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On some definite integrals connecting with sums related to inverse tangent function

Ramesh Kumar Muthumalai

Abstract. Some definite integrals are evaluated through sums related to inverse tangent function that found in chapter 2 of Ramanujan notebooks, Part-I. In addition, integrals involving complicated arguments of inverse tangent function are evaluated through infinite products.

1. Introduction

The chapter 2 of Ramanujan notebooks, Part-I [**1**] contains many elementary formulas and identities on finite and infinite sums. Many of these identities involve sums related to harmonic and inverse tangent functions. In particular, it contains following infinite sums related to inverse tangent function

$$\sum_{k=-\infty}^{\infty} (-1)^k \tan^{-1} \frac{x}{k\pi + a} = \tan^{-1} (\sinh x \csc a), \tag{1.1}$$

$$\sum_{k=-\infty}^{\infty} \tan^{-1} \frac{x}{k\pi + a} = \tan^{-1} (\tanh x \cot a), \tag{1.2}$$

$$\sum_{k=0}^{\infty} (-1)^k \tan^{-1} \frac{x}{2k+1} = \tan^{-1} \left(\tanh \frac{\pi x}{4}\right), \tag{1.3}$$

for $a, x \in R$ [**1**, p. 39-40].

In the present study, some definite integrals that involving various combinations of elementary functions are evaluated through above mentioned infinite series either by direct formulas or infinite products. The solutions of integrals presented here are not found in the classical table of integrals by Gradshteyn and Ryzhik [**2**].

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2. Definite integrals involving elementary functions.

In this section, the direct formulas for integrals involving various combinations of powers and elementary functions are derived through the sums related to inverse tangent function.

2.1. Combinations of powers, trigonometric and hyperbolic function.

Consider the following definite integral for a positive integer m and an arbitrary a

$$\begin{aligned}&\int_0^\infty \frac{\sin^{2m+1} at}{\cosh t}\frac{dt}{t}\\ &= 2\int_0^\infty \frac{\sin^{2m+1} at}{t}\sum_{k=0}^{\infty}(-1)^k e^{-(2k+1)t}dt\\ &= 2\sum_{k=0}^{\infty}(-1)^k\int_0^\infty e^{-(2k+1)t}\frac{\sin^{2m+1} at}{t}dt\end{aligned}$$

Then, using the solution of the integral $\int_0^\infty e^{-(2k+1)t}\frac{\sin^{2m+1} at}{t}dt$ [**2**, p. 494]

$$= \frac{(-1)^m}{2^{2m-1}}\sum_{k=0}^{\infty}(-1)^k\sum_{i=0}^{m}(-1)^i\binom{2m+1}{i}\tan^{-1}\frac{(2m-2i+1)a}{2k+1}.$$

Using the identity (1.3), yields

$$\begin{aligned}&\int_0^\infty \frac{\sin^{2m+1} at}{\cosh t}\frac{dt}{t}\\ &= \frac{(-1)^m}{2^{2m-1}}\sum_{i=0}^{m}(-1)^i\binom{2m+1}{i}\tan^{-1}\left(\tanh(2m-2i+1)\frac{a\pi}{4}\right).\end{aligned} \tag{2.1}$$

Further, the definite integrals involving powers, hyperbolic cosine function with various combinations of $\sin ax$ and $\cos ax$ can be evaluated by differentiating (2.1) with respect to a.

EXAMPLE 2.1. Let $m = 1$ and $a = 2$ in (2.1). Then

$$\int_0^\infty \frac{\sin^3 2t}{\cosh t}\frac{dt}{t} = -\frac{1}{2}\tan^{-1}\left(\tanh\frac{3\pi}{2}\right) + \frac{3}{2}\tan^{-1}\left(\tanh\frac{\pi}{2}\right).$$

Similar integrals involving powers, hyperbolic and trigonometric functions are listed in sections 4.11-4.12 of classical table of integrals [**2**].

