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Analytic continuation formula for a unified family of Euler-type elliptic integrals

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Abstract. Elliptic type integrals have their importance or potential in certain problems in radiation physics and nuclear technology. A number of earlier works on the subject contains several interesting unifications and generalizations of some significant families of elliptic-type integrals. We obtained analytic continuation formula for a unified family of Euler-type elliptic integral and established region of convergence for the same in the present investigation.

1. Introduction

Elliptic integrals occur in a number of physical problems ([3], [5], [14]), and frequently in the form of multiple integrals. For example, the problems dealing with the computation of the radiations field off axis from certain uniform circular disc radiating according to an arbitrary angular distribution law [7], when treated with Legendre polynomials expansion method, give rise to Epstein and Hubbell [4, 5] family of elliptic-type integrals:

$$\Omega_j(k) = \int_0^{\pi} (1 - k^2 \cos\theta)^{-j-\frac{1}{2}}\, d\theta \ (j = 0, 1, 2, \ldots);\ \{0 \leqslant k < 1\}. \tag{1.1}$$

Elliptic integrals (1.1) have been studied and generalized by many authors notably by Kalla [8, 9] and Kalla et al. [11]. Kalla and Al-Saqabi [10], Saxena et al. [20], Kalla et al. [12], Srivastava and Bromberg [22] and others.

Some of these generalizations are as follows:

Kalla [8, 9] introduced the generalization of the form:

$$R_\mu(k, \alpha, \gamma) = \int_0^{\pi} \frac{\cos^{2\alpha-1}(\theta/2)\, \sin^{2\gamma-2\alpha-1}(\theta/2)}{(1 - k^2\cos\theta)^{\mu+\frac{1}{2}}}\, d\theta, \tag{1.2}$$

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where $\{0 \leqslant k < 1\}$; $\{\Re(\gamma) > \Re(\alpha) > 0,\ \Re(\mu) > -\frac{1}{2}\}$.

Results for this generalization are also derived by Glasser and Kalla [6].

Al-Saqabi [1] defined and studied the generalization given by the integral

$$B_\mu(k, m, \nu) = \int_0^\pi \frac{\cos^{2m}(\theta)\,\sin^{2\nu}(\theta)}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}}\,d\theta, \tag{1.3}$$

where $\{0 \leqslant k < 1\}$; $\{m \in \mathcal{N}_0,\ \mu \in \mathcal{C}\}$; $\{\Re(\mu) > -\frac{1}{2}\}$.

Asymptotic expansion of Eq.(1.3) has recently been discussed by Matera et al. [17].

The integral

$$\Lambda_\nu(\alpha, k) = \int_0^\pi \frac{\exp[\alpha\sin^2(\theta/2)]}{(1-k^2\cos\theta)^{\nu+\frac{1}{2}}}\,d\theta, \tag{1.4}$$

where $\{0 \leqslant k < 1\}$; $\{\alpha, \nu \in \mathcal{R}\}$; presents another generalization of Eq.(1.1), given by Siddiqi [21].

Srivastava and Siddiqi [23] have given an interesting unification and extension of the families of elliptic-type integrals in the following form:

$$\Lambda^{(\alpha,\beta)}_{(\lambda,\mu)}(\rho; k) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2)\,\sin^{2\beta-1}(\theta/2)}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}}\left[1-\rho\sin^2\left(\frac{\theta}{2}\right)\right]^{-\lambda} d\theta, \tag{1.5}$$

where $\{0 \leqslant k < 1\}$; $\{\Re(\alpha) > 0,\ \Re(\beta) > 0\}$; $\{\lambda, \mu \in \mathcal{C}\}$; $|\rho| < 1$.

Kalla and Tuan [13] generalized Eq.(1.5) by means of the following integral and also obtained its asymptotic expansion:

(1.6)

$$\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho, \delta; k) = \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)(1-k^2\cos\theta)^{-\mu-\frac{1}{2}} \cdot \left[1-\rho\sin^2\left(\frac{\theta}{2}\right)\right]^{-\lambda}\left[1+\delta\cos^2\left(\frac{\theta}{2}\right)\right] d\theta,$$

$\{0 \leqslant k < 1\}$; $\{\Re(\alpha) > 0,\ \Re(\beta) > 0\}$; $\{\lambda, \mu, \gamma \in \mathcal{C}\}$ and either $|\rho|, |\delta| < 1$ or ρ (or δ) $\in \mathcal{C}$ whenever $\lambda = m$ or $\gamma = -m$, $m \in \mathcal{N}_0$, respectively.

