SCIENTIA
Series A: *Mathematical Sciences*, Vol. 21 (2011), 1–15
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

The Tits alternative for short generalized tetrahedron groups

B. Fine [a] A. Hulpke [b] V. große Rebel [c] G. Rosenberger [d] S. Schauerte [c]

Abstract. A generalized tetrahedron group is defined to be a group admitting the following presentation: $\langle x, y, z \mid x^l = y^m = z^n = W_1^p(x,y) = W_2^q(y,z) = W_3^r(x,z) = 1\rangle$, $2 \leq l, m, n, p, q, r$, where each $W_i(a,b)$ is a cyclically reduced word involving both a and b. These groups appear in many contexts, not least as fundamental groups of certain hyperbolic orbifolds or as subgroups of generalized triangle groups. In this paper, we build on previous work to show that the Tits alternative holds for short generalized tetrahedron groups, that is, if G is a short generalized tetrahedron group then G contains a non-abelian free subgroup or is solvable-by-finite. The term *Tits alternative* comes from the respective property for finitely generated linear groups over a field (see [**Ti**]).

1. Introduction

If T is a tetrahedron in 3-dimensional Euclidean, hyperbolic or spherical space whose dihedral angles are submultiples of π, then the reflections in the faces of T generate a discrete group of isometries. The index 2 subgroup of orientation-preserving isometries in this group is generated by reflections around the edges of any of the faces of T, and has a presentation of the form

$$\langle x, y, z \mid x^l = y^m = z^n = (xy^{-1})^p = (yz^{-1})^q = (zx^{-1})^r = 1\rangle, \tag{1.1}$$

where $2 \leq l, m, n, p, q, r$. These groups are called *ordinary tetrahedron groups*. Coxeter has shown in [**Co**] that an ordinary tetrahedron group of the form 1.1 is finite if and only if the Coxeter matrix

$$C = \begin{pmatrix} 1 & -\cos(\frac{\pi}{l}) & -\cos(\frac{\pi}{m}) & -\cos(\frac{\pi}{n}) \\ -\cos(\frac{\pi}{l}) & 1 & -\cos(\frac{\pi}{p}) & -\cos(\frac{\pi}{r}) \\ -\cos(\frac{\pi}{m}) & -\cos(\frac{\pi}{p}) & 1 & -\cos(\frac{\pi}{q}) \\ -\cos(\frac{\pi}{n}) & -\cos(\frac{\pi}{r}) & -\cos(\frac{\pi}{q}) & 1 \end{pmatrix} \tag{1.2}$$

2000 *Mathematics Subject Classification.* Primary 20F05; Secondary 20E05, 20E07.
Key words and phrases. generalized tetrahedron groups, generalized triangle groups, the Tits alternative, triangle of groups, Fortsetzungssatz.

has positive determinant. Following Vinberg we call a group G defined by a presentation

$$G = \langle x, y, z \mid x^l = y^m = z^n = W_1^p(x,y) = W_2^q(y,z) = W_3^r(x,z) = 1\rangle, \tag{1.3}$$

$2 \le l, m, n, p, q, r$, where each $W_i(a,b)$ is a cyclically reduced word involving both a and b, a *generalized tetrahedron group*. For the following we may always assume that each $W_i(a,b)$ also is not a proper power in the free product on a and b (that only would increase the exponents). A generalized tetrahedron group G is called a Tsaranov generalized tetrahedron group if, in addition, $W_1(x,y) = x^\alpha y^\beta$, $W_2(y,z) = y^\gamma z^\delta$, $W_3(x,z) = x^\varepsilon z^\zeta$, with $1 \le \alpha, \varepsilon < l$, $1 \le \beta, \gamma < m$, $1 \le \delta, \zeta < n$. Here we call a generalized tetrahedron group G a *short* generalized tetrahedron group if all the cyclically reduced words $W_i(a,b)$ have length ≤ 4 in the respective free product on a and b.

Certain operations on presentations of this form 1.3 do not change the groups defined by the presentations. With this in mind, we say that two presentations P_1 and P_2 of the form 1.3 are equivalent if P_2 can be obtained from P_1 by a sequence of operations of the following type:

1. Replace a generator a of order k by a new generator $d = a^\alpha$, where α is coprime to k, and then amend the relations accordingly.
2. Apply a permutation to the generators x, y and z.
3. If $V_i(a,b)$ is a cyclically reduced conjugate of $W_i(a,b)$ in the free product on a and b, then replace the relator $W_i^{k_i}(a,b)$ by $V_i^{k_i}(a,b)$, where $k_i \in \{p,q,r\}$, respectively.
4. Replace the relator $W_i^{k_i}(a,b)$ by $V_i^{k_i}(a,b)$, where $V_i(a,b)$ is the inverse of $W_i(a,b)$ and $k_i \in \{p,q,r\}$, respectively.
5. If a is a generator of order 2, if b is a generator of order k, if α and β are coprime to k, and if we have a relator of the form $W = (ab^\alpha)^2$, then replace W by $(ab^\beta)^2$.

It is clear that, if P_1 and P_2 are equivalent, then P_1 and P_2 define the same group. In the following we often replace a presentation of the form 1.3 by an equivalent one and work in fact up to equivalence. The purpose of this paper is to prove the following

THEOREM 1.1. *Let G be a short generalized tetrahedron group. Then G satisfies the Tits alternative, that is, G contains a non-abelian free subgroup or is solvable-by-finite.*

2. Preliminary results

For the benefit of the reader, we will list here some preliminary definitions and results we will need in this paper. Suppose that G is defined by the presentation $G = \langle x, y, z \mid x^l = y^m = z^n = W_1^p(x,y) = W_2^q(y,z) = W_3^r(x,z) = 1\rangle$ of the form 1.3. Let (after conjugation if necessary) $W_1(x,y) = x^{\alpha_1} y^{\beta_1} \dots x^{\alpha_{k_1}} y^{\beta_{k_1}}$ with $k_1 \ge 1$, $1 \le \alpha_i < l$, $1 \le \beta_i < m$ for each i. We similary take k_2 and k_3 to be half the length of the cyclically reduced words $W_2(y,z)$ and $W_3(x,z)$ respectively. G can be realized as a triangle of groups, that is, as the colimit of the diagram of groups and injective homomorphisms shown in the figure in which