2.2. Combinations of powers, exponentials, trigonometric and hyperbolic functions.

Consider the following definite integral for a positive integer m and $|a| < \pi$

$$2\int_0^\infty \frac{\sinh at\sin^{2m+1} tx}{t\left(e^{\pi t}-1\right)}dt = \int_0^\infty e^{-\pi t}\frac{e^{at}-e^{-at}}{1-e^{-\pi t}}\frac{\sin^{2m+1} tx}{t}dt$$

After simplification, gives

$$= \sum_{k=1}^{\infty} \int_0^{\infty} \frac{\sin^{2m+1} tx}{t} \left(e^{-t(k\pi-a)} - e^{-t(k\pi+a)}\right) dt$$

Then, using solution of the integral $\int_0^{\infty} e^{-px} \sin^{2m+1} ax \frac{dx}{x}, p > 0$[**2**, p. 494]

$$= \frac{(-1)^m}{2^{2m}} \left[\sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} \left(\sum_{k=1}^{\infty} \tan^{-1} \frac{(2m-2i+1)x}{k\pi - a} - \sum_{k=1}^{\infty} \tan^{-1} \frac{(2m-2i+1)x}{k\pi + a}\right)\right].$$

Using the identity (1.2), yields

$$\int_0^{\infty} \frac{\sinh at \sin^{2m+1} tx}{t\left(e^{\pi t}-1\right)} dt = \frac{(-1)^{m+1}}{2^{2m+1}} \sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} \times \left[\tan^{-1}\left(\frac{\tanh(2m-2i+1)x}{\tan a}\right) - \tan^{-1}\left(\frac{(2m-2i+1)x}{a}\right)\right]. \tag{2.2}$$

Similarly, using the identity (1.1), it is find that

$$\int_0^{\infty} \frac{\sinh at \sin^{2m+1} tx}{t\left(e^{\pi t}+1\right)} dt = \frac{(-1)^{m+1}}{2^{2m+1}} \sum_{i=0}^{m} (-1)^i \binom{2m+1}{i} \times \left[\tan^{-1}\left(\frac{\sinh(2m-2i+1)x}{\sin a}\right) - \tan^{-1}\left(\frac{(2m-2i+1)x}{a}\right)\right]. \tag{2.3}$$

Example 2.2. Let $m = 1$ in (2.2) and (2.3). Then

$$\int_0^{\infty} \frac{\sinh at \sin^3 tx}{t\left(e^{\pi t}-1\right)} dt = \frac{1}{8}\left[\tan^{-1}\left(\frac{\tanh 3x}{\tan a}\right) - \tan^{-1}\left(\frac{3x}{a}\right)\right] - \frac{3}{8}\left[\tan^{-1}\left(\frac{\tanh x}{\tan a}\right) - \tan^{-1}\left(\frac{x}{a}\right)\right]$$

and

$$\int_0^{\infty} \frac{\sinh at \sin^3 tx}{t\left(e^{\pi t}+1\right)} dt = \frac{1}{8}\left[\tan^{-1}\left(\frac{\sinh 3x}{\sin a}\right) - \tan^{-1}\left(\frac{3x}{a}\right)\right] - \frac{3}{8}\left[\tan^{-1}\left(\frac{\sinh x}{\sin a}\right) - \tan^{-1}\left(\frac{x}{a}\right)\right].$$

Remark 2.3. The solutions of the following definite integrals can be expressed in finite terms by differentiating (2.2) and (2.3) $2n$ times with respect to a

$$\int_0^{\infty} t^{2n-1} \frac{\sinh at \sin^{2m+1} tx}{e^{\pi t}-1} dt \quad \text{and} \quad \int_0^{\infty} t^{2n-1} \frac{\sinh at \sin^{2m+1} tx}{e^{\pi t}+1} dt.$$

The following integrals can be expressed in finite terms by differentiating (2.2) and (2.3) $2n+1$ times with respect to a

$$\int_0^\infty t^{2n} \frac{\cosh at \sin^{2m+1} tx}{e^{\pi t}-1} dt \quad \text{and} \quad \int_0^\infty t^{2n} \frac{\cosh at \sin^{2m+1} tx}{e^{\pi t}+1} dt.$$

Similarly, differentiating (2.2) and (2.3) with respect to x, the various forms of definite integrals in the combinations powers, exponentials, hyperbolic and trigonometric functions can be expressed in finite terms.

