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$Y\overline{X}$ domination in bipartite graphs

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Abstract. A subset S of X is called a $Y\overline{X}$ dominating set if S is a $Y-$dominating set and $X-S$ is not a $X-$dominating set. A subset S of X is called a minimal $Y\overline{X}$ dominating set if any proper subset of S is not a $Y\overline{X}$ dominating set. The minimum cardinality of a minimal $Y\overline{X}$ dominating set is called the $Y\overline{X}$ domination number of G and is denoted by $\gamma_{Y\overline{X}}(G)$. In this paper some results on $Y\overline{X}$ domination number are obtained.

1. Introduction

Let $G=(X,Y,E)$ be a bipartite graph. The bipartite theory of graphs were introduced in [**1, 2**] and the parameters called $X-$domination number and $Y-$domination number were introduced. Two vertices u,v in X are X-adjacent if they are adjacent to a common vertex in Y. The $X-$neighborhood set of u denoted by $N_Y(u)$ is defined as $N_Y(u)=\{v : u \quad and \quad are \quad X-adjacent\}$. The $X-$degree denoted by $d_Y(u)=|N_Y(u)|$. The minimum and maximum $X-$degree of a graph G denoted by $\delta_Y(G)$ and $\Delta_Y(G)$ is defined as $\delta_Y(G)=min\{d_Y(u) : u\in X\}$ and $\Delta_Y(G)=max\{d_Y(u) : u\in X\}$. A subset D of X is an X-dominating set if every vertex in $X-D$ is X-adjacent to at least one vertex in D. A X-dominating set [**1**] S is a minimal X-dominating set if no proper subset of S is X-dominating set. The minimum cardinality of a minimal X-dominating set is called the X-domination number of G and is denoted by $\gamma_X(G)$. A subset $S\subseteq X$ which dominates all vertices in Y is called a Y-dominating set [**1**] of G. The Y-domination number denoted by $\gamma_Y(G)$ is the minimum cardinality of a Y-dominating set of G. A subset S of X is hyper independent [**1**] if there does not exist a vertex $y\in Y$ such that $N(y)\subseteq S$. The maximum cardinality of a hyper independent set of G is denoted by $\beta_h(G)$. A subset $S\subseteq X$ is hyper $X-$independent if $N_Y(x)\nsubseteq S$, for every $x\in S$. The maximum cardinality of a hyper $X-$independent set of G is called hyper $X-$independence number and is denoted by $\beta_{hX}(G)$.

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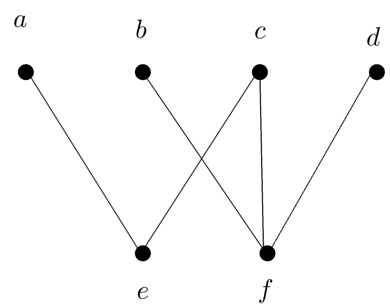
2. $Y\overline{X}$ dominating set

DEFINITION 1. A subset S of X is called a $Y\overline{X}$ dominating set if S is a $Y-$dominating set and $X-S$ is not a $X-$dominating set.

A subset S of X is called a minimal $Y\overline{X}$ dominating set if any proper subset of S is not a $Y\overline{X}$ dominating set. The minimum cardinality of a minimal $Y\overline{X}$ dominating set is called the $Y\overline{X}$domination number of G and is denoted by $\gamma_{Y\overline{X}}(G)$.

EXAMPLE 1.

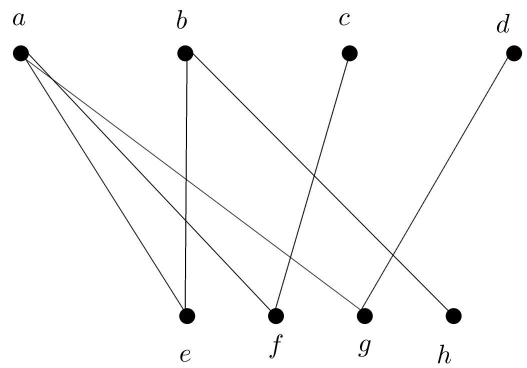


$S=\{c\}$ is a $Y-$dominating set but not $Y\overline{X}-$dominating set. $D=\{a,b,c\}$ is a $Y\overline{X}-$dominating set.

