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Fractional integral operators and the multiindex Mittag-Leffler functions

S.D. Purohit [a], S.L. Kalla [b] and D.L. Suthar [c]

Abstract. The aim of this paper is to study some properties of multiindex Mittag-Leffler type function $E_{(1/\rho_j),(\mu_j)}(z)$ introduced by Kiryakova [V. Kiryakova, J. Comput. Appl. Math. 118 (2000), 241-259]. Here we establish certain theorems which provide the image of this function under the Saigo's fractional integral operators. The results derived are of general character and give rise to a number of known results in the theory of multiindex Mittag-Leffler functions.

1. Introduction

In 1903, Mittag-Leffler [10] defined a function, named after him, in terms of the power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)} \ \Re(\alpha) > 0. \tag{1.1}$$

A further, two-index generalization of this function was given by Wiman [14] as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)} \ \Re(\alpha) > 0, \Re(\beta) > 0. \tag{1.2}$$

Both are entire functions of order $\rho = 1/\alpha$ and type $\sigma = 1$. A detailed account of these functions is available in Erdélyi et al. [2] and Dzrbashjan [1].

Kiryakova [7] has introduced and studied a multiindex Mittag-Leffler function as an extension of the generalized Mittag-Leffler function considered by Dzrbashjan. The multiindex Mittag-Leffler function is defined in Kiryakova [7] by means of the power series:

$$E_{(1/\rho_j),(\mu_j)}(z) = \sum_{k=0}^{\infty} \varphi_k \ z^k = \sum_{k=0}^{\infty} \frac{z^k}{\prod_{j=1}^{m} \Gamma(\mu_j + \frac{k}{\rho_j})} \ , \tag{1.3}$$

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where $m \geqslant 1$ is an integer, $\Re(\rho_1), \cdots, \Re(\rho_m) > 0$ and $\mu_1, \cdots, \mu_m$ are arbitrary parameters.
Some important special cases of this function are enumerated below:

(i) If we set $m = 1$, $\rho_1 = \frac{1}{\alpha}$ and $\mu_1 = \beta$ then (1.3) yields the Mittag-Leffler function given by (1.2).

(ii) For $m = 2$, (1.3) reduces to the generalized Mittag-Leffler function considered by Dzrbashjan [1] denoted by $\Phi_{\rho_1,\rho_2}(z; \mu_1, \mu_2)$ in the following form

$$E_{(1/\rho_1,1/\rho_2),(\mu_1,\mu_2)}(z) = \Phi_{\rho_1,\rho_2}(z; \mu_1, \mu_2) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + \frac{k}{\rho_1})\Gamma(\mu_2 + \frac{k}{\rho_2})}\ , \tag{1.4}$$

and shown to be an entire function of order $\rho = \frac{\rho_1\rho_2}{(\rho_1+\rho_2)}$ and type $\sigma = (\rho_1/\rho)^{\rho/\rho_1}(\rho_2/\rho)^{\rho/\rho_2}$.

(iii) Again, for $m = 2$, $\rho_1 = \frac{1}{\beta}, \rho_2 = 1, \mu_1 = b$ and $\mu_2 = 1$, we have a special case of (1.3) in the form

$$E_{(\beta,1),(b,1)}(z) = \phi(\beta, b; z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b + \beta k)k!}\ , \tag{1.5}$$

with complex $z, b \in \mathbf{C}$ and $\beta \in \mathbf{R}$, known as the Wright function [15]. When $\beta = \delta$, $b = \nu + 1$ and z is replaced by $-z$, the function $\phi(\delta, \nu + 1; -z)$ is denoted by $J_\nu^\delta(z)$:

$$E_{(\delta,1),(\nu+1,1)}(-z) = J_\nu^\delta(z) = \phi(\delta, \nu + 1; -z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{\Gamma(\delta k + \nu + 1)k!}\ , \tag{1.6}$$

and such a function is known as the Bessel-Maitland function, or the Wright generalized Bessel function.

