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Exponential stability of nontrivial solutions of stochastic differential equations

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Abstract. This paper is concerned with the exponential stability of nontrivial solutions of stochastic differential equations. In this paper, we give new criteria for the exponential stability for stochastic differential equations without the trivial solution.

1. Introduction

Stability of stochastic differential equations is one of the most active and important areas in stochastic analysis. Many mathematicians have paid attention to it. Normally, in order to investigate stability for stochastic differential equations of the form

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \tag{1.1}$$

we always suppose that $f(0, t) \equiv g(0, t) \equiv 0$. Then we shall study the stability of the trivial solution. However, it is known that there are many types of stochastic differential equations whose all solutions tend to each other despite they do not have the trivial solution. These equations are worth being interested since the solutions with different initial value has similar large-time properties. In practice, it is therefore sufficient to consider any solution to approximate asymptotic properties of other solutions. In addition, the rate of the convergence tell us how large the error of the approximation is. For this reason, we should also pay attention to estimate decay rate of the distance of two solution with given initial values. The most important decay rate is the exponential one. In the case where $f(0, t) \equiv g(0, t) \equiv 0$, the exponential stability of Equation 1.1 has received quite a lot of attention in the literature. We here mention Arnold [**1**], [**2**], Arnold, Oeljeklaus and Pardoux [**3**], Has'minskii[**4**] among others. We also cite [**5**] for a systematic review. In this paper, we will study the exponential stability of nontrivial solution of Equation (1.1).

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2. Exponential stability of nontrivial solutions

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}-$null sets. Let $B_t = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. We denote $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. Throughout this paper, we always suppose that Equation (1.1) satisfies a sufficient condition under which there is some unique global solution to Equation (1.1) for any $\mathcal{F}_0$-adapted initial value. We can take one of the conditions given by Has'minski in [**4**] and Narita in [**7**]. In this paper, we denote X^x is the solution to Equation (1.1) with the initial value $x \in \mathbb{R}^n$. Relying on the concept of the exponential stability of the trivial solution given in [**6**], we define

Definition 2.1. The solution $X^x(t)$ is said to be exponential stable if for any $y \in \mathbb{R}^n$,

$$\limsup_{t\to\infty} \frac{1}{t} \ln |X^x(t) - X^y(t)| < 0 \text{ a.s.}$$

Denote by $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ the family of all function $V(x,t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+)$ which are continuously twice differentiable in x and once in t. For $V(x,t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ we define LV by

$$LV(x,t) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}\big(g^T(x,t)V_{xx}(x,t)g(x,t)\big)$$

where $V_x = (\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n})$, $V_{xx} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{n\times n}$ and $V_t = \frac{\partial V}{\partial t}$.

Since we consider the difference between two solution of Equation (1.1), we need to introduce a new operator. For any $U \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ we define $\boldsymbol{\mathcal{L}}U : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\begin{aligned}\boldsymbol{\mathcal{L}}U(x,y,t) =& U_t(x-y,t) + U_x(x-y,t)\big(f(x,t) - f(y,t)\big)\\ & \frac{1}{2}\text{trace}\Big(\big(g(x,t) - g(y,t)\big)^T V_{xx}(x,t)\big(g(x,t) - g(y,t)\big)\Big)\end{aligned}$$

If $f(x,t) \equiv g(x,t) \equiv 0$, a sufficient conditions for exponential stability is derived from

Theorem 2.2 (see[**6**]). *Assume that there exists a non-negative function $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ and constants $p > 0, c_1 > 0, c_2 \in \mathbb{R}, c_3 \geq 0$ such that for all $x \neq 0$ and $t \geq 0$,*

(1) $c_1|x|^p \leq V(x,t)$,
(2) $LV(x,t) \leq c_2 V(x,t)$,
(3) $|V_x(x,t)g(x,t)|^2 \geq c_3 V^2(x,t)$.

Then

$$\limsup_{t\to\infty} \frac{1}{t} \ln |X^{x_0}(t)| \leq -\frac{c_3 - 2c_2}{p} \ \textit{a.s.}$$

for all $x_0 \in \mathbb{R}^n$. In particular, if $c_3 > 2c_2$, the trivial solution of Equation (1.1) *is almost surely exponentially stable.*

With a few small modification, we can obtain the following theorem.