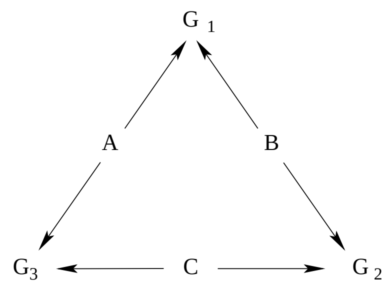

FIGURE 1

$G_1 = \langle x, y \mid x^l = y^m = W_1^p(x, y) = 1\rangle,$
$G_2 = \langle y, z \mid y^m = z^n = W_2^q(y, z) = 1\rangle,$
$G_3 = \langle x, z \mid x^l = z^n = W_3^r(x, z) = 1\rangle,$
$A = \langle x \mid x^l = 1\rangle$, $B = \langle y \mid y^m = 1\rangle$ and $C = \langle z \mid z^n = 1\rangle$. We refer G_1, G_2 and G_3 as vertex groups and A, B, C as edge groups. Groups with a presentation as G_1, G_2 and G_3 are called *generalized triangle groups*. Using the improved Spelling Theorem for generalized triangle groups Howie and Kopteva [**HK**] were able to show the following.

THEOREM 2.1. *Let G be a generalized tetrahedron group of the form 1.3 as above.*
a) *If $1/_{pk_1} + 1/_{qk_2} + 1/_{rk_3} < 1$ then G contains a non-abelian free subgroup.*
b) *If $1/_{pk_1} + 1/_{qk_2} + 1/_{rk_3} = 1$ then G contains a non-abelian free subgroup except in the case of (up to equivalence)*

$$\langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^p = (yz)^q = (zx)^r = 1\rangle$$

with $1/_p + 1/_q + 1/_r = 1$, where G is abelian-by-finite.

Hence, the Tits alternative holds, if the triangle of groups for G is negatively curved or Euclidean. Therefore we are left with the spherical cases (up to equivalence)
$S1 : \langle x, y, z \mid x^l = y^m = z^n = W_1^p(x, y) = (y^\gamma z^\delta)^2 = (x^\varepsilon z^\zeta)^2 = 1\rangle,\ p \geq 2.$
$S2 : \langle x, y, z \mid x^l = y^m = z^n = (x^\alpha y^\beta)^p = (y^\gamma z^\delta)^q = (x^\varepsilon z^\zeta)^r = 1\rangle,\ p \geq 3,$
and $q \geq 3,\ r = 2$ with $1/_p + 1/_q > 1/_2$ or
$q = 2,\ r \geq 3$ with $1/_p + 1/_r > 1/_2$.
$S3 : \langle x, y, z \mid x^l = y^m = z^n = (x^{\alpha_1} y^{\beta_1} x^{\alpha_2} y^{\beta_2})^2 = (y^\gamma z^\delta)^q = (x^\varepsilon z^\zeta)^r = 1\rangle,$
$(q, r) = (2, 3)$ or $(q, r) = (3, 2)$.

We call a generalized tetrahedron group a short spherical generalized tetrahedron group if it is short and spherical.

We also need several preliminary results about linear representations. Let G be a generalized tetrahedron group given by a presentation of the form 1.3. If L is a linear group and $G \to L$ is a representation of G in L, we say that ρ is *essential* if the elements $\rho(x)$, $\rho(y)$, $\rho(z)$, $\rho(W_1(x, y))$, $\rho(W_2(y, z))$ and $\rho(W_3(x, z))$ have orders l, m, n, p, q, r respectively in $\rho(G)$. In this case, we also have essential representations of the three generalized triangle groups G_1, G_2 and G_3, defined above as the vertex groups of the triangle of groups for G. In the following we use the notation G_1, G_2 and G_3 for these vertex groups for G.

THEOREM 2.2. [**FLRR**] *Every generalized tetrahedron group admits an essential representation in* $\mathrm{PSL}(2,\mathbb{C})$.

The next Theorem (see [**FR**] and [**ERST**]) is very useful to prove Theorem 1.1.

THEOREM 2.3 (Fortsetzungssatz). *Let G be the generalized tetrahedron group defined by the presentation $G = \langle x, y, z \mid x^l = y^m = z^n = W_1^p(x,y) = W_2^q(y,z) = W_3^r(x,z) = 1\rangle$ and let G_1 be the generalized triangle group defined by the presentation $G_1 = \langle x, y \mid x^l = y^m = W_1^p(x,y) = 1\rangle$. Suppose that ρ_1 is an essential representation of G_1 into $\mathrm{PSL}(2,\mathbb{C})$ with $X = \rho_1(x)$ and $Y = \rho_1(y)$ and that one of the following two possibilities occurs:*

(1) $\mathrm{tr}([X,Y]) \neq 2$*;*
(2) $(n,q,r) \neq (2,2,2)$ *and $\langle X, Y\rangle$ is an infinite metabelian subgroup of* $\mathrm{PSL}(2,\mathbb{C})$.

Then there is an essential representation $\rho : G \to \mathrm{PSL}(2,\mathbb{C})$ such that $X = \rho(x)$ and $Y = \rho(y)$.
Moreover, if in case (1) *the group $\langle X, Y\rangle$ is non-elementary, then G contains a non-abelian free subgroup in both cases.*

REMARK 2.4. A subgroup of $\mathrm{PSL}(2,\mathbb{C})$ is said to be non-elementary if it is not solvable-by-finite; such a subgroup must contain a free subgroup of rank 2.

If G_1 is as in Theorem 2.3 and $\langle X, Y\rangle$ is an abelian group other than the elementary abelian group of order 4, then there always exists an essential representation $\sigma : G_1 \to \mathrm{PSL}(2,\mathbb{C})$ such that $\langle \sigma(x), \sigma(y)\rangle$ is an infinite metabelian group.

From this we get the following extensions for which we have to look at certain special cases for G_1. A proof is given in [**FR**] together with a little correction in [**EHRT**].

THEOREM 2.5. *Let G be the generalized tetrahedron group defined by the presentation $G = \langle x, y, z \mid x^l = y^m = z^n = W_1^p(x,y) = W_2^q(y,z) = W_3^r(x,z) = 1\rangle$ and suppose $l \le m$ and that $W_1(x,y) = x^{\alpha_1}y^{\beta_1}\dots x^{\alpha_{k_1}}y^{\beta_{k_1}}$, $1 \le \alpha_i < l$, $1 \le \beta_i < m$ for all i, where $k_1 \ge 2$ and $W_1(x,y)$ is not a proper power in the free product on x and y. Assume further that one of the following holds*

1) $m \ge 4$ *and* $p \ge 3$*;*
2) $p \ge 4$*;*
3) $l \ge 3$ *and* $p \ge 3$.

