SCIENTIA
Series A: *Mathematical Sciences*, Vol. 21 (2011), 43–54
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

The integrals in Gradshteyn and Ryzhik. Part 20: Hypergeometric functions

Karen T. Kohl and Victor H. Moll

Abstract. The table of Gradshteyn and Ryzhik contains many integrals that involve the hypergeometric function ${}_pF_q$. Some examples are discussed.

1. Introduction

The hypergeometric function defined by

$$ {}_pF_q\left(a_1,\, a_2, \cdots, a_p; b_1,\, b_2, \cdots, b_q; x\right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!} \tag{1.1} $$

includes, as special cases, many of the elementary special functions. For example,

$$ \begin{aligned} \log(1+x) &= x\ {}_2F_1\left(1,\, 1; 2; -x\right) \\ \sin x &= x\ {}_0F_1\left(-; \tfrac{3}{2}; -x^2/4\right) \\ \cosh x &= \lim_{a,\, b \to \infty} {}_2F_1\left(a, b; \tfrac{1}{2}; x^2/4ab\right). \end{aligned} \tag{1.2} $$

The binomial theorem, for real exponent, can also be expressed in hypergeometric form as

$$ (1-x)^{-a} = {}_1F_0\left(a; -; x\right). \tag{1.3} $$

The goal of this paper is to verify the integrals in [**3**] that involve this function. Due to the large number of entries in [**3**] that can be related to hypergeometric functions, the list presented here represents the first part of these. More entries will appear in a future publication.

The hypergeometric function satisfies a large number of identities. The reader will find in [**1**] the best introduction to the subject. Some elementary identities are

---

2000 *Mathematics Subject Classification.* Primary 33.
Key words and phrases. Integrals, hypergeometric functions.
The second author wishes to acknowledge the partial support of NSF-DMS 0713836. The first author was partially supported, as a graduate student, by the same grant.

described here in detail. For example, if one of the top parameters (the a_i) agrees with a bottom one (the b_i), the function reduces to one with lower indices. The identity

$$ {}_2F_1(a,b;a;x) = {}_1F_0(a;-;x). \tag{1.4}$$

illustrates this point. The binomial theorem identifies the latter as $(1-x)^{-a}$.

2. Integrals over [0, 1]

The first result is a representation of ${}_2F_1$ in terms of the *beta integral*

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt. \tag{2.1}$$

Proposition 2.1. The hypergeometric function ${}_2F_1$ is given by

$$ {}_2F_1\left(a,\, b;\, c;\, x\right) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\,dt. \tag{2.2}$$

PROOF. Expand the term $(1-tx)^{-a}$ by the binomial theorem and integrate term by term. □

This representation appears as **3.197.3** in [**3**]. In order to simplify the replacing of parameters, this entry is also written as

$$\int_0^1 t^b(1-t)^c(1-tx)^a\,dt = B(b+1,c+1)\,{}_2F_1\left(-a,\, b+1;\, b+c+2;\, x\right). \tag{2.3}$$

This is one of the forms in which it will be used here: the integral being the object of primary interest.

Example 2.2. The special case $a=c=1$ in (2.2) appears as **3.197.10** in [**3**]:

$$\int_0^1 \frac{t^{b-1}\,dt}{(1-t)^b\,(1+tx)} = \frac{\pi}{\sin\pi b}(1+x)^{-b}. \tag{2.4}$$

The evaluation is direct. The identity (1.4) gives

$$ {}_2F_1(1,\, b;\, 1;\, -x) = (1+x)^{-b} \tag{2.5}$$

and then use $B(b,1-b) = \Gamma(b)\Gamma(1-b) = \pi/\sin\pi b$ to complete the evaluation.

Example 2.3. Introduce the index r by $r = a-b$ and take $c = b+r$ in (2.2). Then we have

$$\int_0^1 t^{b-1}(1-t)^{r-1}(1-tx)^{-b-r}\,dt = B(b,r)\,{}_2F_1\left(b+r,\, b;\, b+r;\, x\right) \tag{2.6}$$

The identity (1.4) reduces the previous evaluation to

$$\int_0^1 t^{b-1}(1-t)^{r-1}(1-tx)^{-b-r}\,dt = B(b,r)\,(1-x)^{-b}. \tag{2.7}$$

This appears as **3.197.4** in [**3**].