(iv) Also the function $E_{(1/\rho_j),(\mu_j)}(z)$ has the forms:

$$E_{(1/\rho_j),(\mu_j)}(z) = {}_1\Psi_m \left[\begin{matrix} (1,1) \\ (\mu_i, 1/\rho_j)_1^m \end{matrix} ; z \right] \tag{1.7}$$

$$= H_{1,m+1}^{1,1} \left[-z \middle| \begin{matrix} (0,1) \\ (0,1), (1-\mu_i, 1/\rho_j)_1^m \end{matrix} \right] \tag{1.8}$$

$$= \frac{1}{2\pi\omega} \int_L \frac{\Gamma(s)\Gamma(1-s)(-z)^s ds}{\prod_{j=1}^m \Gamma(\mu_j - s/\rho_j)}\ (z \neq 0), \tag{1.9}$$

where ${}_1\Psi_m(.)$ and $H_{1,m+1}^{1,1}(.)$, respectively represent the Wright generalized hypergeometric function and the Fox's H-function. In the Mellin-Barnes type contour integral representation (1.9), L is a suitable contour in $\mathbf{C}$ extending from $-\omega\infty$ to $\omega\infty$ in such a way that the poles $s = 0, -1, -2, \cdots$ of $\Gamma(s)$ lie to the left of L and the poles $s = 0, 1, 2, \cdots$ of $\Gamma(1-s)$ to the right of it.

Besides the Riemann-Liouville definition of fractional operators, several other modifications and generalizations have been studied. Hypergeometric integral operators have been defined and studied by Love [8], Saxena [12], Kalla and Saxena [3,4], Saigo [11], McBride [9] etc. A detailed information is given in the book of Kiryakova [5]. In this paper we consider hypergeometric fractional integral operators defined by Saigo [11].

For complex numbers α, β and η, we begin by considering the fractional integral operator $I_{0,z}^{\alpha,\beta,\eta}$ as: (Saigo [11])

$$I_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} \, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-t/z\right) f(t)dt \ (\Re(\alpha) > 0), \tag{1.10}$$

where the ${}_2F_1(.)$ function occurring in the right-hand side of (1.10) is the familiar Gaussian hypergeometric function defined by

$${}_2F_1\left(a, b; c; z\right) \equiv {}_2F_1\left[\begin{matrix} a, \ b & ; & \\ & & z \\ c & ; & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}. \tag{1.11}$$

The operator $I_{0,z}^{\alpha,\beta,\eta}$ contains both the Riemann-Liouville and the Erdélyi-Kober fractional integral operators, and we have the following relationships:

$$R_{0,z}^{\alpha} f(z) = I_{0,z}^{\alpha,-\alpha,\eta} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} f(t)dt, \tag{1.12}$$

and

$$E_{0,z}^{\alpha,\eta} f(z) = I_{0,z}^{\alpha,0,\eta} f(z) = \frac{z^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} \, t^{\eta} f(t)dt. \tag{1.13}$$

Another class of fractional integrals is defined, for complex numbers α, β and η, by (Saigo [11])

$$J_{z,\infty}^{\alpha,\beta,\eta} f(z) = \frac{1}{\Gamma(\alpha)} \int_z^{\infty} (t-z)^{\alpha-1} \, t^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-z/t\right) f(t)dt \ (\Re(\alpha) > 0), \tag{1.14}$$

which unifies the Weyl and the corresponding Erdélyi-Kober fractional integral operators. Indeed we have

$$W_{z,\infty}^{\alpha} f(z) = J_{z,\infty}^{\alpha,-\alpha,\eta} f(z) = \frac{1}{\Gamma(\alpha)} \int_z^{\infty} (t-z)^{\alpha-1} f(t)dt, \tag{1.15}$$

and

$$K_{z,\infty}^{\alpha,\eta} f(z) = J_{z,\infty}^{\alpha,0,\eta} f(z) = \frac{z^{\eta}}{\Gamma(\alpha)} \int_z^{\infty} (t-z)^{\alpha-1} \, t^{-\alpha-\eta} f(t)dt. \tag{1.16}$$

Since the Mittag-Leffler function provides solutions to certain problems formulated in terms of fractional order differential, integral and difference equations, it has recently become a subject of interest for many authors in the field of fractional calculus and its applications. Motivated by these avenues of applications, a number of workers have made use of the fractional calculus operators in the theory of Mittag-Leffler functions. The aim of the present paper is to obtain certain properties of multiindex Mittag-Leffler type function associated with Saigo's fractional calculus operators.