Then G contains a non-abelian free subgroup.

THEOREM 2.6. *Let G be the generalized tetrahedron group defined by the presentation $G = \langle x, y, z \mid x^l = y^m = z^n = W_1^p(x,y) = W_2^q(y,z) = W_3^r(x,z) = 1\rangle$ and suppose $l \le m$ and that $W_1(x,y) = x^{\alpha_1}y^{\beta_1}\dots x^{\alpha_{k_1}}y^{\beta_{k_1}}$, $k_1 \ge 1$ and $1 \le \alpha_i < l$, $1 \le \beta_i < m$ for all i. Suppose that $1/_l + 1/_m + 1/_p < 1$ and $(n,q,r) \neq (2,2,2)$. Then G contains a non-abelian free subgroup.*

Theorems 2.3, 2.5 and 2.6 hold in a symmetric manner for the vertex groups G_2 and G_3 also.

Using the above preliminary results we were able to prove the following.

THEOREM 2.7. [**FHgRR**] *Let G be a generalized tetrahedron group given by a presentation of the form 1.3. If $(p,q,r) \neq (2,2,2)$ then G satisfies the Tits alternative, that is, G contains a non-abelian free subgroup or is solvable-by-finite.*

From Theorem 2.7 and the result of Howie and Kopteva now the Tits alternative holds unless G has a presentation (up to equivalence)

$$G = \langle x, y, z \mid x^l = y^m = z^n = W_1^2(x,y) = (y^\gamma z^\delta)^2 = (x^\varepsilon z^\zeta)^2 = 1 \rangle, \quad (2.1)$$

with $W_1(x,y) = x^{\alpha_1} y^{\beta_1} \cdots x^{\alpha_k} y^{\beta_k}$, $k \geq 1$, $l,m,n \geq 2$; $1 \leq \alpha_1, \ldots, \alpha_k, \varepsilon < l$, $1 \leq \beta_1, \ldots, \beta_k, \gamma < m$ and $1 \leq \delta, \zeta < n$. We may assume $2 \leq l \leq m$.

For this situation we have a preliminary result.

THEOREM 2.8. [**FR**] *Let G be a generalized tetrahedron group given by a presentation of the form 1.3 with $(p,q,r) = (2,2,2)$. Suppose $l \leq m$ and $1/l + 1/m < 1/2$. Then G has a free subgroup of rank two with the possible exceptions $n = 2$ and $(l,m) = (3,8)$, $(3,10)$, $(4,5)$, $(4,6)$, $(4,8)$ or $(5,6)$.*

Theorem 2.8 holds in a symmetric manner for the vertrex groups G_2 and G_3 also.

We now would like to prove that the Tits alternative holds in general for $(p,q,r) = (2,2,2)$. Unfortunately, all the methods used in [**FHgRR**] do not work analogously in this case. The situation is much more difficult. Hence, to get an impression, we consider in the following short spherical generalized tetrahedron groups.

Let G be a short spherical generalized tetrahedron group. If G is equivalent to a Tsaranov generalized tetrahedron group then G satisfies the Tits alternative.

THEOREM 2.9. [**HgRR**] *Let G be a Tsaranov generalized tetrahedron group. Then G satisfies the Tits alternative, that is, G contains a non-abelian free subgroup or is solvable-by-finite.*

In the following we consider short spherical generalized tetrahedron groups which are not equivalent to Tsaranov generalized tetrahedron groups.

THEOREM 2.10.
Let $A, B, C, D \in \mathrm{SL}(2,\mathbb{C})$ with $A \cdot B = C$ and D arbitrary. Let

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \; B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, \; C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}, \; D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix},$$

$$\vec{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}, \; \vec{r} = \begin{pmatrix} \mathrm{tr}(D) \\ \mathrm{tr}(AD) \\ \mathrm{tr}(BD) \\ \mathrm{tr}(CD) \end{pmatrix} \text{ and } M = \begin{pmatrix} 1 & 0 & 0 & 1 \\ a_1 & a_3 & a_2 & a_4 \\ b_1 & b_3 & b_2 & b_4 \\ c_1 & c_3 & c_2 & c_4 \end{pmatrix}.$$

Then $M \cdot \vec{d} = \vec{r}$ and $\det(M) = \mathrm{tr}([A,B]) - 2$. Moreover, if $\det(M) \neq 0$, that is $\mathrm{tr}([A,B]) \neq 2$, then $\det D = d_1 d_4 - d_2 d_3 = 1$ defines a quadratic polynomial $f(t) \in K[t]$, $K = \mathbb{Q}(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, \mathrm{tr}(D), \mathrm{tr}(AD), \mathrm{tr}(BD))$, with highest coefficient $\alpha_2 = -1/\det M$ and zeros $\mathrm{tr}(ABD)$ and $\mathrm{tr}(BAD)$.

This Theorem may be used in the following manner. Let G be a generalized tetrahedron group with vertex groups G_1, G_2 and G_3 as above. Let $\rho_1 : G_1 \to \mathrm{PSL}(2,\mathbb{C})$, $\rho_1(x) = X$, $\rho_2(y) = Y$, be an essential representation of G_1 into $\mathrm{PSL}(2,\mathbb{C})$. Let $\langle X, Y\rangle$ be non-cyclic and finite, that is, $\langle X, Y\rangle$ is isomorphic to a dihedral group D_{2n}, $n \geq 2$, A_4, S_4 or A_5. We remark that $\mathrm{tr}([X,Y]) \neq 2$. Then we use Theorem 2.10 to construct Z to get an essential representation $\rho : G \to \mathrm{PSL}(2,\mathbb{C})$ with $\rho(x) = X$, $\rho(y) = Y$ and $\rho(z) = Z$. If XYZ has infinite order then $\langle X, Y, Z\rangle$ is non-elementary, and hence, G has a non-abelian free subgroup.