REMARK 2.4. Since $\sinh^{2r+1} x$ can be expressed in finite terms of hyperbolic sine of integral multiples of x, the following integrals can be expressed in finite terms

$$\int_0^\infty \frac{\sinh^{2r+1} at \sin^{2m+1} tx}{t(e^{\pi t}-1)} dt \quad \text{and} \quad \int_0^\infty \frac{\sinh^{2r+1} at \sin^{2m+1} tx}{t(e^{\pi t}+1)} dt.$$

EXAMPLE 2.5. Let $r=1$ and $m=1$ and using integrals in example 2.2. Then

$$\int_0^\infty \frac{\sinh^3 at \sin^3 tx}{t(e^{\pi t}-1)} dt = \frac{1}{32}\left[\tan^{-1}\left(\frac{\tanh 3x}{\tan 3a}\right) - \tan^{-1}\left(\frac{x}{a}\right)\right]$$
$$-\frac{3}{32}\left[\tan^{-1}\left(\frac{\tanh x}{\tan 3a}\right) - \tan^{-1}\left(\frac{x}{3a}\right)\right] - \frac{3}{32}\tan^{-1}\left(\frac{\tanh 3x}{\tan a}\right)$$
$$+\frac{3}{32}\tan^{-1}\left(\frac{3x}{a}\right) + \frac{9}{32}\left[\tan^{-1}\left(\frac{\tanh x}{\tan a}\right) - \tan^{-1}\left(\frac{x}{a}\right)\right].$$

Similar integrals involving powers, exponentials, hyperbolic and trigonometric functions are listed in section 4.13 of classical table of integrals [**2**].

2.3. Combinations of logarithmic gamma and rational functions. Consider the following definite integral [**2**, p. 556] for $b>0$ and $c>0$

$$\int_0^\infty \log\left(1+\frac{x^2}{b^2}\right)\frac{dx}{c^2-x^2} = -\frac{\pi}{c}\tan^{-1}\frac{c}{b}.$$

It can be easily find that for $a \notin N$

$$\int_0^\infty \log \frac{1+\left(\frac{x}{k+a}\right)^2}{1+\left(\frac{x}{k-a}\right)^2}\frac{dx}{c^2-x^2} = -\frac{\pi}{c}\left(\tan^{-1}\frac{c}{k+a} - \tan^{-1}\frac{c}{k-a}\right).$$

Taking summation on both sides $k=1,2,\ldots\infty$ and after simplification

$$\int_0^\infty \log \prod_{k=1}^\infty \frac{1+\left(\frac{x}{k+a}\right)^2}{1+\left(\frac{x}{k-a}\right)^2}\frac{dx}{c^2-x^2} = -\frac{\pi}{c}\sum_{k=1}^\infty\left(\tan^{-1}\frac{c}{k+a} - \tan^{-1}\frac{c}{k-a}\right).$$

Using the following identity [**2**, p. 886] for x, y real and $x \neq 0, -1, -2, \ldots$

$$\prod_{k=0}^\infty\left(1+\frac{y^2}{(x+k)^2}\right) = \left|\frac{\Gamma(x)}{\Gamma(x-iy)}\right|^2.$$

After simplification, yields

$$\int_0^\infty \log \left| \frac{\Gamma(a)\Gamma(-a-ix)}{\Gamma(-a)\Gamma(a-ix)} \right| \frac{dx}{c^2-x^2} = -\frac{\pi}{2c}\left[\tan^{-1}\left(\frac{\tanh c\pi}{\tan a\pi}\right) - \tan^{-1}\left(\frac{c}{a}\right)\right]. \tag{2.4}$$

2.4. Complicated arguments of inverse tangent function. Multiply the identity (1.3) by $x/(p^2+x^2)^2$ and then integrating on $[0,\infty)$, we have

$$\sum_{k=0}^{\infty}(-1)^k \int_0^\infty \frac{x\tan^{-1}\frac{x}{2k+1}}{(p^2+x^2)^2}dx = \int_0^\infty \frac{x\tan^{-1}\left(\tanh\frac{\pi x}{4}\right)}{(p^2+x^2)^2}dx.$$

It is well known that [**2**, p. 599]

$$\int_0^\infty \frac{x\tan^{-1} qx}{(p^2+x^2)^2}dx = \frac{\pi q}{4p}\frac{1}{1+pq}, \qquad p>0, q\geqslant 0. \tag{2.5}$$