2. Multiindex Mittag-Leffler type function and hypergeometric operators

In this section, we obtain the image of the multiindex Mittag-Leffler type function under the hypergeometric fractional integral operators defined earlier by Saigo [11].

Theorem 2.1 *Let $m \geqslant 1$ is an integer, $\Re(\rho_i) > 0$, μ_i $(i = 1, \cdots, m)$ are arbitrary parameters and $I_{0,z}^{\alpha,\beta,\eta}(.)$ be the Saigo's left-sided fractional integral operator (1.10), then the following result holds:*

$$I_{0,z}^{\alpha,\beta,\eta}\left\{z^{\lambda-1}E_{(1/\rho_i),(\mu_i)}(az^\sigma)\right\}$$
$$= z^{\lambda-\beta-1}\sum_{k=0}^{\infty}\frac{(az^\sigma)^k}{\prod\limits_{j=1}^{m}\Gamma(\mu_j+k/\rho_j)}\,\frac{\Gamma(\lambda+\sigma k)\Gamma(\lambda+\eta-\beta+\sigma k)}{\Gamma(\lambda-\beta+\sigma k)\Gamma(\lambda+\alpha+\eta+\sigma k)}. \tag{2.1}$$

The conditions for validity of (2.1) are

(i) α, β, η $(\Re(\alpha) > 0)$ and a are any complex numbers

(ii) λ and σ are arbitrary such that $\Re(\lambda + \sigma k) > 0$ and $\Re(\lambda + \eta - \beta + \sigma k) > 0$.

Proof. Using the definition (1.10) in the left hand side of (2.1), writing the functions in the forms given by (1.3) and (1.11), interchanging the order of integration and summations and evaluating the integral as beta integral, we easily arrive at the result (2.1) under the valid conditions.

For $\lambda = \mu_1$, $\sigma = 1/\rho_1$, $\rho_2 = \rho_1$ and $\mu_2 = \mu_1 + \eta - \beta$ the Theorem 2.1, yields to the following result:

Corollary 2.1 *Let $m \geqslant 1$ is an integer, $\Re(\alpha) > 0$, $\Re(\rho_i) > 0$, μ_i $(i = 1, \cdots, m)$ are arbitrary parameters, then the following result holds:*

$$I_{0,z}^{\alpha,\beta,\eta}\left\{z^{\mu_1-1}E_{(1/\rho_1,1/\rho_1,1/\rho_3,\cdots,1/\rho_m),(\mu_1,\mu_1+\eta-\beta,\mu_3,\cdots,\mu_m)}(az^{1/\rho_1})\right\}$$
$$= z^{\mu_1-\beta-1}E_{(1/\rho_1,1/\rho_1,1/\rho_3,\cdots,1/\rho_m),(\mu_1-\beta,\mu_1+\alpha+\eta,\mu_3,\cdots,\mu_m)}(az^{1/\rho_1}), \tag{2.2}$$

where β, η and a are any complex numbers.

Using the known result due to Saxena, Kalla and Kiryakova [13, p. 369, eqn. (3.1)], namely

$$z^r E_{(1/\rho_i),(\mu_i+r/\rho_i)}(z) = E_{(1/\rho_i),(\mu_i)}(z) - \sum_{h=0}^{r-1}\frac{z^h}{\prod_{j=1}^{m}\Gamma(\mu_j+h/\rho_j)}, \tag{2.3}$$

the result (2.2) reduces to the following corollary:

Corollary 2.2 *Let $m \geqslant 1$ is an integer, $\Re(\alpha) > 0$, $\Re(\rho_i) > 0$, μ_i $(i = 1, \cdots, m)$ are arbitrary parameters, then the following result holds:*

$$a\ I_{0,z}^{\alpha,\beta,\eta}\left\{z^{\mu_1-1}E_{(1/\rho_1,\,1/\rho_1,\,1/\rho_3,\,\cdots,1/\rho_m),\,(\mu_1,\,\mu_1+\eta-\beta,\,\mu_3,\,\cdots,\,\mu_m)}(az^{1/\rho_1})\right\}$$