3. Proof of the Main Theorem 1.1

In the following let G be a short spherical generalized tetrahedron group given by a presentation

$$G = \langle x, y, z \mid x^l = y^m = z^n = (x^{\alpha_1} y^{\beta_1} x^{\alpha_2} y^{\beta_2})^p = (y^\gamma z^\delta)^q = (x^\varepsilon z^\zeta)^r = 1\rangle, \tag{3.1}$$

with $2 \leq l, m, n, p, q, r$ and $1 \leq \alpha_1, \alpha_2, \varepsilon < l$, $1 \leq \beta_1, \beta_2, \gamma < m$ and $1 \leq \delta, \zeta < n$. We may assume $l \leq m$. We have $m \geq 3$ because $W_1(x,y) = x^{\alpha_1} y^{\beta_1} x^{\alpha_2} y^{\beta_2}$ is not a proper power.

In what follows we always use the known fact that the only finite subgroups of $\mathrm{PSL}(2,\mathbb{C})$ are, up to isomorphism, the cyclic groups $\mathbb{Z}_n$ for all $n \in \mathbb{N}$, the dihedral groups D_{2n} for all $n \in \mathbb{N}$, $n \geq 2$, the alternating groups A_k for $k = 4, 5$ and the symmetric group S_4.

In many cases a generalized tetrahedron group G has in an obvious manner an ordinary tetrahedron group as a homomorphic image. If this image has a free subgroup of rank 2, then G also has one. For ordinary tetrahedron groups we have a criterion which only depends on the orders of the relators, that is, on the Coxeter Matrix C.

Theorem 3.1. [**FHgRR**] *Let G be an ordinary tetrahedron group, given by the presentation $G = \langle x, y, z \mid x^l = y^m = z^n = (xy^{-1})^p = (yz^{-1})^q = (zx^{-1})^r = 1\rangle$, where $2 \leq l, m, n, p, q, r$. Let C be the Coxeter matrix 1.2.*

(i) *If $\det C < 0$ then G has a non-abelian free subgroup.*
(ii) *If $\det C = 0$ then G has a non-abelian free subgroup or is abelian-by-finite. In the latter case, G is isomorphic to the Euclidean group $\langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^{f_1} = (yz)^{f_2} = (xz)^{f_3} = 1\rangle$, where $f_1, f_2, f_3 \geq 2$ and $1/f_1 + 1/f_2 + 1/f_3 = 1$.*

Remark 3.2. We already know that G is finite if and only if $\det C > 0$.

Also, quite often we get in an obvious manner as homomorphic images groups of certain SN-type, that is, groups with a presentation $H = \langle a, b, c \mid a^{e_1} = b^{e_2} = c^{e_3} = R_1^{f_1}(a,b) = R_2^{f_2}(a,c) = 1\rangle$, $2 \leq e_1, e_2, e_3, f_1, f_2$, where $R_i(x,y)$ is a cyclically reduced word in x, y involving both x and y for $i = 1, 2$ (see [**FR**]).

Theorem 3.3. *Let H be as above. If at least one of e_2, e_3, f_1, f_2 is greater than 2 then H has a free subgroup of rank 2.*

From now on let G be a short spherical generalized tetrahedron group given by a presentation of the form 3.1. Let, without loss of generality, $l \leq m$. By Theorem 2.7

we know that G satisfies the Tits alternative if $(p,q,r) \neq (2,2,2)$. Hence, from now on let $(p,q,r) = (2,2,2)$.

Theorem 3.4. *Let G be a short spherical generalized tetrahedron group given by a presentation of the form 3.1 with $(p,q,r) = (2,2,2)$. If $1/_l + 1/_m < 1/_2$ or $1/_m + 1/_n < 1/_2$ or $1/_l + 1/_n < 1/_2$ then G has a free subgroup of rank 2.*

Proof. We may assume that $2 \leq l \leq m$. Then $m \geq 3$ because $x^{\alpha_1}y^{\beta_1}x^{\alpha_2}y^{\beta_2}$ is not a proper power.

<u>Case 1:</u> $1/_l + 1/_m < 1/_2$
If $n \geq 3$ then G has a non-abelian free subgroup by Theorem 2.8. Now, let $n = 2$. Also, by Theorem 2.8 we only have to consider the cases

$$(l,m) = (3,8), (3,10), (4,5), (4,6), (4,8) \text{ and } (5,6).$$

If G_1 has a non-elementary image in $\mathrm{PSL}(2,\mathbb{C})$ then G has a non-abelian free subgroup. Now assume that G_1 has no non-elementary image in $\mathrm{PSL}(2,\mathbb{C})$. Then, if $\rho_1 : G_1 \to \mathrm{PSL}(2,\mathbb{C})$ is an essential representation, $\rho_1(G_1)$ is cyclic or infinite metabelian. We may assume that $\rho_1(G_1)$ is cyclic (if $\rho(G_1)$ is infinite metabelian then there exists an essential representation $\rho_1' : G_1 \to \mathrm{PSL}(2,\mathbb{C})$ with $\rho_1'(G_1)$ cyclic). We write $W_1(x,y) = x^{\alpha_1}y^{\beta_1}x^{\alpha_2}y^{\beta_2}$.

(1) $(l,m) = (3,8)$
Then necessarily $\alpha_1 + \alpha_2 \equiv 0 (\mathrm{mod}\ 3)$ and $\beta_1 + \beta_2 \equiv 4 (\mathrm{mod}\ 8)$. We introduce the relation $y^4 = 1$ and get the factor group
$\bar{G} = \langle x, y, z \mid x^3 = y^4 = z^2 = W_1^2(x,y) = (y^\gamma z)^2 = (x^\alpha z)^2 = 1\rangle$ where β_1 and β_2 are reduced mod 4 (if $\gamma = 4$ then we have just $y^4 z = z$ and $z^2 = 1$ in $\bar{G}$). Now $\bar{G_1} = \langle x, y \mid x^3 = y^4 = W_1^2(x,y) = 1\rangle$ has a non-elementary image in $\mathrm{PSL}(2,\mathbb{C})$, see [**R**]. Hence $\bar{G}$, and therefore G, has a free subgroup of rank 2. The cases $(l,m) = (3,10), (4,5)$ and $(5,6)$ are analogous.

(2) $(l,m) = (4,6)$
G has a factor group
$\bar{G} = \langle x, y, z \mid x^4 = y^6 = z^2 = W_1^2(x,y) = (yz)^2 = (xz)^2 = 1\rangle$ which we now consider (if $\bar{G}$ has a non-abelian free subgroup then also G). Then necessarily one of the following three cases holds:
 (a) $\alpha_1 + \alpha_2 \equiv 0 (\mathrm{mod}\ 4)$ and $\beta_1 + \beta_2 \equiv 3 (\ \mathrm{mod}\ 6)$,
 (b) $\alpha_1 + \alpha_2 \equiv 2 (\mathrm{mod}\ 4)$ and $\beta_1 + \beta_2 \equiv 0 (\ \mathrm{mod}\ 6)$,
 (c) $\alpha_1 + \alpha_2 \equiv 2 (\mathrm{mod}\ 4)$ and $\beta_1 + \beta_2 \equiv 3 (\ \mathrm{mod}\ 6)$.