REMARK 1. If X contains an isolated vertex, then any $Y-$dominating set will be a $Y\overline{X}-$dominating set. Therefore, hereafter, by a graph G we mean a bipartite graph $G=(X,Y,E); |X|=p$, without loops, multiple edges and with no isolated vertex in X and Y.

REMARK 2. $\gamma_Y(G)\leqslant\gamma_{Y\overline{X}}(G)$.

OBSERVATION 1. The complement of a minimal $Y\overline{X}-$dominating set need not be a $Y\overline{X}-$dominating set. Consider the graph



$S=\{a,b\}$ is a minimal $Y\overline{X}-$dominating set but $X-S=\{c,d\}$ is not a $Y\overline{X}-$dominating set.

THEOREM 1. Let G be a graph. A $Y-$dominating set S is a $Y\overline{X}-$dominating set if and only if S is not a hyper $X-$independent set.

Proof. Let S be a $Y\overline{X}-$dominating set of G. Then, $X-S$ is not a $X-$dominating set. Therefore, there exists $x \in S$ such that x is not $X-$adjacent to any vertex of $X-S$. Equivalently, $x \in S$ such that $N_Y(x) \subseteq S$. Therefore, S is not hyper $X-$independent set.

Conversely, let S be a $Y-$dominating set which is not a hyper $X-$independent set. That is, there exists $x \in S$ such that $N_Y(x) \subseteq S$. Equivalently, x is not $X-$adjacent to any vertex of $X-S$. Therefore, $X-S$ is not a $X-$dominating set. Hence, S is a $Y\overline{X}$-dominating set. □

THEOREM 2. A subset S of X is a $Y\overline{X}-$ dominating set if and only if
(i) $X-S$ is a hyper independent set.
(ii) S is not a hyper $X-$independent set.

Proof. A subset S of X is a $Y\overline{X}-$dominating set of G. Then, S is a $Y-$dominating set and $X-S$ is not a $X-$dominating set. By Theorem 1, S is not a hyper $X-$independent set. Since S is a $Y-$dominating set, every $y \in Y$ is adjacent to a vertex of S. Therefore, $N(y) \nsubseteq (X-S)$. Hence, $X-S$ is a hyper independent set.

Conversely, conditions (i) and (ii) hold. By the condition (i), $X-S$ is a hyper independent set. Therefore, $N(y) \nsubseteq (X-S)$ for all $y \in Y$. Hence, every $y \in Y$ is adjacent to a vertex of S. Therefore, S is a $Y-$dominating set. By condition (ii) S is not a hyper $X-$independent set. Therefore, by Theorem 1, S is a $Y\overline{X}-$dominating set. □

THEOREM 3. For any graph G, every $\gamma_{Y\overline{X}}-$set intersects with every γ_X-set of G.

Proof. Let S be a γ_X-set of G and let D be a $\gamma_{Y\overline{X}}-$set of G. Suppose $S \cap D = \phi$, then $S \subseteq X-D$, then $X-D$ contains a $X-$dominating set S. Therefore, $X-D$ itself a $X-$dominating set, which is a contradiction. □

THEOREM 4. Let S be a $Y\overline{X}-$dominating set of a graph G. Then S is minimal $Y\overline{X}-$ dominating set if and only if for each vertex $u \in S$ one of the following conditions are satisfied:
(i) u has a private neighborhood.
(ii) $X-(S-\{u\})$ is a X-dominating set of G.

Proof. Let S be a minimal $Y\overline{X}-$dominating set. On the contrary if there exists a vertex $u \in S$ such that u does not satisfy any of the given conditions (i) and (ii), then $S_1 = S-\{u\}$ is a $Y-$dominating set and $X-(S-\{u\})$ is not a $X-$dominating set, a contradiction, to S is a minimal $Y\overline{X}-$dominating set.