3. A linear scaling

In this section integrals obtained from the basic representation (2.3) by the change of variables $y = tp$. This produces

$$\int_0^p y^{b-1}(p-y)^{c-b-1}(p-xy)^{-a}\,dy = p^{c-a-1}B(b,c-b)\,{}_2F_1\left(a,\,b;\,c;x\right). \tag{3.1}$$

Example 3.1. The special case $c = b+1$ produces

$$\int_0^p y^{b-1}(p-xy)^{-a}\,dy = \frac{1}{b}p^{b-a}\,{}_2F_1\left(a,\,b;\,b+1;x\right), \tag{3.2}$$

where we have used $B(b,1) = 1/b$. In order to eliminate the factor p^{-a}, we choose $x = -pr$ to obtain

$$\int_0^p y^{b-1}(1+ry)^{-a}\,dy = \frac{1}{p}u^{p}\,{}_2F_1\left(a,\,b;\,b+1;-rp\right), \tag{3.3}$$

This appears as **3.194.1** in [**3**]. The special case $a = 1$, stating that

$$\int_0^p \frac{y^{b-1}\,dy}{1+ry} = \frac{1}{b}p^{b}\,{}_2F_1\left(1,\,b;\,b+1;-rp\right), \tag{3.4}$$

appears as **3.194.5** in [**3**].

Example 3.2. The table [**3**] contains the formula **3.196.1**:

$$\int_0^u (x+b)^{\nu}(u-x)^{\mu-1}\,dx = \frac{b^{\nu}\,u^{\mu}}{\mu}\,{}_2F_1\left[1,-\nu,1+\mu,-\frac{u}{b}\right]. \tag{3.5}$$

We believe that it is a bad idea to have u and μ in the same formula, so we write this as

$$\int_0^a (x+b)^{\nu}(a-x)^{\mu-1}\,dx = \frac{b^{\nu}\,a^{\mu}}{\mu}\,{}_2F_1\left[1,-\nu,1+\mu,-\frac{a}{b}\right]. \tag{3.6}$$

To prove this, we let $x = at$ to get

$$\int_0^a (x+b)^{\nu}(a-x)^{\mu-1}\,dx = b^{\nu}a^{\mu}\int_0^1 (1+at/b)^{\nu}(1-t)^{\mu-1}\,dt. \tag{3.7}$$

The integral representation (2.3) now gives the result.

4. Powers of linear factors

The hypergeometric function appears in the evaluation of integrals of the form

$$I = \int_a^b L_1(x)^{\mu-1}L_2(x)^{\nu-1}L_3(x)^{\lambda-1}\,dx \tag{4.1}$$

where L_j are linear functions and $L_1(a) = L_2(b) = 0$. For example, **3.198**:

$$\int_0^1 x^{\mu-1}(1-x)^{\nu-1}\left[ax+b(1-x)+c\right]^{-(\mu+\nu)}\,dx = (a+c)^{-\mu}(b+c)^{-\nu}B(\mu,\nu) \tag{4.2}$$

is reduced to the normal form (2.3) by writing

$$I = (b+c)^{-\mu-\nu} \int_0^1 x^{\mu-1}(1-x)^{\nu-1}(1-rx)^{-(\mu+\nu)}\, dx \tag{4.3}$$

with $r = (b-a)/(b+c)$. Then (2.3) gives

$$I = (b+c)^{-\mu-\nu} B(\mu,\nu)\, {}_2F_1\left(\mu+\nu,\, \mu;\, \mu+\nu;\, \frac{b-a}{b+c}\right). \tag{4.4}$$

To produce the stated answer, simply observe the special value of the hypergeometric function

$${}_2F_1(a,\, b;\, a; z) = (1-z)^{-b}. \tag{4.5}$$

Similarly, the evaluation of **3.199**:

$$\int_a^b (x-a)^{\mu-1}(b-x)^{\nu-1}(x-c)^{-\mu-\nu}\, dx = (b-a)^{\mu+\nu-1}(b-c)^{-\mu}(a-c)^{-\nu} B(\mu,\nu), \tag{4.6}$$

is reduced to the interval $[0,1]$ by $t = (x-a)/(b-a)$ and then the result follows from **3.198**.

The specific form of the answer is sometimes simplified due to a special relation of the parameters μ, ν and λ in (4.1). For example, in the evaluation of **3.197.11**:

$$\int_0^1 \frac{x^{p-1/2}\, dx}{(1-x)^p\,(1+qx)^p} = \frac{2}{\sqrt{\pi}} \Gamma\left(p+\tfrac{1}{2}\right)\Gamma(1-p)\, \cos^{2p}(\varphi) \frac{\sin((2p-1)\varphi)}{(2p-1)\sin(\varphi)}, \tag{4.7}$$

with $\varphi = \arctan\sqrt{q}$. The standard reduction of the integral to hypergeometric form is easy. Write

$$I = \int_0^1 x^{p-1/2}(1-x)^{-p}(1+qx)^{-p}\, dx \tag{4.8}$$

and use (2.3) to obtain

$$I = B(p+\tfrac{1}{2}, 1-p)\, {}_2F_1\left(p,\, p+\tfrac{1}{2};\, \tfrac{3}{2};\, -q\right). \tag{4.9}$$

