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Certain properties of fractional calculus operators associated with M-series

Amit Chouhan[a] and Satish Saraswat[b]

Abstract. The present investigation deals with fractional calculus of the generalized M-series which is a further extension of both Mittag-Leffler function and generalized hypergeometric function $_{\mathrm{p}}\mathrm{F}_{\mathrm{q}}$, and these functions have recently found essential applications in solving problems in physics, biology, engineering and applied sciences. Certain relations that exist between M-series and the Riemann-Liouville fractional integrals and derivatives are investigated. It has been shown that the fractional integration and differentiation operators transform such functions with power multipliers into the functions of the same form.

1. Introduction

Sharma and Jain [8] introduced the generalized M-series as the function defined by means of the power series:

$$ {}_{p}^{\alpha}M_{q}^{\beta}\left(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots b_q; z\right) = {}_{p}^{\alpha}M_{q}^{\beta}(z) = {}_{p}^{\alpha}M_{q}^{\beta}\left((a_j)_1^p; (b_j)_1^q; z\right)$$

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{\Gamma(\alpha k + \beta)}; \qquad z, \alpha, \beta \in C, \ R(\alpha) > 0 \tag{1.1}$$

Here $(a_j)_k$, $(b_j)_k$ are the known Pochammer symbols. As usually, the series (1.1) is defined when none of the parameters b_j's, $j = 1, 2, \ldots ,q$, is a negative integer or zero; if any numerator parameter a_j is a negative integer or zero, then the series terminates to a polynomial in z. The series in (1.1) is convergent for all z if $p \leqslant q$, it is convergent for $|z| < \delta = \alpha^{\alpha}$ if $p = q + 1$ and divergent, if $p > q + 1$. When $p = q + 1$ and $|z| = \delta$, the series can converge on conditions depending on the parameters.

Some special cases of the ${}_{\mathrm{p}}^{\alpha}\mathrm{M}_{\mathrm{q}}^{\beta}$ (z)-function are the following:

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- The Mittag-Leffler function ([1],[2],[10]): When there is no upper and lower parameters (p = q = 0), we have

$$E_{\alpha,\beta}(z) = {}_0^{\alpha}M_0^{\beta}(-\;;\;-;z) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+\beta)}z^k\;;\;\mathrm{E}_{\alpha}(\mathrm{z}) = {}_0^{\alpha}M_0(-\;;\;-;z), \tag{1.2}$$

where $\mathrm{E}_{\alpha}(\mathrm{z})$, β= 1 is the one parameter Mittag-Leffler function [3].

- The generalized Mittag-Leffler function, introduced by Prabhakar [4] and studied by Kilbas et al., [1], is obtained from (1.1) for p = q = 1; $a = \gamma \in C$; b = 1:

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!} = \sum_{k=0}^{\infty}\frac{(\gamma)_k}{(1)_k}\frac{z^k}{\Gamma(\alpha k+\beta)} = {}_1^{\alpha}M_1^{\beta}(\gamma;1;z) \tag{1.3}$$

- The generalized M-series can be represented as a special case of the Wright generalized hypergeometric function [1],

$${}_p^{\alpha}M_q^{\beta}\left((a_j)_1^p;(b_j)_1^q;z\right) = k\ {}_{\mathrm{p+1}}\psi_{\mathrm{q+1}}\left[\begin{matrix}(a_1,1),\ldots,(a_p,1),(1,1); \\ (b_1,1),\ldots,(b_q,1),(\beta,\alpha);\end{matrix}\ \mathrm{z}\right];\ k = \frac{\prod_{\mathrm{j=1}}^{\mathrm{q}}\Gamma(\mathrm{b_j})}{\prod_{\mathrm{j=1}}^{\mathrm{p}}\Gamma(\mathrm{a_j})}.$$

The left and right-sided Riemann-Liouville fractional calculus operators are defined by Samko et al., [6]. For $\alpha \in C, (Re(\alpha)\ >\ 0)$

$$\left(I_{0+}^{\alpha}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x\frac{f(t)}{(x-t)^{1-\alpha}}dt;(x>0), \tag{1.4}$$

$$\left(I_{-}^{\alpha}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\infty}\frac{f(t)}{(t-x)^{1-\alpha}}dt;(x>0), \tag{1.5}$$

$$\left(D_{0+}^{\alpha}f\right)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1}\left(I_{a+}^{1-\{\alpha\}}f\right)(x),$$

$$= \frac{1}{\Gamma(1-\{\alpha\})}\left(\frac{d}{dx}\right)^{[\alpha]+1}\int_0^x\frac{f(t)}{(x-t)^{\{\alpha\}}}dt, \tag{1.6}$$

$$\left(D_{-}^{\alpha}f\right)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1}\left(I_{-}^{1-\{\alpha\}}f\right)(x),$$

$$= \frac{1}{\Gamma(1-\{\alpha\})}\left(-\frac{d}{dx}\right)^{[\alpha]+1}\int_x^{\infty}\frac{f(t)}{(t-x)^{\{\alpha\}}}dt, \tag{1.7}$$

where $[\alpha]$ means the maximal integer not exceeding α and $\{\alpha\}$is the fractional part of α.

