants is established.

1. Introduction.

All the rings R considered here are associative with unity and modules M are unital $QTAG$-modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique composition series, $d(M)$ denotes its composition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup\left\{d\left(\frac{yR}{xR}\right) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ and it is h-reduced if it does not contain any h-divisible submodule.

A submodule N of M is h-pure in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geqslant 0$. For a limit ordinal α, $H_\alpha(M) = \bigcap_{\rho<\alpha} H_\rho(M)$, for all ordinals $\rho < \alpha$.

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A submodule $B \subseteq M$ is a basic submodule of M, if B is h-pure in M, $B = \oplus B_i$, where each B_i is the direct sum of uniserial modules of length i and M/B is h-divisible. A fully invariant submodule $L \subseteq M$ is large if $L + B = M$, for every basic submodule B in M. The submodules $H_k(M)$, $k \geqslant 0$ form a neighborhood system of zero, thus a topology known as h-topology arises. Closed modules [2] are also closed with respect to this topology. Thus the closure of $N \subseteq M$ is defined as $\overline{N} = \bigcap_{k=0}^{\infty} (N + H_k(M))$. Therefore the submodule $N \subseteq M$ is closed with respect to h-topology if $\overline{N} = N$ and h-dense in M if $\overline{N} = M$.

The cardinality of the minimal generating set of M is denoted by $g(M)$. For all ordinals α, $f_M(\alpha)$ is the α^{th}-Ulm invariant of M and it is equal to $g(Soc(H_\alpha(M))/Soc(H_{\alpha+1}(M)))$.

2. Small Homomorphisms.

We start with the following:

Definition 2.1. For any arbitrary $x \in M$, $H_M(x) = k$ if $x \in H_k(M)$, $x \notin H_{k+1}(M)$ and $H(x) = \infty$, if $x \in M^1$.
Thus we may also define the heights of the elements which are not uniform.
Definition 2.2. The generalized height of x in M denoted by

$$H^*_M(x) = \begin{cases} \alpha, \text{ if } x \neq 0 \text{ and } \alpha + 1 \text{ is the first ordinal such that } x \notin H_{\alpha+1}(M) \\ \\ \infty, \text{ if } x = 0. \end{cases}$$

For any uniform element $x \in M$, there exist uniform elements $x_1, x_2, \ldots$ such that $xR \supseteq x_1R \supseteq x_2R \supseteq \ldots$ and $d\left(\dfrac{x_iR}{x_{i+1}R}\right) = 1$. Now the Ulm-sequence of x is defined as $U(x) = (H(x),\ H(x_1),\ H(x_2), \ldots)$. This is analogous to the U-sequences in groups [1]. These sequences are partially ordered because $U(x) \leqslant U(y)$ if $H(x_i) \leqslant H(y_i)$ for every i. For a sequence $n(L) = (n_0,\ n_1,\ n_2, \ldots)$ of non negative, non decreasing integers we may consider $L = \{x \mid x \in M, U(x) \geqslant n(L)\}$. Since for any endomorphism f of M, $H(x) \leqslant H(f(x))$, L is fully invariant. Therefore with every large submodule L of M we may associate a sequence $n(L)$.

Definition 2.3. For $QTAG$-modules M and M',a homomorphism $f : M \to M'$ is said to be small if Ker$f$ contains a large submodule of M. The set of all small homomorphisms from M to M', denoted by $Hom_s(M, M')$ is a submodule of $Hom(M, M')$.

For a family $\mathcal{F}$ of $QTAG$-modules, M is a HT-module with respect to $\mathcal{F}$ if $Hom(M, K)$=$Hom_s(M, K)$, for every $K \in \mathcal{F}$. M is said to be a HT-module if $Hom(M, K)$=$Hom_s(M, K)$, for every $QTAG$-module K which is a direct sum of uniserial modules.

Lemma 2.1. Let $\mathcal{F}$ be a family of $QTAG$-modules, $N \in \mathcal{F}$ and K a HT-module with respect to $\mathcal{F}$. If $M = N \oplus K$, then for every h-pure HT-module $Q \subset M$, $Q \cap L \subseteq K$, for some large submodule L of M.