To reduce the answer to the stated form, we employ **9.121.19**:

$${}_2F_1\left(\frac{n+2}{2},\, \frac{n+1}{2};\, \frac{3}{2};\, -\tan^2 z\right) = \frac{\sin nz\, \cos^{n+1} z}{n\, \sin z}.$$

The evaluation of **3.197.12**:

$$\int_0^1 \frac{x^{p-1/2}\, dx}{(1-x)^p\,(1-qx)^p} = \frac{\Gamma(p+\frac{1}{2})\Gamma(1-p)}{\sqrt{\pi}} \frac{\left[(1-\sqrt{q})^{1-2p} - (1+2\sqrt{q})^{1-2q}\right]}{(2p-1)\sqrt{q}}. \tag{4.10}$$

is done in similar form. The reduction to

$$I = B(p+\tfrac{1}{2}, 1-p)\, {}_2F_1\left(p,\, p+\tfrac{1}{2};\, \tfrac{3}{2};\, q\right) \tag{4.11}$$

is direct from (2.3). The stated form now follows from **9.121.4**:

$${}_2F_1\left(-\tfrac{n-1}{2}, -\tfrac{n}{2}+1;\, \tfrac{3}{2};\, \tfrac{z^2}{t^2}\right) = \frac{(t+z)^n - (t-z)^n}{2nzt^{n-1}}.$$

5. Some quadratic factors

The table [**3**] contains several entries of the form

$$I = \int_a^b Q_1(x)^{\mu-1} L_2(x)^{\nu-1} L_3(x)^{\lambda-1}\, dx \tag{5.1}$$

where $Q_1(x)$ is a quadratic polynomial and L_j are linear functions. These are discussed in this section.

Example 5.1. The first entry evaluated here is **3.254.1**

$$\int_0^a x^{\lambda-1}(a-x)^{\mu-1}(x^2+b^2)^{\nu}\, dx = b^{2\nu} a^{\lambda+\mu-1} B(\lambda,\mu) \times \\ {}_3F_2\left(-\nu, \frac{\lambda}{2}, \frac{\lambda+1}{2}; \frac{\lambda+\mu}{2}, \frac{\lambda+\mu+1}{2}; -\frac{a^2}{b^2} \right).$$

The conditions given in [**3**] are $\text{Re}\left(\frac{a}{b}\right) > 0$, $\lambda > 0$, $\text{Re}\,\mu > 0$. This entry appears as entry 186(10) of [**2**] as an example of the Riemann-Liouville transform

$$f(x) \mapsto \frac{1}{\Gamma(\mu)} \int_0^y f(x)(y-x)^{\mu-1}\, dx. \tag{5.2}$$

It is convenient to scale the formula, by the change of variables $x = at$, to the form

$$\int_0^1 t^{\lambda-1}(1-t)^{\mu-1}(1+c^2t^2)^{\nu}\, dt = B(\lambda,\mu)\, {}_3F_2\left(-\nu, \frac{\lambda}{2}, \frac{\lambda+1}{2}; \frac{\lambda+\mu}{2}, \frac{\lambda+\mu+1}{2}; -c^2 \right),$$

with $c = a/b$. The binomial theorem gives

$$(1+c^2t^2)^{\nu} = {}_1F_0(-\nu; -; -c^2t^2) = \sum_{n=0}^{\infty} \frac{(-\nu)_n}{n!} (-1)^n c^{2n} t^{2n} \tag{5.3}$$

that produces

$$\begin{aligned} \int_0^1 t^{\lambda-1}(1-t)^{\mu-1}(1+c^2t^2)^{\nu}\, dt &= \sum_{n=0}^{\infty} \frac{(-\nu)_n}{n!} (-c^2)^n \int_0^1 t^{\lambda+2n-1}(1-t)^{\mu-1}\, dt \\ &= \sum_{n=0}^{\infty} \frac{(-\nu)_n}{n!} (-c^2)^n B(\lambda+2n,\mu). \end{aligned}$$