THEOREM 2.3. *Assume that there exists a function* $U \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ *and constants* $p > 0, c_1 > 0, c_2 \in \mathbb{R}, c_3 \geq 0$ *such that for all* $x \neq y$ *and* $t \geq 0$,

(1) $c_1|x|^p \leq U(x,t)$,
(2) $\mathfrak{L}U(x,y,t) \leq c_2 U(x-y,t)$,
(3) $|U_x(x-y,t)\big(g(x,t)-g(y,t)\big)|^2 \geq c_3 U^2(x-y,t)$.

Then

$$\limsup_{t\to\infty} \frac{1}{t} \ln |X^x(t) - X^y(t)| \leq -\frac{c_3 - 2c_2}{p} \ \ a.s.$$

for all $x, y \in \mathbb{R}^n$. *In particular, if* $c_3 > 2c_2$, *any solution of Equation* (1.1) *is almost surely exponentially stable.*

We see that the theorem demands $\mathfrak{L}U(x,y,t)$ and $|U_x(x-y,t)\big(g(x,t)-g(y,t)\big)|^2$ can be estimated by the function of the difference $x-y$. It seems to be restrictive since $\mathfrak{L}U(x,y,t)$ and $|U_x(x-y,t)\big(g(x,t)-g(y,t)\big)|^2$ may be depend not only on $x-y$ but also on $|x|$ and $|y|$. In other words, there are many situation where the items (2) and (3) of Theorem 2.3 can be satisfied only on each compact subset rather that the entire space $\mathbb{R}^n$. In order to weaken these conditions, we need an additional assumption. We denote by $\mathcal{K}$ the family of all positive continuous functions $\mu : \mathbb{R}^n \to \mathbb{R}_+$ such that $\lim_{|x|\to\infty} \mu(x) = \infty$.

ASSUMPTION 2.1. *Assume that there exists a function* $\mu(\cdot) \in \mathcal{K}$ *and a constant* $M > 0$ *such that for any* $x \in \mathbb{R}^n$,

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \mu\big(X^x(s)\big) ds \leq M \ \ a.s.$$

The following theorem gives a sufficient condition for the boundedness of the average in time of $\mu(X^x(t))$.

THEOREM 2.4. *Suppose that there exist functions* $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, $\mu(x) \in \mathcal{K}$ *and three positive numbers* λ_1, λ_2 *and* c *such that*

$$LV(x,t) + \lambda_1\big(V_x(x,t)g(x,t)\big)^2 + \lambda_2\mu(x) \leq c. \tag{2.1}$$

Then, $\limsup_{t\to\infty} \frac{1}{t} \int_0^t \mu(X^x(s))ds \leq \frac{c}{\lambda_2}$ *a.s. for any* $x \in \mathbb{R}$.

PROOF. By Itô formula, we have

$$\begin{aligned} V(X^x(t),t) =& V(x,0) + \int_0^t \Big(LV(X^x(s),s) + \lambda_1\big(V_x(X^x(s),s)g(X^x(s),s)\big)^2\Big)ds \\ &- \int_0^t \lambda_1\big(V_x(X^x(s),s)g(X^x(s),s)\big)^2 ds + M(t) \end{aligned} \tag{2.2}$$

where $M(t) = \int_0^t V_x(X^x(s),s)g(X^x(s),s)dB(s)$ is a continuous local martingale vanishing at $t=0$. It follows from the exponential martingale inequality (see [**6**, Theorem 7.4]) that for any $n \in \mathbb{N}$

$$\mathbb{P}\Big\{ \sup_{0\leq t\leq n} \Big(M(s) - \lambda_1 \int_0^t \big(V_x(X^x(s),s)g(X^x(s),s)\big)^2 ds\Big) > \frac{\ln n}{\lambda_1} \Big\} \leq \frac{1}{n^2}$$

An application of the Borel-Cantelli lemma yields that for almost all $\omega \in \Omega$, there a random positive integer number $n_0 = n_0(\omega)$ such that for any $n \geq n_0$,

$$\sup_{0\le t\le n}\Big(M(t)-\lambda_1\int_0^t \big(V_x(X^x(s),s)g(X^x(s),s)\big)^2 ds\Big)\le \frac{\ln n}{\lambda_1}$$