Proof. Since $M = N \oplus K$, there exists a projection π of M onto N. If $\alpha = \pi_{|Q}$, then $\alpha \in Hom(Q, N)$. This implies that kerα contains a large submodule L' of Q. Now L' corresponds to a monotonically increasing sequence of positive integers. The same sequence defines a large submodule L in M. Now Q is h-pure in M which contains a large submodule L such that $L' = Q \cap L$. Since ker$\alpha = K \cap Q$, the result follows.

The dual concept of quasi-isomorphism [3], i.e. essential isomorphism is defined as follows:

Definition 2.4. Two $QTAG$-modules M and M' are essentially isomorphic if there exist bounded h-pure submodules N and N' of M and M' respectively such that $M/N \cong M'/N'$.

Remark 2.1. Two $QTAG$-modules M and M' are essentially isomorphic if there exist bounded $QTAG$-modules K and K' such that $M + K \cong M' + K'$.

Theorem 2.1. Let $\mathcal{F}$ be a family of $QTAG$-modules and $K, K' \in \mathcal{F}$. For N, N' the HT-modules with respect to $\mathcal{F}$, if $M = N + K = N' + K'$, then N and K are essentially isomorphic to N' and K' respectively.

Proof. By Lemma 2.1, there exist large submodules L and L' of M such that $N \cap L \subseteq N' \cap L$ and $N' \cap L' \subseteq N \cap L'$. Again $L_0 = L \cap L'$ is a large submodule of M such that $N \cap L_0 = N' \cap L_0$. Since $N \cap L_0$ is large in N and N' both, $N = P + Q$ and $N' = P' + Q'$, where P and P' are bounded and $Soc(Q) = Soc(N \cap L_0) = Soc(N' \cap L_0) = Soc(Q')$. Again Q, Q' are summands of M with the same socle, therefore they are isomorphic. This implies that N and N' are essentially isomorphic.
Again $K + P \cong \dfrac{M}{Q} \cong \dfrac{M}{Q'} \cong K' + P'$, thus K and K' are essentially isomorphic.

A $QTAG$-module M is a closed module if every Cauchy sequence in M converges to a limit.These closed modules coincide with the modules closed with respect to the h-topology[2].Here we prove some results related to these modules.

Theorem 2.2. Let M be an unbounded closed $QTAG$-module and M' a $QTAG$-module without elements of infinite height. Then $Hom(M, M') \neq Hom_s(M, M')$ if and only if M' contains an unbounded closed submodule.

Proof. Suppose there exists a homomorphism $\alpha : M \longrightarrow M'$, containing no large submodule of M. We may select a positive integer n_0 and elements $x_1, x_2, \ldots$ in M such that for every i, $x_i \in H^{n_0}(M)$, $H_M(x_{i+1}) > \max(n_o + H_M(x_i),\ H_{M'}(\alpha(x_i)))$, $\alpha(x_i) \neq 0$ and $\alpha(x_i) \in Soc(M')$. This selection is always possible because M is unbounded and

x_i's may lie in a direct sum of uniserial modules which is h-pure in M. If $n_i = H_M(x_i)$, then there exists $y_i \in M$ such that $d\left(\frac{y_iR}{x_iR}\right) = n_i$. Thus $U = \sum_{i=1}^{\infty} y_iR$ is a h-pure submodule of M. If $z_i = \alpha(x_i)$, then z_i's form the minimal generating subset of $Soc(M')$. Also
$U\cap$ ker $\alpha = \sum_{i=1}^{\infty} x_i'R$, where $d\left(\frac{x_iR}{x_i'R}\right) = 1$.