If (a) holds then we introduce the relation $y^3 = 1$ and get a factor group of $\bar{G}$ which has a non-elementary image in $\mathrm{PSL}(2,\mathbb{C})$, and hence, $\bar{G}$ has a free group of rank 2.

If (b) holds then we introduce the relation $x^2 = 1$ and get the factor group
$\bar{\bar{G}} = \langle x, y, z \mid x^2 = y^6 = z^2 = (xy^{\beta_1}xy^{\beta_2})^2 = (yz)^2 = (xz)^2 = 1\rangle$ with $\beta_1 + \beta_2 \equiv 0 (\ \mathrm{mod}\ 6)$, we have $\beta_1 \neq \beta_2$ and may assume that $\beta_1 \leq \beta_2$, $\beta_1 | 6$. Then $(\beta_1, \beta_2) = (1,5)$ or $(2,4)$. In both cases $\bar{\bar{G}}$ has a non-elementary image in $\mathrm{PSL}(2,\mathbb{C})$, and hence, $\bar{G}$ has a free group of rank 2.

Now, if (c) holds then (up to equivalence) we may assume that $W_1(x,y) = xyxy^2$. Let H be the subgroup of $\bar{G}$ generated by y and y. H has index 2 in $\bar{G}$ and a presentation $H = \langle x, y, \mid x^4 = y^6 = (xyxy^2)^2 = 1\rangle$. H has a free subgroup of rank 2 because $1/_4 + 1/_6 < 1/_2$ (see [**BMS**]). The case $(l,m) = (4,8)$ is analogous.

Case 2: $1/_m + 1/_n < 1/_2$
Again, if $l \geq 3$ then G has a free subgroup of rank 2 by Theorem 2.8, and if $l = 2$ the we have to consider the cases

$$(m,n) = (3,8), (3,10), (4,5), (4,6), (4,8),$$

$$(5,6), (8,3), (10,3), (5,4), (6,4), (8,4) \text{ and } (6,5).$$

(1) $(m,n) = (3,8)$
Up to equivalence, we may assume that
$G = \langle x,y,z \mid x^2 = y^3 = z^8 = W_1^2(x,y) = (yz^\delta)^2 = (xz)^2 = 1\rangle$ with $1 \leq \delta \leq 4$, $\delta|8$. $G_2 = \langle y,z \mid y^3 = z^8 = (yz^\delta)^2 = 1\rangle$ has a non-elementary image in $\mathrm{PSL}(2,\mathbb{C})$. Hence, G has a free subgroup of rank 2. The cases $(m,n) = (3,10), (4,5), (5,6), (8,3), (10,3), (5,4)$ and $(6,5)$ are analogous.

(2) $(m,n) = (4,6)$
Up to equivalence, we may assume that
$G = \langle x,y,z \mid x^2 = y^4 = z^6 = W_1^2(x,y) = (y^\gamma z^\delta)^2 = (xz^\zeta)^2 = 1\rangle$ with $1 \leq \gamma \leq 2$, $\gamma|4$, $1 \leq \delta \leq 3$, $\delta|6$ and $1 \leq \zeta \leq 5$. $G_2 = \langle y,z \mid y^4 = z^6 = (y^\gamma z^\delta)^2 = 1\rangle$ has a non-elementary image in $\mathrm{PSL}(2,\mathbb{C})$ if $\delta \neq 3$ and hence, G has a free subgroup of rank 2 if $\delta \neq 3$. This argument also holds if $\delta = 3$ and $\gamma = 1$. If $\delta = 3$ and $\gamma = 2$ then we introduce the relation $y^2 = 1$ and get a factor group which contains a free subgroup of rank 2 by Theorem 3.3. The cases $(m,n) = (4,8), (6,4)$ and $(8,4)$ are analogous.

Case 3: $1/_l + 1/_n < 1/_2$
Then $m \geq 3$, and G has a free subgroup of rank 2 by Theorem 2.8. □

THEOREM 3.5. *Let G be a short spherical generalized tetrahedron group given by a presentation of the form 3.1 with $(p,q,r) = (2,2,2)$. If $1/_l + 1/_m = 1/_2$ or $1/_m + 1/_n = 1/_2$ or $1/_l + 1/_n = 1/_2$ then G has a free subgroup of rank 2.*

PROOF. We may assume that $l \leq m$ and $m \geq 3$.

Let first $l = 2$. Then $1/_m + 1/_n = 1/_2$ and hence, $(m,n) = (3,6), (6,3)$ or $(4,4)$. Here the results and methods described in Sections 2 and 3, including the reduction to factor groups, work very well to get a free subgroup of rank 2, except in the following case:
$G = \langle x,y,z \mid x^2 = y^6 = z^3 = (xyxy^4)^2 = (xz)^2 = (yz)^2 = 1\rangle$. Let H be the subgroup generated by $a = y$, $b = xyx$ and $c = z$. H has index 2 in G and a presentation $H = \langle a,b,c \mid a^6 = b^6 = c^3 = (ba^4)^2 = (ab^4)^2 = (ac)^2 = (bc^{-1})^2 = 1\rangle$. Now $ba^4 = b(a^{-1})^{-4} = b(a^{-1})^2$ and $(ab^4)^{-1} = b^{-4}a^{-1} = b^2a^{-1}$. If we replace a by a^{-1} and c by c^{-1} we get for H a presentation $H = \langle a,b,c \mid a^6 = b^6 = c^3 = (a^2b)^2 = (ab^2)^2 = (ac)^2 = (bc)^2 = 1\rangle$. Let $A, B \in \mathrm{PSL}(2,\mathbb{C})$ with $\mathrm{tr}A = \mathrm{tr}B = \sqrt{3}$ and $\mathrm{tr}(AB) = 1$.