In particular, for $n-1\le t\le n$,

$$M(t)-\lambda_1\int_0^t \big(V_x(X^x(s),s)g(X^x(s),s)\big)^2 ds\le \frac{\ln n}{\lambda_1}\ \text{a.s},$$

which implies that

$$\frac{M(t)}{t}-\lambda_1\int_0^t \big(V_x(X^x(s),s)g(X^x(s),s)\big)^2 ds\Big)\le \frac{\ln n}{\lambda_1(n-1)}\ \text{a.s.}$$

Letting $n\to\infty$, we obtain

$$\limsup_{t\to\infty}\Big(\frac{M(t)}{t}-\lambda_1\int_0^t \big(V_x(X^x(s),s)g(X^x(s),s)\big)^2 ds\Big)=0\ \text{a.s} \tag{2.3}$$

Combining (2.1), (2.2) and (2.3) yields

$$\limsup_{t\to\infty}\Big(\frac{V(X^x(t),t)}{t}+\frac{\lambda_2}{t}\int_0^t \mu(X^x(s))ds\Big)\le c\ \text{a.s.}$$

Since $V(x,t)\geq 0\,\forall (x,t)\in\mathbb{R}^n\times\mathbb{R}_+$, we get

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mu(X^x(s))ds\le\frac{c}{\lambda_2}\ \text{a.s.}\forall x\in\mathbb{R}^n.$$

□

We now give a criterion for the exponential stability.

Theorem 2.5. *Let Assumption 2.1 holds. Assume that there exists a function $U\in C^{2,1}(\mathbb{R}^n\times\mathbb{R}_+;\mathbb{R}_+)$ and two bounded continuous functions $C_2(r), C_3(r):\mathbb{R}_+\to\mathbb{R}$ and constants $p>0, c_1>0, c_2>0$ such that for all $x\neq y, |x|\vee|y|\le r$ and $t\geq 0$,*

1. $C_2(r)<c_2$ *and* $C_3(r)>2C_2(r)\,\forall r>0$,
2. $c_1|x|^p\le U(x,t)$,
3. $\mathcal{L}U(x,y,t)\le C_2(r)U(x-y,t)$,
4. $|U_x(x-y,t)\big(g(x,t)-g(y,t)\big)|^2\geq C_3(r)U^2(x-y,t)$.

Then there exist $\delta>0$ such that

$$\limsup_{t\to\infty}\frac{1}{t}\ln|X^x(t)-X^y(t)|<-\delta\ \textit{a.s.}$$

for all $x,y\in\mathbb{R}^n$.

Proof. Firstly, we note that if $x\neq y$, then $X^x(t)\neq X^y(t)\,\forall t\geq 0$ almost surely. This claim is rather basic. It can be prove similarly to [**6**, Lemma 3.2]. Thus, $V(X^x(t)-X^y(t))\neq 0\,\forall t\geq 0$ with probability 1. To avoid complicated symbols, we denote $\mathcal{H}U(x,y,t) = \dfrac{U_x(x-y,t)\big(g(x,t)-g(y,t)\big)}{U(x-y,t)}$, for $x \neq y$. By Itô formula, for $t \geq 0$,

$$
\begin{aligned}
\ln U(X^x(t)-X^y(t),t) =& \ln U(x-y,0) + \int_0^t \frac{\mathfrak{L}U(X^x(s),X^y(s),s)}{U(X^x(t)-X^y(t),t)}ds \\
&- \frac{1}{2}\int_0^t \big(\mathcal{H}U(X^x(s),Y^y(s),s)\big)^2 ds \qquad (2.4)\\
&+ \int_0^t \mathcal{H}U(X^x(s),Y^y(s),s)dB(s)
\end{aligned}
$$

With the indication function $\mathbf{1}_{(\cdot)}$, we note that

$$
\limsup_{t\to\infty}\Big(\inf\{\mu(x) : |x| > R\}\cdot\int_0^t \mathbf{1}_{\{|X^x(s)|>R\}}ds\Big) \leq \limsup_{t\to\infty}\int_0^t \mu(X^x(s) \leq M.
$$