Since closed modules coincides with their closures with respect to h-topology, the closure of U i.e. $\overline{U} = \bigcap_{k=0}^{\infty}(U + H_k(M))$. Let $\alpha(U) = V$ and $\alpha(\overline{U}) = V'$. Then $\overline{U}$ is an unbounded closed module. Now $\alpha(U) \cong \frac{U}{U \cap \ker\alpha} \cong \sum U_{n_i+1}$ (U_{n_i+1} is a uniserial module of length $n_i + 1$) and $\frac{\alpha(\overline{U})}{\alpha(U)}$ is h-divisible. Since $\overline{U}\cap$ kerα is the closure of $U\cap$ kerα, $\alpha(U)$ is h-pure in $\alpha(\overline{U})$. Now consider a Cauchy sequence $\{v_i\}$ in $Soc(\alpha(U))$ such that $v_i = \alpha(u_i)$ where $\{u_i\}$ is a Cauchy sequence in $\overline{U}$ and each u_i is contained in a bounded submodule. Now $\{u_i\}$ has a limit $u \in \overline{U}$ and $v = \alpha(u)$ is the limit of the sequence $\{v_i\}$ in $\alpha(\overline{U})$. Thus $\alpha(\overline{U})$ is a closed module which is unbounded because $\alpha(U)$ is unbounded.
The converse is trivial.

Corollary 2.1. Let M be an unbounded closed $QTAG$-module and K a h-dense, h-pure submodule of a closed $QTAG$-module N. If every homomorphism from M to K is not small, then there exists an unbounded summand Q of N such that $Soc(Q) \subseteq K$.

Proof. Let α be a homomorphism from M to K. Every $QTAG$-module M has a basic submodule which is a direct sum of uniserial modules and it is h-pure in M. Since M is unbounded we may select an integer k and x_i's from a direct sum of uniserial modules $\oplus U_i$ which is h-pure in M, such that $x_i \in H^k(M)$,
$H(x_{i+1}) > \max(k + H_M(x_i),\ H_k(\alpha(x_i)))$. If $H_M(x_i) = k_i$, then there exists $y_i \in M$ such that $d\left(\frac{y_iR}{x_iR}\right) = k_i$ and we may consider $U = \sum y_iR$. Now we may choose a h-pure submodule $Q \subset K$ such that $Soc(Q) = Soc(\alpha(U))$. For the closure $\overline{Q} \subset N$, $\overline{Q}$ is an unbounded direct summand of N. Again $\alpha(\overline{U})$ also contains $Soc(\overline{Q})$. Thus $Soc(Q) \subseteq Soc\overline{Q} \subseteq K$.

Theorem 2.3. Let $M(=\overline{B})$ be a closed $QTAG$-module with a basic submodule B. For the homomorphic image N of $\overline{B}$, either N/N^1 is the homomorphic image of B or N/N^1 contains an unbounded closed submodule.

Proof. The homomorphism from M onto N, with the natural homomorphism $N \to N/N^1$, gives rise to a homomorphism $\alpha : M \longrightarrow N/N^1$. If N/N^1 does not contain an unbounded closed submodule, then α is small. This implies that M contains a large submodule L such that N/N^1 is homomorphic image of M/L.

Since $\dfrac{M}{L} = \dfrac{L+B}{L} \cong \dfrac{B}{B \cap L}$, $\dfrac{N}{N^1}$ is a homomorphic image of B.

Theorem 2.4. Let N be a h-dense, h-pure submodule of a closed $QTAG$-module $M = \overline{B}$. If $N = M$ or the minimal generating set of M/N is denumerable, then N is a HT-module.

Proof. If $N = M$, then the result follows from Theorem 2.2. Suppose $g(M/N) = \aleph_0$ and α is a homomorphism of N into a direct sum of uniserial modules. Then $\alpha(N) = K$ is also a direct sum of uniserial modules. If we consider $\overline{K}$, then K is a basic submodule of $\overline{K}$. Since $\overline{N} = \overline{B}$, the homomorphism α may be uniquely extended to $\overline{\alpha} : M \longrightarrow \overline{K}$ and $g\left(\dfrac{\overline{\alpha}(M)}{K}\right) \leqslant \aleph_0$. Again K^1 is $\{0\}$, therefore $\overline{\alpha}(M)$ is also a direct sum of uniserial modules. This implies that $\overline{\alpha}$ is small and $\alpha = \overline{\alpha}_{|N}$ is also small, thus N is a HT-module.