Al-Zamel et al. [2] discussed a generalized family of elliptic-type integrals in the form:

(1.7)

$$Z^{(\alpha,\beta)}_{(\gamma)}(k) = Z^{(\alpha,\beta)}_{(\gamma_1,\ldots,\gamma_n)}(k_1,\ldots,k_n) = \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{x}\right)\prod_{j=1}^n (1-k_j^2\cos\theta)^{-\gamma_j}\,d\theta$$

$$= B(\alpha,\beta)\prod_{j=1}^n (1-k_j^2)^{-n} F_D^{(n)}\left(\beta; \gamma_1,\ldots,\gamma_n; \alpha+\beta; \frac{2k_1^2}{k_1^2-1},\ldots,\frac{2k_n^2}{k_n^2-1}\right),$$

where $\{\Re(\alpha) > 0,\, \Re(\beta) > 0\}$; $|k_j| < 1$; $\{\gamma_j \in \mathcal{C}\}$ $(j = 1, \ldots, n)$; $F_D^{(n)}(\cdot)$ is the Lauricella hypergeometric function of n variables ([16], p.163).

Saxena and Kalla [19] have studied a family of elliptic-type integrals of the form

(1.8)

$$\Omega^{(\alpha,\,\beta)}_{(\sigma_1,\ldots,\sigma_{n-2}:\delta,\,\mu)}(\rho_1,\ldots,\rho_{n-2},\,\delta;\,k) = \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{j=1}^{n-2}\left[1-\rho_j\,\sin^2\left(\frac{\theta}{2}\right)\right]^{-\sigma_j} \cdot\left[1+\delta\,\cos^2\left(\frac{\theta}{2}\right)\right]^{-\gamma}(1-k^2\,\cos\theta)^{-\mu-\frac{1}{2}}\,d\theta,$$

where $\{0 \leqslant k < 1\}$; $\{\Re\,(\alpha) > 0,\, \Re\,(\beta) > 0\}$; $\{\sigma_j\ (j = 1, \ldots, n-2)\}$; $\{\gamma,\, \mu \in \mathcal{C}\}$;

$$\max\left\{|\rho_j|,\ \left|\frac{\delta}{1+\delta}\right|,\ \left|\frac{2k^2}{k^2-1}\right|\right\} < 1.$$

In a recent paper, Saxena and Pathan [18] investigated an extension of Eq.(1.8) in the form:

(1.9)

$$\Omega^{(\alpha,\,\beta)}_{(\sigma_1,\ldots,\sigma_m,\,\gamma;\,\tau_1,\ldots,\tau_n)}(\rho_1,\ldots,\rho_m,\,\delta;\,\lambda_1,\ldots,\lambda_n)$$

$$= \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{i=1}^{m}\left[1-\rho_i\,\sin^2\left(\frac{\theta}{2}\right)\right]^{-\sigma_i} \cdot\left[1+\delta\,\cos^2\left(\frac{\theta}{2}\right)\right]^{-\gamma}\prod_{j=1}^{n}(1-\lambda_j^2\,\cos\theta)^{-\tau_j}\,d\theta,$$

where $\min\{\Re(\alpha),\, \Re(\beta)\} > 0$; $|\lambda_j| < 1$; $\{\sigma_i,\, \gamma,\, \tau_j \in \mathcal{C}\}$;

$$\max\left\{|\rho_j|,\ \left|\frac{2\lambda_j^2}{\lambda_j^2-1}\right|,\ \left|\frac{\delta}{1+\delta}\right|\right\} < 1\ (i = 1, \ldots, m,\ j = 1, \ldots, n).$$

Here we consider a unified and generalized form of a family of elliptic-type integrals:

(1.10)

$$\bar{\Omega}^{(\alpha,\,\beta)}_{(\lambda_i,\,\tau_j)}\{(\rho_i),\,(\delta_i);\,k_j\} = \bar{\Omega}^{(\alpha,\,\beta)}_{\lambda_1,\ldots,\lambda_N,\,\tau_1,\ldots,\tau_M}(\rho_1,\ldots,\rho_N,\,\delta_1,\ldots,\delta_N;\,k_1,\ldots,k_M)$$

$$= \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{i=1}^{N}\left[1+\rho_i\,\sin^2\left(\frac{\theta}{2}\right)+\delta_i\,\cos^2\left(\frac{\theta}{2}\right)\right]^{-\lambda_i}\prod_{j=1}^{M}[1-k_j^2\,\cos\theta]^{-\tau_j}\,d\theta,$$