Since $\inf\{\mu(x) : |x| > R\} \to \infty$ as $R \to \infty$ we imply that for any $0 < \rho < 1$, there exists an $R > 0$ such that

$$
\liminf_{t\to\infty}\int_0^t \mathbf{1}_{\{|X^x(s)|>R\}}ds < \frac{\rho}{2} \text{ and } \liminf_{t\to\infty}\int_0^t \mathbf{1}_{\{|X^y(s)|>R\}}ds < \frac{\rho}{2}.
$$

As a result,

$$
\liminf_{t\to\infty}\frac{1}{t}\int_0^t \mathbf{1}_{\{X^x(s)|\vee|X^y(s)|\leq R\}}ds > 1-\rho \qquad (2.5)
$$

Since $C_3(r) > 2C_2(r)\,\forall r \geq 0$ then $\mathfrak{L}U(u,v,t) - \dfrac{1}{2}\mathcal{H}U(u,v,t) < 0\,\forall u \neq v$. Hence, putting$\lambda = \dfrac{C_3(R)}{2} - C_2(R) > 0$, we have the estimate

(2.6)

$$
\begin{aligned}
&\frac{1}{t}\int_0^t \Big(\frac{\mathfrak{L}U(X^x(s),X^y(s),s)}{U(X^x(s),X^y(s),s)} - \frac{1}{2}\big(\mathcal{H}U(X^x(s),X^y(s),s)\big)^2\Big)ds\\
\leq&\frac{1}{t}\int_0^t \mathbf{1}_{\{|X^x(s)|\vee|X^y(s)|\leq R\}}\Big(\frac{\mathfrak{L}U(X^x(s),X^y(s),s)}{U(X^x(s),X^y(s),s)} - \frac{1}{2}\big(\mathcal{H}U(X^x(s),X^y(s),s)\big)^2\Big)ds\\
\leq&-\frac{\lambda}{t}\int_0^t \mathbf{1}_{\{|X^x(s)|\vee|X^y(s)|\leq R\}}ds.
\end{aligned}
$$

Note that $M(t) = \int_0^t \mathcal{H}U(X^x(s),X^y(s),s)$ is a continuous local martingale with $M(0) = 0$. Hence, employing the manner of the proof of (2.3), we have

$$
\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\Big(M(t) - \frac{\epsilon}{2}\int_0^t \big(\mathcal{H}U(X^x(s),X^y(s),s)\big)^2\Big)ds \qquad (2.7)\\
\leq&\limsup_{t\to\infty}M(t) - \frac{\epsilon}{2}\int_0^t \big(\mathcal{H}U(X^x(s),X^y(s),s)\big)^2 ds \leq 0.
\end{aligned}
$$

On the other hand

$$
\frac{\epsilon}{t}\int_0^t \frac{\mathfrak{L}U(X^x(s),X^y(s),s)}{U(X^x(s)-X^y(s),s)}ds \leq \epsilon c_2 \qquad (2.8)
$$

Combining (2.4), (2.5) (2.6), (2.7) and (2.8), we obtain

$$
\begin{aligned}
&\limsup_{t\to\infty}\frac{\ln U(X^x(t)-X^y(t),t)}{t}\\
\le&\limsup_{t\to\infty}\frac{1-\epsilon}{t}\int_0^t\Big(\frac{\mathcal{L}U(X^x(s),X^y(s),s)}{U(X^x(s)-X^y(s),s)}-\frac{1}{2}\big(\mathcal{H}U(X^x(s),X^y(s),s)\big)^2\Big)ds\\
&+\limsup_{t\to\infty}\frac{\epsilon}{t}\int_0^t\frac{\mathcal{L}U(X^x(s),X^y(s),s)}{U(X^x(s)-X^y(s),s)}ds\\
&+\limsup_{t\to\infty}\frac{1}{t}\Big(M(t)-\frac{\epsilon}{2}\int_0^t\big(\mathcal{H}U(X^x(s),X^y(s),s)\big)^2\Big)ds\\
\le&-(1-\epsilon)(1-\rho)\lambda+\epsilon c_2 \text{ a.s}
\end{aligned}
\tag{2.9}
$$