$$= z^{\mu_1-\beta-1-1/\rho_1}\left\{E_{(1/\rho_1,\,1/\rho_1,\,1/\rho_3,\,\cdots,1/\rho_m),\,(\mu_1-\beta-1/\rho_1,\,\mu_1+\alpha+\eta-1/\rho_1,\,\mu_3-1/\rho_3,\,\cdots,\,\mu_m-1/\rho_m)}(az^{1/\rho_1})\right.$$

$$\left.-\frac{1}{\Gamma(\mu_1-\beta-1/\rho_1)\Gamma(\mu_1+\alpha+\eta-1/\rho_1)\Gamma(\mu_3-1/\rho_3)\cdots\Gamma(\mu_m-1/\rho_m)}\right\}, \qquad (2.4)$$

where β, η and $a \neq 0$ are any complex numbers.

Theorem 2.2 *Let $m \geqslant 1$ is an integer, $\Re(\rho_i) > 0$, μ_i $(i = 1, \cdots, m)$ are arbitrary parameters and $J_{z,\infty}^{\alpha,\beta,\eta}(.)$ be the Saigo's right-sided fractional integral operator (1.14), then the following result holds:*

$$J_{z,\,\infty}^{\alpha,\beta,\eta}\left\{z^{\lambda}E_{(1/\rho_i),\,(\mu_i)}(az^{-\sigma})\right\}$$

$$= z^{\lambda-\beta}\sum_{k=0}^{\infty}\frac{(az^{-\sigma})^k}{\prod_{j=1}^{m}\Gamma(\mu_j+k/\rho_j)}\,\frac{\Gamma(\beta-\lambda+\sigma k)\Gamma(\eta-\lambda+\sigma k)}{\Gamma(-\lambda+\sigma k)\Gamma(\alpha+\beta+\eta-\lambda+\sigma k)}. \qquad (2.5)$$

The conditions for validity of (2.5) are

(i) α, β, η ($\Re(\alpha) > 0$) and a are any complex numbers

(ii) λ and σ are arbitrary such that $\Re(\beta - \lambda + \sigma k) > 0$ and $\Re(\eta - \lambda + \sigma k) > 0$.

Proof. Using the definition (1.14) in the left hand side of (2.5), writing the functions in the forms given by (1.3) and (1.11), interchanging the order of integration and summations and evaluating the integral as beta integral, we easily arrive at the result (2.5) under the valid condition.

For $\lambda = \beta - \mu_1$, $\sigma = 1/\rho_1$, $\rho_2 = \rho_1$ and $\mu_2 = \mu_1 + \eta - \beta$ the Theorem 2.2, also reduces to the following result:

Corollary 2.3 *If $m \geqslant 1$ is an integer, $\Re(\alpha) > 0$, $\Re(\rho_i) > 0$, μ_i $(i = 1, \cdots, m)$ are arbitrary parameters, then the following result holds:*

$$J_{z,\infty}^{\alpha,\beta,\eta}\left\{z^{\beta-\mu_1}E_{(1/\rho_1,\,1/\rho_1,\,1/\rho_3,\,\cdots,1/\rho_m),\,(\mu_1,\,\mu_1+\eta-\beta,\,\mu_3,\,\cdots,\,\mu_m)}(az^{-1/\rho_1})\right\}$$

$$= z^{-\mu_1}E_{(1/\rho_1,\,1/\rho_1,\,1/\rho_3,\,\cdots,1/\rho_m),\,(\mu_1-\beta,\,\mu_1+\alpha+\eta,\,\mu_3,\,\cdots,\,\mu_m)}(az^{-1/\rho_1}), \qquad (2.6)$$

where β, η and a are complex numbers.

On making use of the identity (2.3), the result (2.6) reduces to the following corollary:

Corollary 2.4 *Let $m \geqslant 1$ is an integer, $\Re(\alpha) > 0$, $\Re(\rho_i) > 0$, μ_i $(i = 1, \cdots, m)$ are arbitrary parameters, then there holds the formula:*

$$a\ J_{z,\infty}^{\alpha,\beta,\eta}\left\{z^{\beta-\mu_1}E_{(1/\rho_1,\,1/\rho_1,\,1/\rho_3,\,\cdots,1/\rho_m),\,(\mu_1,\,\mu_1+\eta-\beta,\,\mu_3,\,\cdots,\,\mu_m)}(az^{1/\rho_1})\right\}$$