$\{\lambda_i,\, \tau_j \in \mathcal{C}\}$;

$$\max\left\{|\rho_i|,\,|\delta_i|,\ \left|\frac{2k_j^2}{{k_j}^2-1}\right|,\ \left|\frac{\delta_i-\rho_i}{1+\delta_i}\right|\right\} < 1\ (i = 1, \ldots, N,\ j = 1, \ldots, M).$$

For $\lambda_i = \ldots = \lambda_N = 0$, Eq.(1.10) reduces to Eq.(1.7). Further if we set $\tau_1 = \mu + 1/2$, $\tau_2 = \ldots = \tau_M = 0$ and for $i = 1, \ldots, N-2$, $\rho_i = -\rho_i$, $\delta_i = 0$ with $\rho_{N-1} = \rho_N = \delta_{N-1} = 0$, $\delta_N = \delta$, Eq.(1.10) yields the family of elliptic-type integrals introduced by Saxena and Kalla [19]. Also if we set $\rho_i = -\rho_i$, $\delta_i = 0$ (for $i = 1, \ldots, N-1$) with

$\rho_N = 0$, $\delta_N = \delta$, Eq.(1.10) reduces to Eq.(1.9). So elliptic type integrals given by Eq.(1.10) includes all the generalizations discussed in Eq.(1.7), Eq.(1.8) and Eq.(1.9).

2. Relations with other families of elliptic-type integrals

On comparing our result (1.10) with the definitions (1.2) - (1.6) we get the following relationships:

(2.1)

$$\bar{\Omega}^{(\alpha,\,\gamma-\alpha)}_{(\underbrace{0,\ldots,0}_{N},\,\mu+1/2,\,\underbrace{0,\ldots,0}_{M-1})}(\rho_1,\ldots,\rho_N,\,\delta_1,\ldots,\delta_N;\,k,\,\underbrace{0,\ldots,0}_{M-1})$$

$$=\;\bar{\Omega}^{(\alpha,\,\gamma-\alpha)}_{(\lambda_1,\ldots,\lambda_N,\,\mu+1/2,\,\underbrace{0,\ldots,0}_{M-1})}(\underbrace{0,\ldots,0}_{N},\,\underbrace{0,\ldots,0}_{N},\,\underbrace{0,\ldots,0}_{M-1})\;=\;R_\mu\,(k,\,\alpha,\,\gamma)$$

where $\{0 \leqslant k < 1\}$; $\{\Re\,(\gamma),\,\Re\,(\alpha) > 0\}$; $\{\mu \in \mathcal{C}\}$.

(2.2)

$$\bar{\Omega}^{(\nu+1/2,\,\nu+1/2)}_{(-2m,\,\underbrace{0,\ldots,0}_{N-1},\,\mu+1/2,\,\underbrace{0,\ldots,0}_{M-1})}(-2,\,\rho_2,\ldots,\rho_N,\,0,\,\delta_2,\ldots,\delta_N;\,k,\,\underbrace{0,\ldots,0}_{M-1})$$

$$=\;\bar{\Omega}^{(\nu+1/2,\,\nu+1/2)}_{(-2m,\,\lambda_2,\ldots,\lambda_N,\,\mu+1/2,\,\underbrace{0,\ldots,0}_{M-1})}(-2,\,\underbrace{0,\ldots,0}_{N-1},\,\underbrace{0,\ldots,0}_{N};\,k,\,\underbrace{0,\ldots,0}_{M-1})\;=\;2^{-2\nu}\,B_\mu(k,\,m,\,\nu)$$

where $\{0 \leqslant k < 1\}$; $\{m \in \mathcal{N}_0,\,\mu \in \mathcal{C}\}$; $\{\Re(\nu) > -\frac{1}{2}\}$.

(2.3)

$$\lim_{\lambda\to\infty}\left[\bar{\Omega}^{(1/2,\,1/2)}_{(\lambda,\,\underbrace{0,\ldots,0}_{N-1},\,\mu+1/2,\,\underbrace{0,\ldots,0}_{M-1})}(-\rho/\lambda,\,\underbrace{0,\ldots,0}_{N-1},\,\underbrace{0,\ldots,0}_{N};\,k,\,\underbrace{0,\ldots,0}_{M-1})\right]\;=\;\Lambda_\mu(\rho;\,k)$$

where $\{0 \leqslant k < 1\}$; $\{\lambda\}$ and $\{\mu \in \mathcal{C}\}$.