Let $\epsilon\to 0$ we get

$$\limsup_{t\to\infty}\frac{\ln U(X^x(t)-X^y(t),t)}{t}\le -(1-\rho)\lambda \text{ a.s.}$$

Since $c_1(X^x(t)-X^y(t))^p\le U(X^x(t)-X^y(t),t)$, it is easy to see that

$$\limsup_{t\to\infty}\frac{\ln\big|X^x(t)-X^y(t),t\big|}{t}\le-\frac{(1-\rho)\lambda}{p} \text{ a.s.}$$

The proof is complete. $\square$

Example 2.6. Let $f(x)=(x+\sin x)^3$ and consider the following equation

$$dX(t)=\Big(a(t)+\frac{b(t)}{e^{X(t)}+1}+\frac{\alpha^2}{2}X(t)-c(t)f(X(t))\Big)dt+\big(\alpha X(t)+\sigma(t)\big)dB(t)\tag{2.10}$$

where $a(t),b(t),c(t),\sigma(t)$ are bounded continuous functions, $\hat b:=\inf\{b(t):t\ge 0\}>0,\hat c:=\inf\{c(t):t\ge 0\}>0$ and α is a constant. Obviously, we can find out a positive constant θ such that $xf(x)\ge\theta x^4\,\forall x\in\mathbb{R}$. Let $V(x,t)=x^2$ and $\mu(x)=x^2$ also. We have

$$
\begin{aligned}
LV(x,t)=&2a(t)x+\frac{2b(t)x}{e^x+1}+\alpha^2x^2-2c(t)xf(x)+\big(\alpha x+\sigma(t)\big)^2\\
=&\sigma^2(t)+2\big(a(t)+\alpha\sigma(t)+\frac{b(t)}{e^x+1}\big)x+2\alpha x^2-2c(t)xf(x)\\
=&\sigma^2(t)+2\big(a(t)+\alpha\sigma(t)+\frac{b(t)}{e^x+1}\big)x+2\alpha x^2-2c(t)xf(x)+2\alpha^2x^2\\
&+\frac{\hat c\theta}{\alpha^2}x^2\big(\alpha x+\sigma(t)\big)^2+x^2\\
&-\frac{\hat c\theta}{4\alpha^2}4x^2\big(\alpha x+\sigma(t)\big)^2-x^2.
\end{aligned}
\tag{2.11}
$$

Using the inequality $c(t)xf(x) \geq \hat{c}\theta x^4 \, \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}_+$, we imply that

$$M = \sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}_+} \Big\{\sigma^2(t) + 2\big(a(t) + \alpha\sigma(t) + \frac{b(t)}{e^x+1}\big)x + 2\alpha x^2 \\ - 2c(t)xf(x) + \frac{\hat{c}}{\alpha^2}x^2\big(\alpha x + \sigma(t)\big)^2 + x^2\Big\} < \infty.$$

As a result,

$$LV \leq M - \frac{\hat{c}}{2\alpha^2}4x^2\big(\alpha x + \sigma(t)\big)^2 - x^2.$$

It follows from this inequality that this equation satisfies the Has'minskii condition (see [**4**]). Moreover, applying Theorem 2.4 yields $\frac{1}{t}\int_0^t \big(X^x(s)\big)^2 ds \leq M \text{a.s} \, \forall x \in \mathbb{R}^2$.

Now, we calculate

$$\Big(\mathcal{H}V(x,y,t)\Big)^2 = 4\alpha^2 \, \forall x \neq y, t \geq 0, \tag{2.12}$$

and

$$\begin{aligned} \mathcal{L}V(x,y,t) =& 2b(t)(x-y)\big(\frac{1}{e^x+1} - \frac{1}{e^y+1}\big) + \alpha^2(x-y)^2 \\ & - 2c(t)(x-y)\big(f(x)-f(y)\big) + \alpha^2(x-y)^2. \end{aligned} \tag{2.13}$$

Since $x^2f(x)$ is a non-decreasing function, $-2c(t)(x-y)\big(f(x)-f(y)\big) \leq 0 \, \forall x,t$. We also have

$$2b(t)(x-y)\big(\frac{1}{e^x+1} - \frac{1}{e^y+1}\big) = -2\frac{(x-y)(e^x-e^y)}{e^x+e^y} \leq 0.$$