$$= z^{-\mu_1+1/\rho_1}\left\{E_{(1/\rho_1,\,1/\rho_1,\,1/\rho_3,\,\cdots,1/\rho_m),\,(\mu_1-\beta-1/\rho_1,\,\mu_1+\alpha+\eta-1/\rho_1,\,\mu_3-1/\rho_3,\,\cdots,\,\mu_m-1/\rho_m)}(az^{1/\rho_1})\right.$$

$$\left.-\frac{1}{\Gamma(\mu_1-\beta-1/\rho_1)\Gamma(\mu_1+\alpha+\eta-1/\rho_1)\Gamma(\mu_3-1/\rho_3)\cdots\Gamma(\mu_m-1/\rho_m)}\right\}, \quad (2.7)$$

where β, η and $a \neq 0$ are any complex quantities.

3. Erdélyi-Kober operators with Mittag-Leffler type function

We now give images of the multiindex Mittag-Leffler type function under the fractional integral operators of Erdélyi-Kober type. By taking $\beta = 0$ and making use of the relations (1.13) and (1.16), we derive the following corollaries (3.1) to (3.6), of the Theorems 2.1 and 2.2, and Corollaries 2.1 to 2.4 in terms of Erdélyi-Kober type fractional calculus results involving the multiindex Mittag-Leffler function.

Corollary 3.1 *Let $m \geqslant 1$ is an integer, $\Re(\alpha) > 0$, $\Re(\rho_i) > 0$, μ_i $(i = 1, \cdots, m)$ are arbitrary parameters and $E_{0,z}^{\alpha,\eta}(.)$ be the Erdélyi-Kober fractional integral operator (1.13), then the following result holds:*

$$E_{0,z}^{\alpha,\eta}\left\{z^{\lambda-1}E_{(1/\rho_i),\,(\mu_i)}(az^{\sigma})\right\} = z^{\lambda-1}\sum_{k=0}^{\infty}\frac{(az^{\sigma})^k}{\prod_{j=1}^{m}\Gamma(\mu_j+k/\rho_j)}\frac{\Gamma(\lambda+\eta+\sigma k)}{\Gamma(\lambda+\alpha+\eta+\sigma k)}, \quad (3.1)$$

where η and a are any complex numbers and $\Re(\lambda+\eta+\sigma k) > 0$.

Corollary 3.2 *Let $m \geqslant 1$ is an integer, $\Re(\alpha) > 0$, $\Re(\rho_i) > 0$, μ_i $(i = 1, \cdots, m)$ are arbitrary parameters, then the following result holds:*

$$E_{0,z}^{\alpha,\eta}\left\{z^{\mu_1-1}E_{(1/\rho_i),\,(\mu_1+\eta,\,\mu_2,\,\cdots,\,\mu_m)}(az^{1/\rho_1})\right\} = z^{\mu_1-1}E_{(1/\rho_i),\,(\mu_1+\alpha+\eta,\,\mu_2,\,\cdots,\,\mu_m)}(az^{1/\rho_1}), \quad (3.2)$$

where η and a are complex numbers.

Corollary 3.3 *If $m \geqslant 1$ is an integer, $\Re(\alpha) > 0$, $\Re(\rho_i) > 0$, $\mu_i 0$ $(i = 1, \cdots, m)$ are arbitrary parameters, then the following result holds:*

$$a\ E_{0,z}^{\alpha,\eta}\left\{z^{\mu_1-1}E_{(1/\rho_i),\,(\mu_1+\eta,\,\mu_2,\,\cdots,\,\mu_m)}(az^{1/\rho_1})\right\}$$

$$= z^{\mu_1-1-1/\rho_1}\left\{E_{(1/\rho_i),\,(\mu_1+\alpha+\eta-1/\rho_1,\,\mu_2-1/\rho_2,\,\cdots,\,\mu_m-1/\rho_m)}(az^{1/\rho_1})\right.$$

$$\left.-\frac{1}{\Gamma(\mu_1+\alpha+\eta-1/\rho_1)\Gamma(\mu_2-1/\rho_2)\cdots\Gamma(\mu_m-1/\rho_m)}\right\}, \quad (3.3)$$

where η and $a \neq 0$ are arbitrary complex quantities.