Using the identity in Ref[**2**, p. 897], it gives

$$\int_0^\infty \frac{x\tan^{-1}\left(\tanh\frac{\pi x}{4}\right)}{(p^2+x^2)^2}dx = \frac{\pi}{4p}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1+p} = \frac{\pi}{8p}\beta\left(\frac{p+1}{2}\right). \tag{2.6}$$

where $\beta(x) = \frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right]$ [**2**, p. 896]. Replacing x by $x\pi$ and a by $a\pi$ in identity (1.2), gives

$$\tan^{-1}\frac{x}{a} + \sum_{k=1}^{\infty}\tan^{-1}\frac{x}{k+a} - \sum_{k=1}^{\infty}\tan^{-1}\frac{x}{k-a} = \tan^{-1}(\tanh x\pi \cot a\pi).$$

Multiply by $x/(p^2+x^2)^2$, integrating on $[0,\infty)$ and using (2.5), then

$$\int_0^\infty \frac{x\tan^{-1}(\tanh x\pi\cot a\pi)}{(p^2+x^2)^2}dx = \frac{\pi}{4p(a+p)} + \frac{\pi}{4p}\sum_{k=1}^{\infty}\left(\frac{1}{k+p+a} - \frac{1}{k+p-a}\right).$$

Using an identity of psi function ψ in Ref[**2**, p. 893] and after simplification

$$\int_0^\infty \frac{x\tan^{-1}(\tanh x\pi\cot a\pi)}{(p^2+x^2)^2}dx = \frac{\pi}{4p}\left[\psi(p+1-a) - \psi(p+a)\right]. \tag{2.7}$$

Similarly, using the identity (1.1), yields

$$\int_0^\infty \frac{x\tan^{-1}(\sinh \pi x\csc a\pi)}{(p^2+x^2)^2}dx = \frac{\pi}{4p}\left[\beta(p+1-a) + \beta(p+a)\right]. \tag{2.8}$$

A variety of integrals involving inverse tangent function is listed in section 4.5 of Ref [**2**].

REMARK 2.6. The collection of integrals given in this section cannot be evaluated using a symbolic language. For instance, the current version of Mathematica 8.0 cannot compute the integrals in examples 2.1, 2.2 and 2.5.

3. Definite integrals connecting with infinite products.

In this section, definite integrals involving complicated arguments of inverse tangent function are evaluated by connecting with infinite products.

Divide both sides of identity (1.3) by $x(p^2+x^2)$ and integrate on $(0,\infty]$, then

$$\sum_{k=0}^{\infty}(-1)^k\int_0^{\infty}\frac{\tan^{-1}\frac{x}{2k+1}}{x(p^2+x^2)}dx=\int_0^{\infty}\frac{\tan^{-1}\left(\tanh\frac{\pi x}{4}\right)}{x(p^2+x^2)}dx.$$

Using the following integral [**2**, p. 599]

$$\int_0^{\infty}\frac{\tan^{-1}qx}{x(p^2+x^2)}dx=\frac{\pi}{2p^2}\log(1+pq),\qquad p>0,q\geqslant 0. \tag{3.1}$$

After simplification, gives

$$\int_0^{\infty}\frac{\tan^{-1}\left(\tanh\frac{\pi x}{4}\right)}{x(p^2+x^2)}dx=\frac{\pi}{2p^2}\log\prod_{k=0}^{\infty}\left(1+\frac{p}{2k+1}\right)^{(-1)^k}.$$

Similarly, using the identity (1.3) and following integrals [**2**, p. 599]

$$\int_0^{\infty}\frac{\tan^{-1}qx}{x(1-p^2x^2)}dx=\frac{\pi}{4}\log\left(1+\frac{q^2}{p^2}\right),\qquad p>0,q\geqslant 0. \tag{3.2}$$

$$\int_0^{1}\frac{\tan^{-1}qx}{x\sqrt{1-x^2}}dx=\frac{\pi}{2}\log\left(q+\sqrt{1+q^2}\right),\qquad q\geqslant 0. \tag{3.3}$$