Now write the beta term as

$$\begin{aligned} B(\lambda+2n,\mu) &= \frac{\Gamma(\lambda+2n)\,\Gamma(\mu)}{\Gamma(\lambda+2n+\mu)} \\ &= \Gamma(\mu) \frac{2^{\lambda+2n-1}\Gamma(\frac{\lambda}{2}+n)\Gamma(\frac{\lambda+1}{2}+n)}{2^{\lambda+2n+\mu-1}\Gamma(\frac{\lambda+\mu}{2}+n)\Gamma(\frac{\lambda+\mu+1}{2}+n)} \end{aligned}$$

where the duplication formula for the gamma function

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x)\Gamma(x+\tfrac{1}{2}) \tag{5.4}$$

has been employed. The relation $\Gamma(x+m) = (x)_m\Gamma(x)$ now yields

$$\int_0^1 t^{\lambda-1}(1-t)^{\mu-1}(1+c^2t^2)^\nu\,dt = \frac{\Gamma(\mu)\Gamma(\frac{\lambda}{2})\Gamma(\frac{\lambda+1}{2})}{2^\mu\,\Gamma(\frac{\lambda+\mu}{2})\Gamma(\frac{\lambda+\mu+1}{2})}\,{}_3F_2\left(-\nu, \frac{\lambda}{2}, \frac{\lambda+1}{2}; \frac{\lambda+\mu}{2}, \frac{\lambda+\mu+1}{2}; -c^2\right).$$

Now simplify the gamma factors to produce the result.

Example 5.2. The next entry contains a typo in the *7th*-edition of [**3**]. The correct version of **3.254.2** states that

$$\int_a^\infty x^{-\lambda}(x-a)^{\mu-1}(x^2+b^2)^\nu\,dx = a^{\mu-\lambda+2\nu}B(\mu,\lambda-\mu-2\nu)\,{}_3F_2\left(-\nu, \frac{\lambda-\mu}{2}-\nu, \frac{1+\lambda-\mu}{2}-\nu; \frac{\lambda}{2}-\nu, \frac{1+\lambda}{2}-\nu; -\frac{b^2}{a^2}\right) \tag{5.5}$$

that follows directly from Example 5.1 by the change of variables $y = a^2/x$. It is convenient to scale this entry to the form

$$\int_1^\infty t^{-\lambda}(t-1)^{\mu-1}(t^2+c^2)^\nu\,dt = B(\mu,\lambda-\mu-2\nu)\,{}_3F_2\left(-\nu, \frac{\lambda-\nu}{2}-\nu, \frac{1+\lambda-\mu}{2}-\nu; \frac{\lambda}{2}-\nu, \frac{1+\lambda}{2}-\nu; -c^2\right). \tag{5.6}$$

6. A single factor of higher degree

In this section we conside entries in [**3**] of the

$$I = \int_a^b H_1(x)^{\mu-1}L_2(x)^{\nu-1}L_3(x)^{\lambda-1}\,dx \tag{6.1}$$

where $H_1(x)$ is a polynomial of degree $h \geqslant 2$ and L_j are linear functions.

Example 6.1. Entry **3.259.2** of [**3**] states that

$$\int_0^a x^{\nu-1}(a-x)^{\mu-1}(x^m+b^m)^\lambda\,dx = b^{m\lambda}a^{\mu+\nu-1}B(\mu,\nu) \times {}_{m+1}F_m\left(-\lambda, \frac{\nu}{m}, \frac{\nu+1}{m}, \cdots, \frac{\nu+m-1}{m}; \frac{\mu+\nu}{m}, \frac{\mu+\nu+1}{m}, \cdots, \frac{\mu+\nu+m-1}{m}; -\frac{a^m}{b^m}\right).$$

The scaling $t = x/a$ transforms this entry into

$$\int_0^1 t^{\nu-1}(1-t)^{\mu-1}(1+c^mt^m)^\lambda\,dt = B(\mu,\nu) \times {}_{m+1}F_m\left(-\lambda, \frac{\nu}{m}, \frac{\nu+1}{m}, \cdots, \frac{\nu+m-1}{m}; \frac{\mu+\nu}{m}, \frac{\mu+\nu+1}{m}, \cdots, \frac{\mu+\nu+m-1}{m}; -c^m\right)$$

with $c = a/b$. This is established next using a technique developed by Euler in his proof of the integral representation of ${}_2F_1$.

Start with

$$I = \int_0^1 t^{\nu-1}(1-t)^{\mu-1}\left(c^m t^m + 1\right)^{\lambda} dt$$
$$= \int_0^1 t^{\nu-1}(1-t)^{\mu-1}\, {}_1F_0\left(-\lambda; -; -c^m t^m\right)\, dt$$

using the elementary identity (1.3). This gives

$$I = \int_0^1 t^{\nu-1}(1-t)^{\mu-1} \sum_{n=0}^{\infty} \frac{(-\lambda)_n}{n!} \left(-c^m t^m\right)^n dt$$
$$= \sum_{n=0}^{\infty} \frac{(-\lambda)_n}{n!} \left(-c^m\right)^n \int_0^1 t^{\nu+mn-1}(1-t)^{\mu-1} dt.$$