Now we shall prove the following result to establish the relationship between large submodule and Ulm-invariants.

Theorem 2.5. Let L be a large submodule of a $QTAG$-module M determined by the sequence $n(L) = (n_0, n_1, \dots)$. Then a monomorphism α of L into a $QTAG$-module N can be extended to an isomorphism of M onto N if and only if

(*i*) $f_M(n) = f_N(n)$ for $n \leqslant n_0$
(*ii*) $H_M(x) = H_N(\alpha(x))$, for every $x \in L$ and
(*iii*) $\alpha(L)$ is large in N.

Proof. Suppose these conditions hold and $B = \bigoplus\limits_{i=1}^{\infty} B_i$ is a basic submodule of M. Now for every $n > n_0$, there exists a monotonically increasing sequence of non-negative integers $k_1, k_2, \dots$ with $k_n \leqslant n-1$ and $B \cap L = \sum\limits_{n=n_0+1}^{\infty} H_{k_n}(B_n)$. Since $f_M(n) = f_N(n)$ for $n \leqslant n_0$, we may assume $B_n = 0$. Otherwise if $B_n = \sum x_i^n R$ and $d\left(\dfrac{x_i^n R}{y_i^n R}\right) = k_n$, then $H_{k_n}(B_n) = \sum y_i^n R$ and $\alpha(H_{k_n}(B_n)) = \sum z_i^n R$, where $z_i^n = \alpha(y_i^n)$ and $H_N(z_i^n) = k_n$. Now we may select $u_i^n \in N$ such that $d\left(\dfrac{z_i^n R}{u_i^n R}\right) = k_n$. If we put $Q_n = \sum u_i^n R$, $Q = \sum\limits_{n=n_0+1}^{\infty} Q_n$, then there is an isomorphism $\beta : B \longrightarrow Q$ such that $\beta(x_i^n) = u_i^n$ and $\beta_{|B \cap L} = \alpha_{|B \cap L}$. Now Q is h-pure in N and $\alpha(L)$ is fully invariant in N, $\alpha(L) \cap Q_n = H_j(Q_n)$ for some $j \leqslant k_n$, which is a contradiction because $H_M(x) = H_N(\alpha(x))$, for every $x \in L$. Therefore $\alpha(L) \cap Q = \sum\limits_{n=n_0+1}^{\infty} \alpha(L) \cap Q = \sum\limits_{n=n_0+1}^{\infty} H_{k_n}(Q_n) = \alpha(B \cap L)$.

Again Q is a basic submodule of N, otherwise we may find an integer m and an element $u \in H^m(N)$ such that $Q_m + yR$ is a summand of N, contradicting the fact that $\alpha(L)$ is large in N.

As $M = L + B$ we may consider a map $\overline{\alpha} : M \longrightarrow N$ such that $\overline{\alpha}(x) = \overline{\alpha}(l + b) = \alpha(l) + \beta(b)$. This $\overline{\alpha}$ is well defined monomorphism because $N = \alpha(L) + Q$. Now $\overline{\alpha}$ is an isomorphism of M onto N such that $\overline{\alpha}_{|_L} = \alpha$.

We conclude with the following remarks:

Remark 2.2. Each automorphism of a large submodule which preserves heights is induced by an automorphism of the containing $QTAG$-module. Therefore every automorphism of $H_k(M)$, $k = 0, 1, 2, \ldots \infty$ is induced by an automorphism of M.

Remark 2.3. For the h-pure, h-dense submodules N and K of a $QTAG$-module M, if $N \cap L = K \cap L$ for some large submodule L of M, then $N \cong K$.

In the last we would like to state an open problem which is yet to be explored.

Problem. If a large submodule L of a $QTAG$-module M is totally projective, then every large submodule of M is totally projective.

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Department of Mathematics,
Aligarh Muslim University,
Aligarh 202002,
India
E-mail address: alveera _mehdi@rediffmail.com, sabah _kaled@yahoo.com,ayaz.maths@gmail.com