Generalization of the Riemann-Liouville and Erdlyi-Kober fractional integral operators has been introduced by Saigo [5] in terms of Gauss hypergeometric function as given below. Let $\alpha,\beta,\gamma \in C$ and $x \in R_+$,

$$\left(I_{0+}^{\alpha,\beta,\gamma}f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}{}_2F_1(\alpha+\beta,-\gamma;\alpha;1-\frac{t}{x})f(t)dt, \mathrm{Re}(\alpha)>0. \tag{1.8}$$

2. Fractional calculus of generalized M-series

Theorem 2.1 *Let $\alpha > 0, \beta > 0, \gamma > 0,\ a \in R$ and I_{0+}^{α} be the left sided operator of Riemann-Liouville fractional integral (1.4).* Then there holds the formula

$$\left(I_{0+}^{\alpha}\left[t^{\gamma-1}\ {}_{p}^{\beta}M_{q}^{\gamma}\left((a_j)_1^p;\ (b_j)_1^q; at^{\beta}\right)\right]\right)(x) = x^{\alpha+\gamma-1}\ {}_{p}^{\beta}M_{q}^{\alpha+\gamma}\left((a_j)_1^p; (b_j)_1^q; ax^{\beta}\right) \quad (2.1)$$

Proof. By using definition of generalised M-series (1.1) and fractional integral formula (1.4), we obtained

$$K \equiv \left(I_{0+}^{\alpha}\left[t^{\gamma-1}{}_{p}^{\beta}M_{q}^{\gamma}\left((a_j)_1^p; (b_j)_1^q; at^{\beta}\right)\right]\right)(x)$$

$$= \frac{1}{\Gamma\alpha}\int_0^x (x-t)^{\alpha-1}t^{\gamma-1}\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\ \frac{a^n t^{\beta n}}{\Gamma(\beta n+\gamma)}dt$$

$$= \frac{1}{\Gamma\alpha}\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\ \frac{a^n}{\Gamma(\beta n+\gamma)}\ \int_0^x (x-t)^{\alpha-1}t^{\gamma+\beta n-1}dt$$

$$= x^{\alpha+\gamma-1}\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\ \frac{\left(ax^{\beta}\right)^n}{\Gamma(\beta n+\alpha+\gamma)}$$

$$= x^{-\gamma}{}_{p}^{\beta}M_{q}^{\alpha+\gamma}\left((a_j)_1^p; (b_j)_1^q; ax^{-\beta}\right).$$

Remark 1. If we put $p = q = 1,\ a = \delta \in C,\ b = 1$ in (2.1), we obtained

$$\left(I_{0+}^{\alpha}\ \left[t^{\gamma-1}\ E_{\beta,\gamma}^{\delta}\left(at^{\beta}\right)\right]\right)(x) = x^{\alpha+\gamma-1}E_{\beta,\alpha+\gamma}^{\delta}\left(ax^{\beta}\right),$$

which is well derived result [7, p.145, Eq.14] by Saxena and Saigo.

Theorem 2.2. *Let $\alpha > 0, \beta > 0, \gamma > 0,\ a \in R$ and I_{0-}^{α} be the right sided operator of Riemann-Liouville fractional integral (1.5). Then there holds the formula*

$$\left(I_{-}^{\alpha}\left[t^{-\alpha-\gamma}\ {}_{p}^{\beta}M_{q}^{\gamma}\left((a_j)_1^p\ ; (b_j)_1^q; at^{-\beta}\right)\right]\right)(x) = x^{-\gamma}\ {}_{p}^{\beta}M_{q}^{\alpha+\gamma}\left((a_j)_1^p; (b_j)_1^q; ax^{-\beta}\right)$$