Conversely, suppose that S is a $Y\overline{X}-$dominating set and for each vertex $u \in S$, one of the two conditions holds. Suppose S is not a minimal $Y\overline{X}-$dominating set. That is there exists $u \in S$ such that $S-\{u\}$ is a $Y\overline{X}-$ dominating set of G. Therefore, $S-\{u\}$ is a $Y-$dominating set of G. Every $y \in Y$ is adjacent to a vertex $u_1 \in S-\{u\}$, that is condition (i) does not hold. Since $S-\{u\}$ is a $Y\overline{X}-$dominating set of G,thus $X-(S-\{u\})$ is not a $X-$dominating set, a contradiction to (ii). □

3. Bounds for $Y\overline{X}-$domination number

THEOREM 5. For any graph G, $\delta_Y(G)+1 \leqslant \gamma_{Y\overline{X}}(G) \leqslant \gamma_Y(G)+\delta_Y(G)$.

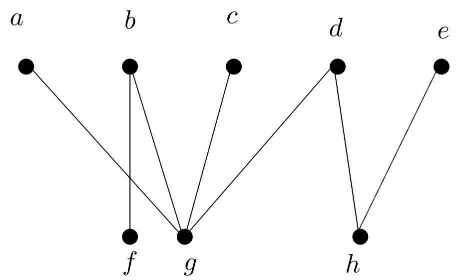
Proof. Let $v \in X$ be a vertex with $d_Y(v) = \delta_Y(G)$. Then every $Y\overline{X}-$dominating set must contain v and $X-$neighborhood of v. Therefore, $\delta_Y(G)+1 \leqslant \gamma_{Y\overline{X}}(G)$.

Let S be a γ_Y-set of G. Let $u \in X$ be such that $d_Y(u) = \delta_Y(G)$. Then at least one vertex $u_1 \in N_Y[u]$ belongs to S. Therefore, $S \cup (N_Y[u] - \{u_1\})$ is a $Y\overline{X}-$dominating set of G. Therefore, $\gamma_{Y\overline{X}}(G) \leqslant |S \cup (N_Y[u] - \{u_1\})| = \gamma_Y(G) + \delta_Y(G)$. □

THEOREM 6. For any graph G, if $\gamma_{Y\overline{X}}(G) = 2$, then $\Delta_Y(G) = p-1$ and $\delta_Y(G) = 1$.

Proof. Let us assume $\gamma_{Y\overline{X}}(G) = 2$. By Theorem 5, $\delta_Y(G) \leqslant \gamma_{Y\overline{X}}(G) - 1 = 2 - 1 = 1$. But $\delta_Y(G) \geqslant 1$. Hence, $\delta_Y(G) = 1$. Since, $\gamma_Y(G) \leqslant \gamma_{Y\overline{X}}(G) = 2$. Therefore, $\gamma_Y(G) \leqslant 2$. If $\gamma_Y(G) = 1$ then $\Delta_Y(G) = p-1$. Let $\gamma_Y(G) = 2$. Let x_1, x_2 be the two vertices which dominate Y. Let us assume $\Delta_Y(G) \neq p-1$. Any vertex in X cannot be a $X-$dominating set. Every $Y-$dominating set is a $X-$dominating set. Therefore, $\{x_1, x_2\}$ is a minimal $X-$dominating set. Complement of a minimal $X-$dominating set is a $X-$dominating set. Therefore, $\gamma_{Y\overline{X}} \geqslant 3$, a contradiction. Hence, $\Delta_Y(G) = p-1$. □

OBSERVATION 2. Converse of the above need not be true. Consider the graph



$\Delta_Y(G) = 4 = p-1$, $\delta_Y(G) = 1$ and $\gamma_{Y\overline{X}}(G) = 3 \neq 2$.

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