Then $A^6 = B^6 = (A^2B)^2 = (BA^2)^2 = 1$ and $\mathrm{tr}[A,B] = 2$. We may choose A and B so that $< A, B >$ is infinite metabelian. By the Fortsetungssatz 2.3 we now may construct $C \in \mathrm{PSL}(2,\mathbb{C})$ with $C^3 = (AC)^2 = (BC)^2 = 1$ which defines an essential representation $\rho_H : H \to \mathrm{PSL}(2,\mathbb{C})$, $a \mapsto A$, $b \mapsto B$, $c \mapsto C$ such that $\rho_H(H)$ is non-elementary. Hence, G has a free subgroup of rank 2.

Now, let $l \geq 3$. If $m \geq 6$ or $n \geq 6$ the G has a free subgroup of rank 2 by Theorem 2.3 and Theorem 3.4. Now, let $m < 6$ and $n < 6$. If Then $1/_l + 1/_m < 1/_2$ or $1/_l + 1/_n < 1/_2$ or $1/_m + 1/_n < 1/_2$ then G has a free subgroup of rank 2 by Theorem 3.4. Let $1/_l + 1/_m \geq 1/_2$ and $1/_l + 1/_n \geq 1/_2$ and $1/_m + 1/_n \geq 1/_2$.

Case 1: $1/_l + 1/_m = 1/_2$
Then $l = m = 4$ because $l \leq m < 6$. If $n \geq 5$ then G has a free subgroup of rank 2 by Theorem 3.4. Now, let $2 \leq n \leq 4$.

(1) $n = 2$
Then G has the factor group
$\bar{G} = \langle x, y, z \mid x^4 = y^4 = z^2 = W_1^2(x,y) = (yz)^2 = (xz)^2 = 1\rangle$ with $W_1(x,y) = x^{\alpha_1}y^{\beta_1}x^{\alpha_2}y^{\beta_2}$ because $(x^\alpha z)^2 = 1$ and $(y^\gamma z)^2 = 1$ are consequences of $z^2 = (xz)^2 = 1$ and $z^2 = (yz)^2 = 1$, respectively. Let H be the subgroup of $\bar{G}$ generated by x and y. H has index 2 in $\bar{G}$ and a presentation $H = \langle x, y \mid x^4 = y^4 = W_1^2(x,y) = W^2(x^{-1}, y^{-1}) = 1\rangle$. If $\alpha_1 = \alpha_2 = 2$ then we may write H as a free product $H = H_1 *_A H_2$ with amalgamation, where $H_1 = \langle x \mid x^4 = 1\rangle$, $H_2 = \langle x^2, y \mid (x^2)^2 = y^4 = (x^2y^{\beta_1}x^2y^{\beta_2})^2 = 1\rangle$ and $A = \langle x^2 \mid (x^2)^2 = 1\rangle$. Recall that $(x^2y^{-\beta_1}x^2y^{-\beta_2})^2 = 1$ is a consequence of $(x^2)^2 = (x^2y^{\beta_1}x^2y^{\beta_2})^2 = 1$. H, and hence G, has a free subgroup of rank 2. Analogously we may handle the case $\beta_1 = \beta_2 = 2$. Now, let $\gcd(\alpha_1, \alpha_2) = \gcd(\beta_1, \beta_2) = 1$. We may assume that $\alpha_1 = \beta_1 = 1$. If $(\alpha_2, \beta_2) = (1, 3)$ then $H = \langle x, y \mid x^4 = y^4 = (xyxy^3)^2 = 1\rangle$, and H, and hence G, has a free subgroup of rank 2, see [**R**]. Analogously we may handle the case $(\alpha_2, \beta_2) = (3, 1)$. In all the other possibilities for (α_2, β_2) we get a non-elementary image for H in $\mathrm{PSL}(2,\mathbb{C})$, and hence G has a free subgroup of rank 2.

(2) $n = 3$
We remark that a group $K = \langle a, b \mid a^3 = b^4 = (ab^2)^2 = 1\rangle$ has a non-elementary image in $\mathrm{PSL}(2,\mathbb{C})$. Therefore, the results and methods described in Section 2 and 3 work (up to equivalence) exept in the case $G = \langle x, y, z \mid x^4 = y^4 = z^3 = (xyxy^3)^2 = (yz)^2 = (xz)^2 = 1\rangle$. We introduce the relation $(xy)^2 = 1$ and get the factor group
$\bar{G} = \langle x, y, z \mid x^4 = y^4 = z^3 = (xy)^2 = (yz)^2 = (xz)^2 = 1\rangle$ which has a free subgroup of rank 2 [**HgRR**].

(3) $n = 4$
Then
$G = \langle x, y, z \mid x^4 = y^4 = z^4 = (x^{\alpha_1}y^{\beta_1}x^{\alpha_2}y^{\beta_2})^2 = (y^\gamma z^\delta)^2 = (x^\varepsilon z^\zeta)^2 = 1\rangle$ with $1 \leq \alpha_1, \alpha_2, \varepsilon \leq 3$, $1 \leq \beta_1, \beta_2, \gamma \leq 3$, $1 \leq \delta, \zeta \leq 3$. We write again $W_1(x,y) = x^{\alpha_1}y^{\beta_1}x^{\alpha_2}y^{\beta_2}$. Let first $\alpha_1 = \alpha_2 = 2$. Then we may assume that

$\beta_1 = 1$ because $W_1(x, y)$ is not a proper power. If $\beta_2 = 2$ then G_1 has a non-elementary image in $\mathrm{PSL}(2, \mathbb{C})$, and hence G has a free subgroup of rank 2. If $\beta_2 = 3$ then we introduce the relation $(x^2y)^2 = 1$ and get the factor group $\bar{G} = \langle x, y, z \mid x^4 = y^4 = z^4 = (x^2y)^2 = (y^\gamma z^\delta)^2 = (x^\varepsilon z^\zeta)^2 = 1\rangle$, and $\bar{G}$ has a free subgroup of rank 2 by [**HgRR**]. This handles the case $\alpha_1 = \alpha_2 = 2$. Analogously we get the result if $\beta_1 = \beta_2 = 2$.

Now, let $\gcd(\alpha_1, \alpha_2) = \gcd(\beta_1, \beta_2) = 1$. We may assume that $\alpha_1 = \beta_1 = 1$ With respect to case 1 (1), without loss of generality, we may assume that $W_1(x, y) = xyxy^3$ (otherwise G has a free subgroup of rank 2). Then we introduce the relation $(xy)^2 = 1$ and get the factor group $\bar{\bar{G}} = \langle x, y, z \mid x^4 = y^4 = z^4 = (xy)^2 = (y^\gamma z^\delta)^2 = (x^\varepsilon z^\zeta)^2 = 1\rangle$ which has a free subgroup of rank 2 by [**HgRR**].