Corollary 3.4 *Let $m \geqslant 1$ is an integer, $\Re(\rho_i) > 0$, μ_i $(i = 1, \cdots, m)$ are arbitrary parameters and $K_{z,\infty}^{\alpha,\eta}(.)$ be the Erdélyi-Kober fractional integral operator (1.16), then the following result holds:*

$$K_{z,\infty}^{\alpha,\eta}\left\{z^{\lambda} E_{(1/\rho_i),(\mu_i)}(az^{-\sigma})\right\} = z^{\lambda-\beta} \sum_{k=0}^{\infty} \frac{(az^{-\sigma})^k}{\prod_{j=1}^{m} \Gamma(\mu_j + k/\rho_j)} \frac{\Gamma(\eta - \lambda + \sigma k)}{\Gamma(\alpha + \eta - \lambda + \sigma k)}, \quad (3.4)$$

where α, η and a $(\Re(\alpha) > 0)$ are complex numbers and $\Re(\eta - \lambda + \sigma k) > 0$.

Corollary 3.5 *Let $m \geqslant 1$ is an integer, $\Re(\alpha) > 0$, $\Re(\rho_i) > 0$, μ_i $(i = 1, \cdots, m)$ are arbitrary parameters, then*

$$K_{z,\infty}^{\alpha,\eta}\left\{z^{-\mu_1} E_{(1/\rho_i),(\mu_1+\eta,\mu_2,\cdots,\mu_m)}(az^{-1/\rho_1})\right\} = z^{-\mu_1} E_{(1/\rho_i),(\mu_1+\alpha+\eta,\mu_2,\cdots,\mu_m)}(az^{-1/\rho_1}), \quad (3.5)$$

where η and a are complex numbers.

Corollary 3.6 *If $m \geqslant 1$ is an integer, $\Re(\alpha) > 0$, $\Re(\rho_i) > 0$, μ_i $(i = 1, \cdots, m)$ are arbitrary parameters, then the following result holds:*

$$a\ K_{z,\infty}^{\alpha,\eta}\left\{z^{-\mu_1} E_{(1/\rho_i),(\mu_1+\eta,\mu_2,\cdots,\mu_m)}(az^{-1/\rho_1})\right\}$$

$$= z^{-\mu_1+1/\rho_1}\left\{E_{(1/\rho_i),(\mu_1+\alpha+\eta-1/\rho_1,\mu_2-1/\rho_2,\cdots,\mu_m-1/\rho_m)}(az^{-1/\rho_1}) - \frac{1}{\Gamma(\mu_1+\alpha+\eta-1/\rho_1)\Gamma(\mu_2-1/\rho_2)\cdots\Gamma(\mu_m-1/\rho_m)}\right\}, \quad (3.6)$$

where η and $a \neq 0$ are any complex numbers.

4. Special cases

Special values of the parameters reduces the Saigo's operators to some well known classical operators. For example if we set $\alpha + \beta = 0$, in Theorems 2.1 and 2.2, we get the following results (4.1) to (4.6) involving Riemann-Liouville operators of fractional calculus.

Corollary 4.1 *Let $m \geqslant 1$ is an integer, $\Re(\rho_i) > 0$, μ_i $(i = 1, \cdots, m)$ are arbitrary parameters and $R_{0,z}^{\alpha}(.)$ be the Riemann-Liouville fractional integral operator (1.12), then the following result holds:*

$$R_{0,z}^{\alpha}\left\{z^{\lambda-1} E_{(1/\rho_i),(\mu_i)}(az^{\sigma})\right\} = z^{\lambda+\alpha-1} \sum_{k=0}^{\infty} \frac{(az^{\sigma})^k}{\prod_{j=1}^{m} \Gamma(\mu_j + k/\rho_j)} \frac{\Gamma(\lambda + \sigma k)}{\Gamma(\lambda + \alpha + \sigma k)}, \quad (4.1)$$

where α *and* a $(\Re(\alpha) > 0)$ *are complex numbers and* $\Re(\lambda + \sigma k) > 0$.