The integral is recognized as a beta function value, therefore

$$\begin{aligned}
I &= \sum_{n=0}^{\infty} \frac{(-\lambda)_n}{n!} \left(-c^m\right)^n \frac{\Gamma(\nu+mn)\Gamma(\mu)}{\Gamma(\nu+mn+\mu)} \\
&= \sum_{n=0}^{\infty} \frac{(-\lambda)_n}{n!} \left(-c^m\right)^n \frac{\Gamma(m(\frac{\nu}{m}+n))\Gamma(\mu)}{\Gamma(m(\frac{\nu+\mu}{m}+n))} \\
&= \Gamma(\mu) \sum_{n=0}^{\infty} \frac{(-\lambda)_n(-c^m)^n}{n!} \frac{m^{m(\nu/m+n)-1/2}\Gamma(\frac{\nu}{m}+n)\cdots\Gamma(\frac{\nu+m-1}{m}+n)}{m^{m(\frac{\nu+\mu}{m}+m)-1/2}\Gamma(\frac{\nu+\mu}{m}+n)\cdots\Gamma(\frac{\nu+\mu+m-1}{m}+n)} \\
&= \frac{\Gamma(\mu)}{m^{\mu}} \frac{\Gamma(\frac{\nu}{m})\cdots\Gamma(\frac{\nu+m-1}{m})}{\Gamma(\frac{\nu+\mu}{m})\cdots\Gamma(\frac{\nu+\mu+m-1}{m})} \times \sum_{n=0}^{\infty} \frac{(-\lambda)_n(\frac{\nu}{m})_n\cdots(\frac{\nu+m-1}{m})_n}{(\frac{\nu+\mu}{m})_n\cdots(\frac{\nu+\mu+m-1}{m})} \frac{(-c^m)^n}{n!} \\
&= \frac{\Gamma(\mu)}{m^{\mu}} \frac{\Gamma(\frac{\nu}{m})\cdots\Gamma(\frac{\nu+m-1}{m})}{\Gamma(\frac{\nu+\mu}{m})\cdots\Gamma(\frac{\nu+\mu+m-1}{m})} \times \\
&\quad \times\ {}_{m+1}F_m\left(-\lambda, \frac{\nu}{m}, \ldots, \frac{\nu+m-1}{m}; \frac{\nu+\mu}{m}, \ldots, \frac{\nu+\mu+m-1}{m}; -c^m\right).
\end{aligned}$$

This is the evaluation presented in entry **3.259.2**.

7. Integrals over a half-line

This section considers integrals over a half-line that can be expressed in terms of the hypergeometric function.

Example 7.1. To write (3.3) as an integral over an infinite half-line, make the change of variables $w = 1/y$ to obtain

$$\int_{1/u}^{\infty} w^{a-b-1}(1+w/r)^{-a}\, dw = \frac{u^b r^a}{b}\, {}_2F_1\left(a, b; b+1; -ru\right), \tag{7.1}$$

Now replace u by $1/u$ and r by $1/r$ to produce

$$\int_{u}^{\infty} w^{a-b-1}(1+rw)^{-a}\, dw = \frac{1}{bu^b r^a}\, {}_2F_1\left(a, b; b+1; -\frac{1}{ru}\right). \tag{7.2}$$

Finally let $b = a - s$ to obtain

$$\int_u^\infty w^{s-1}(1+rw)^{-a}\,dw = \frac{1}{(a-s)u^{a-s}r^a}\,{}_2F_1\left(a,\, a-s;\, a-s+1; -\frac{1}{ru}\right). \tag{7.3}$$

This appears as **3.194.2** in [**3**].

Example 7.2. The change of variable $y = 1/t$ converts (2.3) into **3.197.6**:

$$\int_1^\infty y^{a-c}(y-1)^{c-b-1}(\alpha y-1)^{-a}\,dy = \alpha^{-a}B(b, c-b)\,{}_2F_1\,(a,\, b;\, c; 1/\alpha) \tag{7.4}$$

where we have labelled $\alpha = 1/x$.

Example 7.3. The change of variables $y = t/(1-t)$ converts (2.3) into **3.197.5**:

$$\int_0^\infty y^{b-1}(1+y)^{a-c}(1+\alpha y)^{-a}\,dy = B(b, c-b)\,{}_2F_1\,(a,\, b;\, c; 1-\alpha) \tag{7.5}$$

where we have labelled $\alpha = 1 - x$. If we now replace α by $1/\alpha$ we obtain

$$\int_0^\infty y^{b-1}(1+y)^{a-c}(y+\alpha)^{-a}\,dy = \alpha^a B(b, c-b)\,{}_2F_1\,(a,\, b;\, c;\, 1-1/\alpha)\,. \tag{7.6}$$