(2.4)

$$\bar{\Omega}^{(\alpha,\,\beta)}_{(\lambda,\,\underbrace{0,\ldots,0}_{N-1},\,\mu+1/2,\,\underbrace{0,\ldots,0}_{M-1})}(-\rho,\,\rho_2,\ldots,\rho_N,\,0,\,\delta_2,\ldots,\delta_N;\,k,\,\underbrace{0,\ldots,0}_{M-1})$$

$$=\;\bar{\Omega}^{(\alpha,\,\beta)}_{(\lambda,\,\lambda_2,\ldots,\lambda_N,\,\mu+1/2,\,\underbrace{0,\ldots,0}_{M-1})}(-\rho,\,\underbrace{0,\ldots,0}_{N-1},\,\underbrace{0,\ldots,0}_{N};\,k,\,\underbrace{0,\ldots,0}_{M-1}\;=\;\Lambda^{(\alpha,\,\beta)}_{(\lambda,\,\mu)}\,(\rho;\,k)$$

where $\{0 \leqslant k < 1\}$; $\{\Re(\alpha) > 0,\,\Re(\beta) > 0\}$; $\{\lambda,\,\mu \in \mathcal{C}\}$; $|\rho| < 1$.

(2.5)

$$\bar{\Omega}^{(\alpha,\,\beta)}_{(\lambda,\,\underbrace{0,\ldots,0}_{N-2},\,\gamma;\,\mu+1/2,\,\underbrace{0,\ldots,0}_{M-1})}(-\rho,\,\rho_2,\ldots,\rho_N,\,\delta_1,\,\delta_2,\ldots,\delta_{N-1},\,\delta;\,k,\,0,\ldots,0)$$

$$=\;\bar{\Omega}^{(\alpha,\,\beta)}_{(\lambda,\,\lambda_2,\ldots,\lambda_{N-1},\,\gamma;\,\mu+1/2,\,\underbrace{0,\ldots,0}_{M-1})}(-\rho,\,\underbrace{0,\ldots,0}_{N-1},\,\underbrace{0,\ldots,0}_{M-1},\,\delta;\,k,\,\underbrace{0,\ldots,0}_{M-1}\;=\;\Lambda^{(\alpha,\,\beta)}_{(\lambda,\,\gamma,\,\mu)}\,(\rho,\,\delta;\,k)$$

where $\{0 \leqslant k < 1\}$; $\{\Re(\alpha) > 0,\, \Re(\beta) > 0\}$; $\{\lambda,\, \mu,\, \gamma \in \mathcal{C}\}$; either $|\rho| < 1$, $|\delta| < 1$ or ρ (or δ) $\in \mathcal{C}$, whenever $\lambda = -m$ (or $\gamma = -m$); $m \in \mathcal{N}_0$.

3. Analytic continuation of the generalized family elliptic-type integrals

In the section 1, we have considered a new family of unified and generalized elliptic-type integrals given as:

(3.1)
$$\bar{\Omega}^{(\alpha,\,\beta)}_{(\lambda_i,\,\tau_j)}\,\{(\rho_i),\,(\delta_i);\,k_j\} \;=\; \bar{\Omega}^{(\alpha,\,\beta)}_{\lambda_1,\ldots,\lambda_N,\,\tau_1,\ldots,\tau_M}\,(\rho_1,\ldots,\rho_N,\,\delta_1,\ldots,\delta_N;\,k_1,\ldots,k_M)$$
$$= \int_0^{\pi} \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\,\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\prod_{i=1}^{N}\left[1{+}\rho_i\,\sin^2\left(\frac{\theta}{2}\right){+}\delta_i\,\cos^2\left(\frac{\theta}{2}\right)\right]^{-\lambda_i}\prod_{j=1}^{M}[1{-}k_j^2\,\cos\theta]^{-\tau_j}\,d\theta,$$
where $\min(\{\Re\,(\alpha),\,\Re\,(\beta)\} > 0$, $|k_j| < 1$; $\lambda_i,\,\tau_j \in \mathcal{C}$;
$$\max\left\{|\rho_i|,\,|\delta_i|,\,\left|\frac{2k_j^2}{{k_j}^2-1}\right|,\,\left|\frac{\delta_i-\rho_i}{1+\delta_i}\right|\right\} < 1\ (i=1,\ldots,N \text{ and } j=1,\ldots,M),$$