Now, let $1/l + 1/m > 1/2$. Then $l = 3$ because $3 \le l \le m < 6$. Then also $1/l + 1/n > 1/2$ because $l = 3$ and $n < 6$.

Case 2: $1/m + 1/n = 1/2$

Then $m = n = 4$ because $m, n < 6$. Here the results and methods described in Section 2 and 3 work to get a free subgroup of rank 2.

□

THEOREM 3.6. *Let G be a short spherical generalized tetrahedron group given by a presentation of the form 3.1 with $(p, q, r) = (2, 2, 2)$. If $1/l + 1/m > 1/2$, $1/m + 1/n > 1/2$ and $1/l + 1/n > 1/2$ then G is either finite, infinite solvable or has a free subgroup of rank 2.*

PROOF. All finite generalized tetrahedron groups are described (up to equivalence) in [**FHHgRRS**], especially the short spherical ones. Now, let G be infinite. Again, we may assume $l \le m$ and $3 \le m$. Here the results and methods described in Sections 2 and 3, including the reductions to factor groups or, especially, ordinary tetrahedron groups, work very well exept in the following cases (up to equivalence):

(1) $l = n = 2$, $m = 4$ and $G = \langle x, y, z \mid x^2 = y^4 = z^2 = (xyxy^2)^2 = (yz)^2 = (xz)^2 = 1\rangle$. Let H be the subgroup of G generated by x and y. H has index 2 in G and a presentation $H = \langle x, y, \mid x^2 = y^4 = (xyxy^2)^2 = 1\rangle$ which is infinite solvable. Hence, G is infinite solvable.

(2) $l = 2$, $m = n = 4$ and $G = \langle x, y, z \mid x^2 = y^4 = z^4 = (xyxy^3)^2 = (yz)^2 = (xz^2)^2 = 1\rangle$. Let H be the subgroup of G generated by $a = y$, $b = xyx$, $c = z$ and $d = xzx$. H has index 2 in G and a presentation $H = \langle a, b, c, d \mid a^4 = b^4 = c^4 = d^4 = (ab^3)^2 = c^2d^2 = (ac)^2 = (bd)^2 = 1\rangle$. We introduce in H the relations $c^2 = d^2 = 1$ and get the factor group $\bar{H} = \langle a, b, c, d \mid a^4 = b^4 = c^2 = d^2 = (ab)^2 = (ac)^2 = (bd)^2 = 1\rangle$. $\bar{H}$ can be written as a free product $\bar{H} = H_1 *_A H_2$ with amalgamation with $H_1 = \langle a, b, c \mid a^4 = b^4 = c^2 = (ab)^2 = (ac)^2 = 1\rangle$, $H_2 = \langle a, b, d \mid a^4 = b^4 = d^2 = (ab)^2 = (bd)^2 = 1\rangle$ and $A = \langle a, b \mid a^4 = b^4 = (ab)^2 = 1\rangle$. H_1 has a free subgroup of rank 2 [**FR**], and hence also G.

(3) $l = 2$, $m = n = 4$ and $G = \langle x, y, z \mid x^2 = y^4 = z^4 = (xyxy^3)^2 = (yz)^2 = (xz)^2 = 1\rangle$. Let H be the subgroup of G generated by $a = y$, $b = xyx$ and

$c = z$. H has index 2 in G and a presentation $H = \langle a, b, c \mid a^4 = b^4 = c^4 = (ab^3)^2 = (bc^3)^2 = (ac)^2 = 1\rangle$ which has a free subgroup of rank 2 by Theorem 3.5, and hence also G.

(4) $l = 2$, $m = 4$, $n = 3$ and $G = \langle x, y, z \mid x^2 = y^4 = z^3 = (xyxy^3)^2 = (yz)^2 = (xz)^2 = 1\rangle$. Let H be the subgroup of G generated by $a = y$, $b = xyx$ and $c = z$. H has index 2 in G and a presentation $H = \langle a, b, c \mid a^4 = b^4 = c^3 = (ab^3)^2 = (bc^2)^2 = (ac)^2 = 1\rangle$ which has a free subgroup of rank 2 by Theorem 3.5, and hence also G.

(5) $l = 2$, $m = 4$, $n = 2$ and $G = \langle x, y, z \mid x^2 = y^4 = z^2 = (xyxy^3)^2 = (y^\gamma z)^2 = (xz)^2 = 1\rangle$, $1 \le \gamma \le 2$. Let H be the subgroup of G generated by $a = y$, $b = xyx$ and $c = z$. H has index 2 in G and a presentation $H = \langle a, b, c \mid a^4 = b^4 = c^2 = (ab^3)^2 = (b^\gamma c)^2 = (a^\gamma c)^2 = 1\rangle$ which has a free subgroup of rank 2 by Theorem 3.5, and hence also G.

(6) $l = 2$, $m = 5$, $n = 3$ and $G = \langle x, y, z \mid x^2 = y^5 = z^3 = (xyxy^2)^2 = (yz)^2 = (xz)^2 = 1\rangle$. The polynomial for $xyxy^2$ is $\lambda(t^2 - 1)$. We choose $X, Y \in \mathrm{PSL}(2, \mathbb{C})$ with $\mathrm{tr}X = 0$, $\mathrm{tr}Y = \lambda = 2\cos\frac{\pi}{5}$ and $\mathrm{tr}(XY) = 1$. Using Theorem 2.10 we construct $Z \in \mathrm{PSL}(2, \mathbb{C})$ with $\mathrm{tr}Z = -1$, $\mathrm{tr}(XZ) = \mathrm{tr}(YZ) = 0$. Then $x \mapsto X$, $y \mapsto Y$, $z \mapsto Z$ defines an essential representation $\rho : G \to \mathrm{PSL}(2, \mathbb{C})$ with non-real $\mathrm{tr}(XYZ)$. Hence, G has a free subgroup of rank 2.

(7) $l = 2$, $m = 6$, $n = 2$ and $G = \langle x, y, z \mid x^2 = y^6 = z^2 = (xyxy^{\beta_2})^2 = (yz)^2 = (xz)^2 = 1\rangle$, $2 \le \beta_2 \le 5$. Let H be the subgroup of G generated by x and y. H has index 2 in G and a presentation $H = \langle x, y \mid x^2 = y^6 = (xyxy^{\beta_2})^2 = 1\rangle$. If $\beta_2 = 3$ or 5 then H has a free subgroup of rank 2 [**R**], and hence also G. If $\beta = 2$ or 4 then H is infinite solvable [**R**] and hence also G.