Use the identity

$${}_2F_1\,(a,\, b;\, c;\, 1-1/\alpha) = (1-\alpha)^a\,{}_2F_1\,(a,\, c-b;\, c;\, \alpha) \tag{7.7}$$

to produce **3.197.9**:

$$\int_0^\infty y^{b-1}(1+y)^{a-c}(y+\alpha)^{-a}\,dy = \alpha^a B(b, c-b)\,{}_2F_1\,(a,\, c-b;\, c;\, 1-\alpha)\,. \tag{7.8}$$

Example 7.4. The change of variables $y = tu$ converts (2.3), with $-x$ instead of x, into **3.197.8**:

$$\int_0^u y^{b-1}(u-y)^{c-b-1}(y+\alpha)^{-a}\,dy = \alpha^{-a}u^{c-1}B(b, c-b)\,{}_2F_1\,(a,\, b;\, c; -u/\alpha) \tag{7.9}$$

where we have labelled $\alpha = u/x$.

Example 7.5. The change of variables $y = st/(1-t)$ converts (2.3) into

$$\int_0^\infty y^{b-1}(y+s)^{a-c}(y+r)^{-a}\,dy = r^{-a}s^{a+b-c}B(b, c-b)\,{}_2F_1\left(a,\, b;\, c;\, 1-\frac{s}{r}\right), \tag{7.10}$$

where $r = s/(1-x)$. This is **3.197.1** in [**3**]. The special case $a = c - 1$ produces **3.227.1**:

$$\int_0^\infty \frac{y^{b-1}(y+r)^{1-c}}{y+s}\,dy = r^{1-c}s^{b-1}B(b, c-b)\,{}_2F_1\left(c-1,\, b;\, c;\, 1-\frac{s}{r}\right). \tag{7.11}$$

Example 7.6. Now shift the lower limit of integration via $x = y + u$ to produce

$$\int_u^\infty (x-u)^{b-1}(x-u+s)^{a-c}(x-u+r)^{-a}\,dx = r^{-a}u^{a+b-c}B(b, c-b)\,{}_2F_1\left(a,\, b;\, c;\, 1-\frac{s}{r}\right).$$

Choose $s = u$ and introduce the parameter v by $v = r - u$ to get

$$\int_u^\infty x^{a-c}(x-u)^{b-1}(x+v)^{-a}\,dx = (v+u)^{-a}u^{a+b-c}B(b,c-b)\,{}_2F_1\left(a,\,b;\,c;\,\frac{v}{v+u}\right).$$

Introduce new parameters via $a = -p$, keeping b and $c = q - p$. This yields

$$\begin{aligned}\int_u^\infty x^{-q}(x-u)^{b-1}(x+v)^{p}\,dx &= (v+u)^{p}u^{b-q}B(b,c-b-p)\,{}_2F_1\left(-p,\,b;\,q-p;\,\frac{v}{v+u}\right)\\ &= (v+u)^{p}u^{b-q}B(b,c-b-p)\,{}_2F_1\left(b,\,-p;\,q-p;\,\frac{v}{v+u}\right)\end{aligned}$$

where the symmetry of the hypergeometric function in its two variables has been used.

This result is transformed using **9.131.1**:

$$(7.12)\qquad {}_2F_1\,(a,\,b;\,c;\,z\,) = (1-z)^{-a}\,{}_2F_1\,(a,\,c-b;\,c;\,z/(z-1)\,)\,,$$

that gives

$$\int_u^\infty x^{-q}(x-u)^{b-1}(x+v)^{p}\,dx = (v+u)^{b+p}u^{b-q}B(b,q-p-b)\,{}_2F_1\left(b,\,q;\,q-p;\,-\frac{v}{u}\right).$$

This is the form that is found in **3.197.2**.

8. An exponential scale

The change of variables $t = e^{-r}$ in (2.3) produces

$$(8.1)\qquad {}_2F_1\,(a,\,b;\,c;\,x) = \frac{1}{B(b,c-b)}\int_0^\infty e^{-br}(1-e^{-r})^{c-b-1}(1-xe^{-r})^{-a}\,dr.$$

The parameters are relabeled by $a = \rho$, $b = \mu, c = \nu + \mu, x = \beta$ to produce **3.312.3**:

$$(8.2)\qquad \int_0^\infty (1-e^{-x})^{\nu-1}(1-\beta e^{-x})^{-\rho}e^{-\mu x}\,dx = B(\mu,\nu)\,{}_2F_1\,(\rho,\,\mu;\,\mu+\nu;\,\beta)\,.$$

9. A more challenging example

The evaluation of **3.197.7**

$$(9.1)\qquad \int_0^\infty x^{\mu-1/2}(x+s)^{-\mu}(x+r)^{-\mu}\,dx = \sqrt{\pi}(\sqrt{r}+\sqrt{s})^{1-2\mu}\frac{\Gamma(\mu-1/2)}{\Gamma(\mu)}$$

requires some more properties of the hypergeometric function.