which can be also explicitly represented in the form

(3.2)
$$\bar{\Omega}^{(\alpha,\,\beta)}_{(\lambda_1,\ldots,\lambda_N;\,\tau_1,\ldots,\tau_M)}\,(\rho_1,\ldots,\rho_N,\,\delta_1,\ldots,\delta_N;\,k_1,\ldots,k_M) \;=\; B\,(\alpha,\,\beta)\prod_{j=1}^{M}(1{-}k_j^2)^{-\tau_j}\prod_{i=1}^{N}[(1{+}\delta_i)^{-\lambda_i}]$$
$$\cdot\, F_D^{(M+N)}\left[\beta,\,\tau_1,\ldots,\tau_M,\,\lambda_1,\ldots,\lambda_N;\,\alpha{+}\beta;\,\frac{2k_1^2}{k_1^2-1},\ldots,\frac{2k_M^2}{k_M^2-1},\,\frac{\delta_1-\rho_1}{1+\delta_1},\ldots,\frac{\delta_N-\rho_N}{1+\delta_N}\right],$$
where $F_D^{M+N}(\cdot)$ is the Lauricella function of $(M+N)$ variables. Valid under the conditions $\min\{\Re\,(\alpha),\,\Re\,(\beta)\} > 0$, $|k_j| < 1$;
$$\max\left\{|\rho_i|,\,|\delta_i|,\,\left|\frac{2k_j^2}{{k_j}^2-1}\right|,\,\left|\frac{\delta_i-\rho_i}{1+\delta_i}\right|\right\} < 1\ (i=1,\ldots,N,\ j=1,\ldots,M),$$

On employing the following formula ([16], p.163):

(3.3)
$$\frac{\Gamma\,(\alpha)\,\Gamma\,(\gamma-\alpha)}{\Gamma\,(\gamma)}\,F_D^{(n)}\,[\alpha,\,\beta_1,\ldots,\beta_n;\,\gamma;\,x_1,\ldots,x_n]$$
$$= \int_0^1 u^{\alpha-1}\,(1{-}u)^{\gamma-\alpha-1}\,(1{-}ux_1)^{-\beta_1}\ldots(1{-}ux_n)^{-\beta_n}\,du,$$
where $\{\Re\,(\gamma) > 0,\,\Re\,(\alpha) > 0\}$, in Eq.(3.1), we get the following useful form involving Euler-type elliptic integral:

(3.4)
$$\bar{\Omega}^{(\alpha,\,\beta)}_{(\lambda_1,\ldots,\lambda_N;\,\tau_1,\ldots,\tau_M)}\,(\rho_1,\ldots,\rho_N,\,\delta_1,\ldots,\delta_N;\,k_1,\ldots,k_M) \;=\; \prod_{j=1}^{M}(1{-}k_j^2)^{-\tau_j}\prod_{i=1}^{N}(1{+}\delta_i)^{-\lambda_i}$$

$$\cdot \int_0^1 \omega^{\beta-1}(1-\omega)^{\alpha-1} \prod_{j=1}^{M} \left[1-\frac{2\omega k_j^2}{k_j^2-1}\right]^{-\tau_j} \prod_{i=1}^{N} \left[1-\frac{(\delta_i-\rho_i)\,\omega}{1+\delta_i}\right]^{-\lambda_i} d\omega,$$

$$\{0 \leqslant k_j^2 < 1\};\ \{\Re(\alpha) > 0,\ \Re(\beta) > 0\},\ \left|\frac{2k_j^2}{k_j^2-1}\right| < 1,\ \left|\frac{\delta_i-\rho_i}{1+\delta_i}\right| < 1,$$

for $i = 1, \ldots, N,\ j = 1, \ldots, M$.