Corollary 4.2 *Let* $m \geqslant 1$ *is an integer,* $\Re(\alpha) > 0$, $\Re(\rho_i) > 0$, μ_i $(i = 1, \cdots, m)$ *are arbitrary parameters, then we obtain the following result due to Saxena, Kalla and Kiryakova [13, p. 372, eqn. (4.1)]:*

$$R^{\alpha}_{0,z}\left\{z^{\mu_1-1}E_{(1/\rho_i),(\mu_i)}(az^{1/\rho_1})\right\} = z^{\alpha+\mu_1-1}E_{(1/\rho_i),(\mu_1+\alpha,\mu_2,\cdots,\mu_m)}(az^{1/\rho_1}), \quad (4.2)$$

where a *is any complex number.*

Corollary 4.3 *For* $\Re(\alpha) > 0$, $\Re(\rho_i) > 0$, μ_i $(i = 1, \cdots, m)$ *are arbitrary parameters, we obtain the following result given by Saxena, Kalla and Kiryakova [13, p. 373, eqn. (4.6)]:*

$$a\ R^{\alpha}_{0,z}\left\{z^{\mu_1-1}E_{(1/\rho_i),(\mu_1+\eta,\mu_2,\cdots,\mu_m)}(az^{1/\rho_1})\right\}$$

$$= z^{\mu_1+\alpha-1-1/\rho_1}\left\{E_{(1/\rho_i),(\mu_1+\alpha-1/\rho_1,\mu_2-1/\rho_2,\ ,\cdots,\mu_m-1/\rho_m)}(az^{1/\rho_1}) - \frac{1}{\Gamma(\mu_1+\alpha-1/\rho_1)\Gamma(\mu_2-1/\rho_2)\cdots\Gamma(\mu_m-1/\rho_m)}\right\}, \quad (4.3)$$

where $m \geqslant 1$ *is an integer and* $a \neq 0$ *is any complex number.*

Corollary 4.4 *Let* $m \geqslant 1$ *is an integer,* $\Re(\rho_i) > 0$, μ_i $(i = 1, \cdots, m)$ *are arbitrary parameters and* $W^{\alpha}_{z,\infty}(.)$ *be the Weyl fractional integral operator (1.15), then the following result holds:*

$$W^{\alpha}_{z,\infty}\left\{z^{\lambda}E_{(1/\rho_i),(\mu_i)}(az^{-\sigma})\right\} = z^{\lambda-\beta}\sum_{k=0}^{\infty}\frac{(az^{-\sigma})^k}{\prod_{j=1}^{m}\Gamma(\mu_j+k/\rho_j)}\frac{\Gamma(\sigma k-\alpha-\lambda)}{\Gamma(\sigma k-\lambda)}, \quad (4.4)$$

where α $(\Re(\alpha) > 0)$ *and* a *are complex numbers and* $\Re(\sigma k - \alpha - \lambda) > 0$.

Corollary 4.5 *Let* $m \geqslant 1$ *is an integer,* $\Re(\alpha) > 0$, $\Re(\rho_i) > 0$, μ_i $(i = 1, \cdots, m)$ *are arbitrary parameters, then there holds the formula:*

$$W^{\alpha}_{z,\infty}\left\{z^{-\alpha-\mu_1}E_{(1/\rho_i),(\mu_i)}(az^{-1/\rho_1})\right\} = z^{-\mu_1}E_{(1/\rho_i),(\mu_1+\alpha,\mu_2,\cdots,\mu_m)}(az^{-1/\rho_1}), \quad (4.5)$$

where a *is any complex number.*

We may mention that (4.5) is a known result due to Saxena, Kalla and Kiryakova [13, p. 374, eqn. (4.9)].

Corollary 4.6 *If* $m \geqslant 1$ *is an integer,* $\Re(\alpha) > 0$, $\Re(\rho_i) > 0$, μ_i $(i = 1, \cdots, m)$ *are arbitrary parameters, then we get the following result given by Saxena, Kalla and Kiryakova [13, pp. 375-376, eqn. (4.10)]:*

$$a\ W^{\alpha}_{z,\infty}\left\{z^{-\alpha-\mu_1}E_{(1/\rho_i),(\mu_i)}(az^{-1/\rho_1})\right\}$$

$$= z^{-\mu_1+1/\rho_1}\left\{E_{(1/\rho_i),(\mu_1+\alpha-1/\rho_1,\mu_2-1/\rho_2,\cdots,\mu_m-1/\rho_m)}(az^{-1/\rho_1})\right.$$

$$-\frac{1}{\Gamma(\mu_1+\alpha+\eta-1/\rho_1)\Gamma(\mu_2-1/\rho_2)\cdots\Gamma(\mu_m-1/\rho_m)}\Bigg\}, \tag{4.6}$$

where η and $a \neq 0$ are any complex quantities.