Proof. Using (1.1) and fractional integral formula (1.5), we obtained

$$K \equiv \left(I_{-}^{\alpha}\left[t^{-\alpha-\gamma}\ {}_{p}^{\beta}M_{q}^{\gamma}\left((a_j)_1^p; (b_j)_1^q; at^{-\beta}\right)\right]\right)(x)$$

$$= \frac{1}{\Gamma\alpha}\ \int_x^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\gamma}\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{a^n t^{-\beta n}}{\Gamma(\beta n+\gamma)}dt$$

$$= \sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{a^n x^{-\gamma-\beta n}}{\Gamma(\alpha+\beta n+\gamma)}$$

$$= x^{-\gamma}\ {}_{p}^{\beta}M_{q}^{\alpha+\gamma}\left((a_j)_1^p; (b_j)_1^q; ax^{-\beta}\right).$$

Remark 2. If we put $p = q = 1,\ a = \delta \in C,\ b = 1$ in (2.2), we obtained

$$\left(I_{-}^{\alpha}\left[t^{-\alpha-\gamma}E_{\beta,\gamma}^{\delta}\left(at^{-\beta}\right)\right]\right)(x) = x^{-\gamma}\ E_{\beta,\alpha+\gamma}^{\delta}\left(ax^{-\beta}\right),$$

which is well derived result [7, p.147, Eq.23] by Saxena and Saigo.

Theorem 2.3. *Let* $\alpha > 0, \beta > 0, \gamma > 0,\ a \in R$ *and* D_{0+}^{α} *be the left sided operator of Riemann-Liouville fractional derivative (1.6). Then there holds the formula*

$$\left(D_{0+}^{\alpha}\left[t^{\gamma-1}\ {}_{p}^{\beta}M_{q}^{\gamma}\left((a_j)_1^p;(b_j)_1^q;at^{\beta}\right)\right]\right)(x) = x^{\gamma-\alpha-1}\ {}_{p}^{\beta}M_{q}^{\gamma-\alpha}\left((a_j)_1^p;(b_j)_1^q;ax^{\beta}\right) \quad (2.3)$$

Proof. Using (1.1) and fractional integral formula (1.6), we obtained

$$K \equiv \left(D_{0+}^{\alpha}\left[t^{\gamma-1}\ {}_{p}^{\beta}M_{q}^{\gamma}\left((a_j)_1^p;(b_j)_1^q;at^{\beta}\right)\right]\right)(x)$$

$$= \left(\frac{d}{dx}\right)^{k}\left(I_{0+}^{k-\alpha}\ \left[t^{\gamma-1}\ {}_{p}^{\beta}M_{q}^{\gamma}\left((a_j)_1^p;(b_j)_1^q;at^{\beta}\right)\right]\right)(x),$$

where $k = [\alpha] + 1$,

$$= \frac{1}{\Gamma(k-\alpha)}\left(\frac{d}{dx}\right)^{k}\int_0^x (x-t)^{k-\alpha-1}\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\ \frac{a^n t^{\beta n+\gamma-1}}{\Gamma(\beta n+\gamma)}dt$$

$$= \sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{\Gamma(\beta n+\gamma)\,a^n}{\Gamma(\beta n+\gamma+k-\alpha)}\left(\frac{d}{dx}\right)^{k}x^{\beta n+\gamma-\alpha+k-1}$$

$$= \sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{a^n x^{\beta n+\gamma-\alpha-1}}{\Gamma(\beta n+\gamma-\alpha)}$$

$$= x^{\gamma-\alpha-1}\ {}_{p}^{\beta}M_{q}^{\gamma-\alpha}\left((a_j)_1^p;(b_j)_1^q;ax^{\beta}\right).$$

Remark 3. If we put $p = q = 1,\ a = \delta \in C,\ b = 1$ in (2.3), we obtained the well established result [7, p.149, Eq.29].

Theorem 2.4. *Let* $\alpha > 0, \beta > 0, \gamma > 0,$ *with* $\gamma - \alpha + \{\alpha\} > 1$ *and* $a \in R$ *and let* D_{-}^{α} *be the right sided operator of Riemann-Liouville fractional derivative (1.7). Then there holds the formula*

$$\left(D_{-}^{\alpha}\left[t^{\alpha-\gamma}\ {}_{p}^{\beta}M_{q}^{\gamma}\left((a_j)_1^p;(b_j)_1^q;at^{-\beta}\right)\right]\right)(x) = x^{-\gamma}\ {}_{p}^{\beta}M_{q}^{\gamma-\alpha}\left((a_j)_1^p;(b_j)_1^q;ax^{-\beta}\right) \quad (2.4)$$