Hence,

$$\mathcal{L}V(x,y,t) \leq 2\alpha^2 V(x-y,t) \, \forall x \neq y, t \geq 0.$$

However, we are able to choose a constant $c_2 < 2\alpha^2$ such that $\mathcal{L}V(x,y,t) \leq c_2 V(x-y,t) \, \forall x \neq y, t \geq 0$. Indeed, since $(2k+1)\pi, k = 0, \pm 1, \ldots$ are stationary points of $f(x)$, for each $\epsilon > 0$, there exist $\delta = \delta(k,\epsilon) > 0$ such that

$$f\big((2k+1)\pi + \delta\big) - f\big((2k+1)\pi\big) \geq -\epsilon\delta.$$

On the other hand,

$$\lim_{y\to+\infty} \frac{(x-y)(e^x-e^y)}{(e^x+1)(e^y+1)} = \lim_{y\to+\infty} \frac{l(e^{y+l}-e^y)}{(e^{y+l}+1)(e^y+1)} = 0.$$

For this reason, we are unable to prove the exponential stability of this equation by employing Theorem 2.3 for $V(x,t) = x^2$. However, for any $r > 0$, applying the mean value theorem, there exists an $\alpha_r > 0$ such that

$$-2b(t)\frac{(x-y)(e^x-e^y)}{(e^x+1)(e^y+1)} \leq -2\hat{b}\frac{(x-y)(e^x-e^y)}{(e^x+1)(e^y+1)} \leq -\alpha_r(x-y)^2 \, \forall |x| \vee |y| \leq r.$$

Consequently, for any x, y such that $|x| \vee |y| \leq r$,

$$\mathcal{L}V(x,y,t) \leq (-\alpha_r + 2\alpha^2)(x-y)^2,$$

while

$$\big(\mathcal{H}V(x,y,t)\big)^2 = 4\alpha^2 \, \forall x \neq y, t \geq 0.$$

Thus, it follows from Theorem 2.5 that the solution $X^x(t)$ is exponential stable for any $x \in \mathbb{R}$.

We now give another criterion for the exponential stability.

THEOREM 2.7. *Suppose there is a function $\mu \in \mathcal{K}$ satisfying Assumption 2.1. If there exists a function $V(x,t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+;\mathbb{R}_+)$ and a constant $c_r > 0$ for each $r > 0$ such that*

$$\frac{\mathfrak{L}U(x,y,t)}{U(x-y,t)} - \frac{1}{2}\big(\mathcal{H}U(x,y,t)\big)^2 \leq -c_r. \tag{2.14}$$

where $\mathcal{H}U(x,y,t)$ is defined in the proof of Theorem 2.5. Moreover, there is a $C > 0$ such that

$$\big(\mathcal{H}U(x,y,t)\big)^2 \leq C\big(\mu(x) + \mu(y)\big). \tag{2.15}$$

Then any solution $X^x(t)$ of Equation 1.1 is exponential stable.

PROOF. By Itô formula, for $x \neq y$, we have

$$\begin{aligned} &\ln U(X^x(t) - X^y(t), t) = \ln U(x-y, 0) \\ &+ \int_0^t \left(\frac{\mathfrak{L}U(X^x(s), X^y(s), s)}{U(X^x(s) - X^y(s), s)} - \frac{1}{2}\Big(\mathcal{H}U(X^x(s), X^y(s), t)\Big)^2\right) ds \\ &+ \int_0^t \mathcal{H}U(X^x(s), X^y(s), s) dB(s). \end{aligned} \tag{2.16}$$

It follows from (2.7) and (2.15) that, for any $\epsilon > 0$,

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathcal{H}U(X^x(s), X^y(s), s) dB(s) \\ \leq &\limsup_{t\to\infty} \frac{\epsilon C}{t}\int_0^t \Big(\mu(X^x(s)) + \mu(X^y(s))\Big) ds \leq \epsilon C M \text{ a.s.} \end{aligned}$$

Let $\epsilon \to 0$, we get

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathcal{H}U(X^x(s), X^y(s), s) dB(s) \leq 0 \text{ a.s.} \tag{2.17}$$

On the other hand, employing the inequality (2.14) and the estimation (2.4), we conclude that, there exists $\rho < 1$ and $R > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \left(\frac{\mathfrak{L}U(X^x(s), X^y(s), s)}{U(X^x(s) - X^y(s), s)} - \frac{1}{2}\Big(\mathcal{H}U(X^x(s), X^y(s), t)^2\Big)\right) ds \leq -(1-\rho)c_R. \tag{2.18}$$

Combining (2.16), (2.17) and (2.18) we get the assertion of this theorem.