5. Concluding Remark

The Multiindex Mittag-Leffler functions due to Kiryakova [7] is an elegant unification of the various special functions viz. the Mittag-Leffler function (1.2), the generalized Mittag-Leffler function (1.4), the Wright function (1.5) and the Bessel-Maitland function (1.6). By making suitable specialization of the parameters $\rho_i,\ \mu_i\ (i = 1, 2, \cdots, m)$ and m, one can deduce numerous fractional calculus results involving the various types of Mittag-Leffler functions as an application of the theorems and corollaries of the preceding sections. Hence, the results obtained in this paper are useful in preparing some tables of Riemann-Liouville operator, Weyl operator, Erdélyi-Kober operators and Saigo's operators of fractional integration involving various types of Mittag-Leffler functions. Results obtained here may be useful in solution of certain fractional integro-differential equations.

References

[1] M.M. Dzrbashajan, *Integral Transforms and Representations of Functions in the Complex Domain.* (In Russian), Nauka, Moscow, 1966.

[2] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions.* Vol. 3, McGraw-Hill, New York, 1955.

[3] S.L. Kalla and R. K. Saxena, Integral operators involving hypergeometric functions. *Math. Zeitschr.* **108** (1969), 231-234.

[4] S.L. Kalla and R. K. Saxena, Integral operators involving hypergeometric functions-II. *Univ. Nac. Tucuman, Rev. Ser.*, **A24** (1974), 31-36.

[5] V. Kiryakova, *Generalized Fractional Calculus and Applications.* Longman Scientific & Tech., Essex, 1994.

[6] V. Kiryakova, Multiindex Mittag-Leffler functions, related Gelfond-Leontiev operators and Laplace type integral transforms. *Fract. Calc. Appl. Anal.*, **2(4)** (1999), 445-462.

[7] V. Kiryakova, Multiple (multiindex) Mittag-Leffler functions and relations to generalized fractional calculus. *J. Comput. Appl. Math.*, **118** (2000), 241-259.

[8] E.R. Love, Some integral equations involving hypergeometric functions. *Proc. Edin. Math. Soc.*, **15(3)** (1997), 169-198.

[9] A.C. McBride, Fractional power of a class of ordinary differential operators. *Proc. London Math. Soc.(III)*, **45**(1982), 519-546.

[10] G.M. Mittag-Leffler, Sur la nouvelle function $E_\alpha(x)$. *C.R. Acad. Sci. Paris*, **137** (1903), 554-558.

[11] M. Saigo, A remark on integral operators involving the Guass hypergeometric function. *Rep. College General Ed., Kyushu Univ.*, **11** (1978), 135-143.

[12] R.K. Saxena, On fractional integral operators. *Math Zeitschr.* **96**(1967), 288-291.

[13] R.K. Saxena, S.L. Kalla and V. Kiryakova, Relations connecting multiindex Mittag-Leffler functions and Riemann-Liouville fractional calculus. *Algebras Groups Geom.*, **20** (2003), 363-386.
[14] A. Wiman, *Ü*ber den Fundamental satz in der Theorie der Functionen $E_\alpha(x)$. *Acta Math.*, **29** (1905), 191-201.
[15] E.M. Wright, The asymptotic expansion of the generalized hypergeometric functions. *J. London Math. Soc.*, **10** (1935), 286-293.

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[a]Department of Basic-Sciences (Mathematics),
College of Technology and Engineering,
M.P. University of Agriculture and Technology,
Udaipur-313001,
INDIA.
E-mail address: sunil_a_purohit@yahoo.com

[b] Institute of Mathematics,
Vyas Institute of Higher Education,
Jodhpur-342008,
INDIA.
E-mail address: shyamkalla@gmail.com

[c] Alwar Institute of Engineering & Technology,
North Ext., Matsya Industrial Area,
Alwar-301030,
INDIA.
E-mail address: dd_suthar@gmail.com