To carry out the analytic continuation we need to show that the above integral in Eq.(3.4) has meaning and gives analytic function of ρ_i, δ_i and k_j (for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$) in the domain where ρ_i in the complex plane cut along $[-1, \infty)$ and δ_i is in the complex plane cut along $[-1, \infty)$. Or ρ_i in the complex plane cut along $(-\infty, -1]$ and δ_i is in the complex plane cut along $(-\infty, -1]$ and k_j^2 is in the complex plane cut along the intervals $(-\infty, -1]$ and $[1, \infty)$. Suppose that ρ_i, δ_i (for $i = 1, 2, \ldots, N$) and k_j (for $j = 1, 2, \ldots, M$) are belonging to the closed domain defined by

(3.5) $$\rho_{i1} \leqslant |\rho_i + 1| \leqslant \rho_{i2},\ |arg(1+\rho_i)| \leqslant \pi - \rho_{i3}$$

(3.6) $$\delta_{i1} \leqslant |\delta_i + 1| \leqslant \delta_{i2},\ |arg(1+\delta_i)| \leqslant \pi - \delta_{i3}$$

(3.7) $$k_{j1}^{+} \leqslant |k_j + 1| \leqslant k_{j2}^{+},\ |arg(1+k_j)| \leqslant \pi - k_{j3}^{+}$$

(3.8) $$k_{j1}^{-} \leqslant |k_j - 1| \leqslant k_{j2}^{-},\ |arg(1-k_j)| \leqslant \pi - k_{j3}^{-};$$

where ρ_{i1}, ρ_{i3}, δ_{i1}, δ_{i3}, $k_{j1}^{\pm}$, $k_{j3}^{\pm}$ are arbitrary small positive numbers and ρ_{i2}, δ_{i2}, $k_{j2}^{\pm}$ are arbitrary large positive numbers (for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$). Furthermore, If $0 < \omega < 1$, then the integrand

(3.9) $$\omega^{\beta-1}(1-\omega)^{\alpha-1} \prod_{i=1}^{N} \left[1-\frac{(\delta_i-\rho_i)\,\omega}{1+\delta_i}\right]^{-\lambda_i} \prod_{j=1}^{M} \left[1-\frac{2\omega k_j^2}{k_j^2-1}\right]^{-\tau_j}$$

is continuous in ω and analytic in each of ρ_i, δ_i and k_j, and we need only to show that the integral is uniformly convergent in the indicated region [15], section (9.1). But this follows at once from the estimate

(3.10)
$$\left|\omega^{\beta-1}(1-\omega)^{\alpha-1} \prod_{i=1}^{N} \left[1-\frac{(\delta_i-\rho_i)\,\omega}{1+\delta_i}\right]^{-\lambda_i} \prod_{j=1}^{M} \left[1-\frac{2\omega k_j^2}{k_j^2-1}\right]^{-\tau_j}\right| \leqslant M\omega^{\Re(\beta)-1}(1-\omega)^{\Re(\alpha)-1},$$

where M is the maximum value of the continuous function

(3.11) $$\left|\prod_{i=1}^{N} \left[1-\frac{(\delta_i-\rho_i)\,\omega}{1+\delta_i}\right]^{-\lambda_i} \prod_{j=1}^{M} \left[1-\frac{2\omega k_j^2}{k_j^2-1}\right]^{-\tau_j}\right|,$$

for $\omega \in [0, 1]$, and ρ_i, δ_i and k_j in the domain defined by (3.5)-(3.8). In addition to that, the integral

$$M \int_0^1 \omega^{\Re(\beta)-1}(1-\omega)^{\Re(\alpha)-1} d\omega,$$

converges for $\{\Re(\alpha) > 0, \Re(\beta) > 0\}$. Thus the conditions stated in Eq.(3.1), Eq.(3.2) and Eq.(3.4) can be dropped and the desired analytic continuation of the elliptic-type integral is given by the formula

(3.12)

$$\bar{\Omega}^{(\alpha,\beta)}_{(\lambda_1,\ldots,\lambda_N;\,\tau_1,\ldots,\tau_M)}(\rho_1,\ldots,\rho_N,\,\delta_1,\ldots,\delta_N;\,k_1,\ldots,k_M) = \prod_{j=1}^{M}(1-k_j^2)^{-\tau_j}\prod_{i=1}^{N}(1+\delta_i)^{-\lambda_i}$$

$$\cdot\int_0^1 \omega^{\beta-1}(1-\omega)^{\alpha-1}\prod_{j=1}^{M}\left[1-\frac{2\omega k_j^2}{k_j^2-1}\right]^{-\tau_j}\prod_{i=1}^{N}\left[1-\frac{(\delta_i-\rho_i)\,\omega}{1+\delta_i}\right]^{-\lambda_i}d\omega$$

with the conditions $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\left|arg(1+k_j^2)\right| < \pi$, $|arg(1+\rho_i)| < \pi$, $|arg(1+\delta_i)| < \pi$ (for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$).

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