□

EXAMPLE 2.8. We consider the following equation

$$\begin{aligned} dX(t) = &\Big(a(t) - \alpha\,\mathrm{sgn}(X(t)) \ln\big(1 + |X(t)|\big)\Big) dt \\ &+ \big(b(t)X^2(t) + c(t)X(t) + \sigma(t)\big) dB(t) \end{aligned} \tag{2.19}$$

where $a(t), b(t), c(t), \sigma(t)$ are bounded continuous functions, $\hat{b} := \inf\{b(t) : t \geq 0\} > 0$ and α is a constant. Let $V(x,t) = \ln(1+x^2)$. We have

$$LV(x,t) = -2\alpha\frac{|x|\ln(1+|x|)}{1+x^2} + \frac{2xa(t)}{1+x^2} + \frac{(1-x^2)\big(b(t)x^2+c(t)x+\sigma(t)\big)^2}{(1+x^2)^2}$$

value $x(0)$, there is a unique solution to Equation (2.10). Since $b(t) \geq \hat{b}\,\forall t \geq 0$,

$$\sup_{(x,t)\in\mathbb{R}\times\mathbb{R}_+}\Big\{\frac{(1-x^2)\big(b(t)x^2+c(t)x+\sigma(t)\big)^2}{(1+x^2)^2} - \hat{b}^2x^2\Big\} < \infty.$$

Consequently, there exists two positive constant K_1, K_2 such that

$$LV(x,t) \leq K_1 + V(x,t) \quad \forall (x,t) \in \mathbb{R}\times\mathbb{R}_+ \tag{2.20}$$

and

$$LV(x,t) + \frac{\hat{b}^2}{2}x^2 + \frac{1}{8}\Big(\frac{2x}{1+x^2}\big(b(t)x^2+c(t)x+\sigma(t)\big)\Big)^2 \leq K_2\,\forall (x,t) \in \mathbb{R}\times\mathbb{R}_+. \tag{2.21}$$

The inequality (2.20) guarantees the existence and uniqueness of global solution to Equation (2.19). Meanwhile, it follows from Theorem 2.4 and the inequality (2.21) that

$$\frac{1}{t}\int_0^t \big(X^x(s)\big)^2 ds \leq \frac{2K_2}{\hat{b}^2} \text{ a.s } \forall x \in \mathbb{R}.$$

Now we let $U(x,t) = x^2$ and get

$$\frac{\mathcal{L}U(x,y,t)}{U(x-y,t)} = -2\alpha\frac{\operatorname{sgn} x\ln(1+|x|) - \operatorname{sgn} y\ln(1+|y|)}{x-y} + b^2(t)(x+y)^2$$

and

$$\Big(\frac{\mathcal{H}U(x,y,t)}{U(x-y,t)}\Big)^2 = 4b^2(t)(x+y)^2 \leq 8\sup_{t\geq 0}\{b^2(t)\}(x^2+y^2).$$

It is easy to show that for any $r > 0$, there exists a $c_r > 0$ such that for any $x \neq y$ and $\max\{|x|,|y|\} \leq r$,

$$\frac{\operatorname{sgn} x\ln(1+|x|) - \operatorname{sgn} y\ln(1+|y|)}{x-y} \geq c_r.$$

Hence,

$$\frac{\mathcal{L}U(x,y,t)}{U(x-y,t)} - \frac{1}{2}\big(\mathcal{H}U(x,y,t)\big)^2 \leq -2\alpha c_r.$$

Employing Theorem 2.7, we conclude that $X^x(t)$ is exponential stable for all $x \in \mathbb{R}$.

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