The scaling $x = rt$ produces

$$(9.2)\qquad I = s^{-\mu}\sqrt{r}\int_0^\infty t^{\mu-1/2}(1+t)^{-\mu}(1+rt/s)^{\mu}\,dt$$

and using **3.197.5** we have

$$(9.3)\qquad I = s^{-\mu}\sqrt{r}B\left(\mu+\tfrac{1}{2},\mu-\tfrac{1}{2}\right)\,{}_2F_1\left(\mu,\mu+\tfrac{1}{2},2\mu;\,z\right)$$

where $z = 1 - r/s$. To simplify this expression we employ the relation

$$\begin{aligned} {}_2F_1(\alpha,\beta;\gamma;z) &= \frac{(1-z)^{-\alpha}\,\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\,\Gamma(\gamma-\alpha)}{}_2F_1(\alpha,\gamma-\beta;\alpha-\beta+1;\frac{1}{1-z})+ \\ &+ \frac{(1-z)^{-\beta}\,\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)}{}_2F_1(\beta,\gamma-\alpha;\beta-\alpha+1;\frac{1}{1-z}) \end{aligned}$$

to produce

$$\begin{aligned} {}_2F_1\left(\mu,\mu+\tfrac{1}{2},2\mu;\,z\right) &= \frac{(1-z)^{-\mu}\Gamma(2\mu)\Gamma(1/2)}{\Gamma(\mu+1/2)\,\Gamma(\mu)}{}_2F_1\left(\mu,\mu-\tfrac{1}{2}\,\tfrac{1}{2};\,\frac{1}{1-z}\right) \\ &+ \frac{(1-z)^{-\mu-1/2}\Gamma(2\mu)\Gamma(-1/2)}{\Gamma(\mu-1/2)\,\Gamma(\mu)}{}_2F_1\left(\mu,\mu+\tfrac{1}{2}\,\tfrac{3}{2};\,\frac{1}{1-z}\right). \end{aligned}$$

The binomial theorem shows that

$$(9.4)\qquad {}_2F_1\left(-\frac{n}{2},-\frac{n-1}{2};\,\frac{1}{2};\,\frac{z^2}{t^2}\right) = \frac{1}{2t^n}\left((t+z)^n+(t-z)^n\right),$$

that appears as **9.121.2** in [**3**]. Thus

$${}_2F_1\left(\mu,\,\mu-\tfrac{1}{2};\,\tfrac{1}{2};\,\frac{1}{1-z}\right) = \frac{1}{2(1-z)^{1/2-\mu}}\left((1+\sqrt{1-z})^{1-2\mu}+(-1+\sqrt{1-z})^{1-2\mu}\right).$$

Similarly, **9.121.4** states that

$$(9.5)\qquad {}_2F_1\left(-\frac{n-1}{2},-\frac{n-2}{2};\,\frac{3}{2};\,\frac{z^2}{t^2}\right) = \frac{1}{2nzt^{n-1}}\left((t+z)^n-(t-z)^n\right),$$

to produce

$${}_2F_1\left(\mu,\,\mu-\tfrac{1}{2};\,\tfrac{3}{2};\,\frac{1}{1-z}\right) = \frac{1}{2(1-2\mu)(1-z)^{-\mu}}\left((1+\sqrt{1-z})^{1-2\mu}-(-1+\sqrt{1-z})^{1-2\mu}\right).$$

Replacing these values in (9.3) produces the result.

10. One last example: a combination of algebraic factors and exponentials

Entry **3.389.1** presents an analytic expression for the integral

$$(10.1)\qquad I := \int_0^a x^{2\nu-1}(a^2-x^2)^{\rho-1}e^{\mu x}dx.$$

The evaluation begins with an elementary scaling to obtain

$$\begin{aligned} I =& a^{2(\rho-1)}\int_0^1 x^{2\nu-1}(1-\frac{x^2}{a^2})^{\rho-1}e^{\mu x}dx \\ =& \tfrac{1}{2}a^{2\rho-1}\int_0^1 (ay^{1/2})^{2\nu-1}(1-y)^{\rho-1}e^{\mu a y^{1/2}}y^{-1/2}dy. \end{aligned}$$

Now use ${}_0F_0(;;x) = e^x$ to obtain

$$\begin{aligned}
I &= \frac{a^{2\rho+2\nu-2}}{2}\int_0^1 y^{\nu-1}(1-y)^{\rho-1}\,{}_0F_0(;;\mu a y^{1/2})\,dy \\
&= \frac{a^{2\rho+2\nu-2}}{2}\int_0^1 y^{\nu-1}(1-y)^{\rho-1}\sum_{n=0}^{\infty}\frac{(\mu a y^{1/2})^n}{n!}dy \\
&= \frac{a^{2\rho+2\nu-2}}{2}\sum_{n=0}^{\infty}\frac{(\mu a)^n}{n!}\int_0^1 y^{\nu+n/2-1}(1-y)^{\rho-1}dy.
\end{aligned}$$