Proof. Using (1.1) and fractional integral formula (1.7), we obtained

$$K \equiv \left(D_{-}^{\alpha}\left[t^{\alpha-\gamma}\ {}_{p}^{\beta}M_{q}^{\gamma}\left((a_j)_1^p;(b_j)_1^q;at^{-\beta}\right)\right]\right)(x)$$

$$= \left(-\frac{d}{dx}\right)^{k}\left(\mathrm{I}_{-}^{k-\alpha}\left[t^{\alpha-\gamma}\ {}_{p}^{\beta}M_{q}^{\gamma}\left((a_j)_1^p;(b_j)_1^q;at^{-\beta}\right)\right]\right)(x),$$

where $k = [\alpha] + 1$,

$$= \frac{1}{\Gamma(k-\alpha)}\left(-\frac{d}{dx}\right)^{k}\int_x^{\infty}(t-x)^{k-\alpha-1}t^{\alpha-\gamma}\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{a^n t^{-\beta n}}{\Gamma(\beta n+\gamma)}dt$$

$$= \sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{\Gamma(\beta n+\gamma-k)\,a^n}{\Gamma(\beta n+\gamma-\alpha)\,\Gamma(\beta n+\gamma)}\left(-\frac{d}{dx}\right)^{k}x^{-\beta n-\gamma+k}$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{a^n x^{-\beta n-\gamma}}{\Gamma(\beta n+\gamma-\alpha)}$$

$$= x^{-\gamma} \, {}_p^{\beta}M_q^{\gamma-\alpha}\left((a_j)_1^p; (b_j)_1^q; ax^{-\beta}\right).$$

Remark 4. If we put $p = q = 1,\ a = \delta \in C,\ b = 1$ in (2.4), we obtained the well established result [7, p.150, Eq.35].

Theorem 2.5. *Let* $\alpha > 0, \beta > 0, \gamma > 0, \upsilon > 0, \rho > 0,\ Re(\alpha+\gamma) \geqslant 0$ *and* $I_{0+}^{\alpha,\beta,\gamma}$ *be the left sided operator of Riemann-Liouville fractional integral (1.8). Then there holds the formula*

$$\left(I_{0+}^{\alpha,\beta,\gamma}\left[\,{}_p^{\upsilon}M_q^{\rho}\left((a_j)_1^p; (b_j)_1^q; t\right)\right]\right)(x)$$

$$= \frac{x^{-\beta}}{\Gamma(1-\beta)} \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma-\beta+1)} {}_{p+2}^{\upsilon}M_{q+2}^{\rho}\left((a_j)_1^p, \gamma-\beta+1, 1; (b_j)_1^q, \alpha+\gamma+1,\ 1-\beta; x\right) \tag{2.5}$$

Proof. Using (1.1) and fractional integral formula (1.8), we obtained

$$K \equiv \left(I_{0+}^{\alpha,\beta,\gamma}\left[{}_p^{\upsilon}M_q^{\rho}\left((a_j)_1^p; (b_j)_1^q; t\right)\right]\right)(x)$$

$$= \frac{x^{-\alpha-\beta}}{\Gamma\alpha} \int_0^x (x-t)^{\alpha-1} {}_2\mathrm{F}_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\mathrm{t}}{\mathrm{x}}\right) \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{t^k}{\Gamma(\upsilon k+\rho)} dt$$

By the use of Gaussian hypergeometric series [9], interchanging the order of integration and summations and evaluating the inner integral by the use of the known formula of Beta integral,

$$= x^{-\beta} \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{(\alpha+\beta)_n(-\gamma)_n}{(\alpha+k+1)_n \ \mathrm{n}!} \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)} \frac{x^k}{\Gamma(\upsilon k+\rho)}$$

Finally by the virtue of Gauss summation theorem, we have

$$= \frac{x^{-\beta}}{\Gamma(1-\beta)} \frac{\Gamma(\alpha+\gamma+1)}{\Gamma(\gamma-\beta+1)} {}_{p+2}^{\upsilon}M_{q+2}^{\rho}\left((a_j)_1^p, \gamma-\beta+1, 1;\ (b_j)_1^q, \alpha+\gamma+1,\ 1-\beta; x\right).$$

Remark 5. If we put $\alpha = -\beta$ in our result (2.5), we arrive at the result [8, p.451,Eq.10] given by Sharma and Jain.

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[a]Department of Mathematics, JIET-SETG,
JIET Group of Institutions,
Jodhpur 342002,
India
E-mail address: amit.chouhan@indiatimes.com

[b]Department of Mathematics,
Govt. College Kota,
Kota 324001,
India
E-mail address: satish_ktt@yahoo.co.in