The following integrals can be expressed in terms of infinite products as follows

$$\int_0^{\infty}\frac{\tan^{-1}\left(\tanh\frac{\pi x}{4}\right)}{x(1-p^2x^2)}dx=\frac{\pi}{4}\log\prod_{k=0}^{\infty}\left(1+\frac{1}{(2k+1)^2p^2}\right)^{(-1)^k}.$$

$$\int_0^{1}\frac{\tan^{-1}(\tanh\frac{\pi x}{4})}{x\sqrt{1-x^2}}dx=\frac{\pi}{2}\log\prod_{k=0}^{\infty}\left(\frac{1}{2k+1}+\sqrt{1+\frac{1}{(2k+1)^2}}\right)^{(-1)^k}.$$

The following infinite products representations of integrals are obtained by using the identity (1.2) and integrals (3.1), (3.2) and (3.3). For $|a| \notin W$,

$$\int_0^\infty \frac{\tan^{-1}(\tanh \pi x \cot a\pi)}{x(p^2+x^2)}dx = \frac{\pi}{2p^2}\log \frac{\prod_{k=0}^{\infty}\left(1+\frac{p}{k+a}\right)}{\prod_{k=1}^{\infty}\left(1+\frac{p}{k-a}\right)}.$$

$$\int_0^\infty \frac{\tan^{-1}(\tanh \pi x \cot a\pi)}{x(1-p^2x^2)}dx = \frac{\pi}{4}\log \frac{\prod_{k=0}^{\infty}\left(1+\frac{1}{p^2(k+a)^2}\right)}{\prod_{k=1}^{\infty}\left(1+\frac{1}{p^2(k-a)^2}\right)}.$$

$$\int_0^1 \frac{\tan^{-1}(\tanh \pi x \cot a\pi)}{x\sqrt{1-x^2}}dx = \frac{\pi}{2}\log \frac{\prod_{k=0}^{\infty}\left(\frac{1}{k+a}+\sqrt{1+\frac{1}{(k+a)^2}}\right)}{\prod_{k=1}^{\infty}\left(\frac{1}{k-a}+\sqrt{1+\frac{1}{(k-a)^2}}\right)}.$$

Similarly, the following integrals are derived from the identity (1.1) and the integrals given in (3.1), (3.2) and (3.3).

$$\int_0^\infty \frac{\tan^{-1}(\sinh \pi x \csc a\pi)}{x(p^2+x^2)}dx = \frac{\pi}{2p^2}\log \frac{\prod_{k=0}^{\infty}\left(1+\frac{p}{k+a}\right)^{(-1)^k}}{\prod_{k=1}^{\infty}\left(1+\frac{p}{k-a}\right)^{(-1)^k}}.$$

$$\int_0^\infty \frac{\tan^{-1}(\sinh \pi x \csc a\pi)}{x(1-p^2x^2)}dx = \frac{\pi}{4}\log \frac{\prod_{k=0}^{\infty}\left(1+\frac{1}{p^2(k+a)^2}\right)^{(-1)^k}}{\prod_{k=1}^{\infty}\left(1+\frac{1}{p^2(k-a)^2}\right)^{(-1)^k}}.$$

$$\int_0^1 \frac{\tan^{-1}(\sinh \pi x \csc a\pi)}{x\sqrt{1-x^2}}dx = \frac{\pi}{2}\log \frac{\prod_{k=0}^{\infty}\left(\frac{1}{k+a}+\sqrt{1+\frac{1}{(k+a)^2}}\right)^{(-1)^k}}{\prod_{k=1}^{\infty}\left(\frac{1}{k-a}+\sqrt{1+\frac{1}{(k-a)^2}}\right)^{(-1)^k}}.$$

4. Conclusion.

The integrals presented in this paper are evaluated through sums related to inverse tangent function [**1**] either by direct formulas or through infinite products. The solutions of these integrals are not found in the classical tables by Gradshteyn and Ryzhik [**2**]. The collection presented here will give a valuable addition to the existing list of integrals.

References

[1] B. C. Bernt, *Ramanujan notebooks, Part-I*, Springer-Verlag, New York, 1985.
[2] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products, 6 Ed*, Academic Press, New York, 2000.

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Department of Mathematics,
D.G. Vaishnav college,
Chennai-600106, Tamil Nadu, India.
E-mail address: ramjan_80@yahoo.com.