The integral is now recognized as a beta value to conclude that

$$\begin{aligned}
I =& \frac{a^{2\rho+2\nu-2}}{2}\sum_{n=0}^{\infty}\frac{(\mu a)^n}{n!}B(\nu+n/2,\rho) \\
=& \frac{a^{2\rho+2\nu-2}\Gamma(\rho)}{2}\sum_{n=0}^{\infty}\frac{(\mu a)^n}{n!}\frac{\Gamma(\nu+n/2)}{\Gamma(\nu+n/2+\rho)} \\
=& \frac{a^{2\rho+2\nu-2}\Gamma(\rho)\Gamma(\nu)}{2\Gamma(\nu+\rho)}\sum_{k=0}^{\infty}\frac{(\mu a)^{2k}(\nu)_k}{\Gamma(2k+1)(\nu+\rho)_k} + \frac{a^{2\rho+2\nu-2}\Gamma(\rho)}{2}\sum_{k=0}^{\infty}\frac{(\mu a)^{2k+1}\Gamma(\nu+k+1/2)}{(2k+1)!\Gamma(\nu+\rho+k+1/2)}
\end{aligned}$$

and combining the gamma factors to produce the beta function yields

$$\begin{aligned}
I =& \frac{1}{2}a^{2\rho+2\nu-2}B(\rho,\nu)\sum_{k=0}^{\infty}\frac{(\mu^2a^2)^k(\nu)_k}{(2k)\Gamma(2k)(\nu+\rho)_k} + \\
&+ \frac{1}{2}a^{2\rho+2\nu-1}\mu\Gamma(\rho)\sum_{k=0}^{\infty}\frac{(\mu a)^{2k}}{\Gamma(2k+2)}\frac{(\nu+1/2)_k\Gamma(\nu+1/2)}{(\nu+\rho+1/2)_k\Gamma(\nu+\rho+1/2)}.
\end{aligned}$$

This can be reduced to

$$\begin{aligned}
2I =& a^{2\rho+2\nu-2}B(\rho,\nu)\sum_{k=0}^{\infty}\frac{(\nu)_k(\mu^2a^2)^k}{(\nu+\rho)_k(2k)}\frac{2^{1-2k}\sqrt{\pi}}{\Gamma(k)\Gamma(k+1/2)} + \\
&+ a^{2\rho+2\nu-1}\mu B(\rho,\nu+1/2)\sum_{k=0}^{\infty}\frac{(\nu+1/2)_k}{(\nu+\rho+1/2)_k}\frac{(\mu^2a^2)^k2^{1-2(k+1)}\sqrt{\pi}}{\Gamma(k+1)\Gamma(k+\frac{3}{2})} \\
=& a^{2\rho+2\nu-2}B(\rho,\nu)\sum_{k=0}^{\infty}\frac{(\nu)_k}{(\nu+\rho)_k(\frac{1}{2})_k k!}\left(\frac{\mu^2a^2}{4}\right)^k + \\
&+ a^{2\rho+2\nu-1}\mu B(\rho,\nu+1/2)\sum_{k=0}^{\infty}\frac{(\nu+1/2)_k}{(\nu+\rho+1/2)_k\left(\frac{3}{2}\right)_k}\left(\frac{\mu^2a^2}{4}\right)^k \\
=& a^{2\rho+2\nu-2}B(\rho,\nu)\,{}_1F_2\left(\nu;\nu+\rho,\frac{1}{2};\frac{\mu^2a^2}{4}\right) + \\
&+ a^{2\rho+2\nu-1}\mu B(\rho,\nu+1/2)\,{}_1F_2\left(\nu+1;\nu+\rho+1/2,\frac{3}{2};\frac{\mu^2a^2}{4}\right).
\end{aligned}$$

There are many other entries of [**3**] that can be evaluated in terms of hypergeometric functions. A second selection of examples is in preparation.

References

[1] G. Andrews, R. Askey, and R. Roy. *Special Functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, New York, 1999.
[2] A. Erdélyi. *Tables of Integral Transforms*, volume II. McGraw-Hill, New York, 1st edition, 1954.
[3] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products*. Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 7th edition, 2007.

Received 23 12 2010 revised 13 04 2011

Department of Mathematics,
Tulane University,
New Orleans, LA 70118, USA.
E-mail address: vhm@math.tulane.edu
E-mail address: kkohl@math.tulane.edu