(8) $l = 2$, $m = 6$, $n = 2$ and $G = \langle x, y, z \mid x^2 = y^6 = z^2 = (xy^{\beta_1}xy^{\beta_2})^2 = (y^3z)^2 = (xz)^2 = 1\rangle$, $(\beta_1, \beta_2) = (1, 2)$, $(1, 4)$ or $(2, 4)$. Let H be the subgroup of G generated by $a = x$, $b = y$ and $c = zyz$. H has index 2 in G and a presentation $H = \langle a, b, c \mid a^2 = b^6 = c^6 = (ab^{\beta_1}ab^{\beta_2})^2 = (ac^{\beta_1}ac^{\beta_2})^2 = b^3c^3 = 1\rangle$. If we introduce the relations $b^3 = c^3 = 1$ in H then we get the factor group $\bar{H} = \langle a, b, c \mid a^2 = b^3 = c^3 = (ab^{\bar{\beta}_1}ab^{\bar{\beta}_2})^2 = (ac^{\bar{\beta}_1}ac^{\bar{\beta}_2})^2 = 1\rangle$, $(\bar{\beta}_1, \bar{\beta}_2) = (1, 2), (1, 1)$ or $(2, 1)$. $\bar{H}$ has a free subgroup of rank 2 by Theorem 3.3, and hence also G.

(9) $l = 2$, $m = 6$, $n = 2$ and $G = \langle x, y, z \mid x^2 = y^6 = z^2 = (xy^2xy^4)^2 = (yz)^2 = (xz)^2 = 1\rangle$. Let H be the subgroup of G generated by x and y. H has index 2 and a presentation $H = \langle x, y \mid x^2 = y^6 = (xy^2xy^4)^2 = 1\rangle$. H has a free subgroup of rank 2 [**R**] and hence also G.

(10) $l = 2$, $m \ge 7$, $n = 2$ and $G = \langle x, y, z \mid x^2 = y^m = z^2 = (xy^{\beta_1}xy^{\beta_2})^2 = (y^\gamma z)^2 = (xz)^2 = 1\rangle$. G has the factor group $\bar{G} = \langle x, y, z \mid x^2 = y^m = z^2 = (xy^{\beta_1}xy^{\beta_2})^2 = (yz)^2 = (xz)^2 = 1\rangle$ because $(y^\gamma z)^2 = 1$ is a consequence of $z^2 = (yz)^2 = 1$. Let H be the subgroup of $\bar{G}$ generated by x and y. H has index 2 in $\bar{G}$ and a presentation $H = \langle x, y, \mid x^2 = y^m = (xy^{\beta_1}xy^{\beta_2})^2 = 1\rangle$. H has a free subgroup of rank 2 [**R**], and hence also G.

(11) $l = m = 3$, $n = 5$ and $G = \langle x, y, z \mid x^3 = y^3 = z^5 = (xyx^2y^2)^2 = (yz^3)^2 = (xz)^2 = 1\rangle$. The subgroup H generated by $a_1 = x$, $a_2 = z$, $a_3 = yxy^{-1}$, $a_4 = yzy^{-1}zy^{-1}$ and $a_5 = y^{-1}xyz^3yxy^{-1}$ has index 6 in G and a presentation

$$\begin{aligned} H = \ & \langle a_1, a_2, a_3, a_4, a_5 \mid a_1^3 = a_2^5 = a_3^3 = a_4^2 = (a_1a_2)^2 = \\ & (a_5a_4a_3a_2^{-1})^2 = (a_3a_2^{-2}a_3a_5^{-1})^2 = (a_5a_3^{-1}a_2^2a_1^{-1})^2 = \\ & a_5a_3^{-1}a_2^2a_5a_3^{-1}a_5^{-1}a_2^{-2}a_3^{-1}a_2^2 = \\ & a_4a_2^{-1}a_5a_3^{-1}a_4a_3^{-1}a_4a_2^{-1}a_5a_3^{-1}a_4a_3^{-1} = \\ & a_5a_2^{-1}a_5a_3^{-1}a_4a_2^{-1}a_5a_2^{-1}a_5a_3^{-1}a_4a_2^{-1} = 1\rangle. \end{aligned}$$

H can be written as a non-trivial free product $H_1 *_A H_2$ with amalgamation with $H_1 = \langle a_1, a_2, a_3, a_5\rangle$, $H_2 = \langle a_2, a_3, a_4, a_5\rangle$ and $A = \langle a_2, a_3, a_5\rangle$. H, and hence G, has a free subgroup of rank two because $\mid H_1 : A \mid \geq 3$ for the index of A in H_1.

(12) $l = 3$, $m = 5$, $n = 3$ and $G = \langle x, y, z \mid x^3 = y^5 = z^3 = (xyx^2y^2)^2 = (y^2z)^2 = (xz)^2 = 1\rangle$. This case can be handled in a similar, computationally more complicated manner as case (11). We found a subgroup of index 21 with 9 generators and 27 relations which can be written as a non-trivial free product with amalgamation and contains a free group of rank two.

(13) $l = m = 3$, $n = 5$ and $G = \langle x, y, z \mid x^3 = y^3 = z^5 = (xyxy^2)^2 = (yz)^2 = (xz)^2 = 1\rangle$. We choose $X, Y \in \mathrm{PSL}(2, \mathbb{C})$ with $\mathrm{tr}X = \mathrm{tr}Y = 1$, $\mathrm{tr}(XY) = \lambda = 2\cos\frac{\pi}{5}$. Now using Theorem 2.10 we construct $Z \in \mathrm{PSL}(2, \mathbb{C})$ with $\mathrm{tr}Z = -\lambda$, $\mathrm{tr}(XZ) = \mathrm{tr}(YZ) = 0$. Then $x \mapsto X$, $y \mapsto Y$, $z \mapsto Z$ defines an essential representation $\rho : G \to \mathrm{PSL}(2, \mathbb{C})$ with non-real $\mathrm{tr}(XYZ)$. Hence G has a free subgroup of rank 2.

(14) $l = m = 3$, $n = 5$ and $G = \langle x, y, z \mid x^3 = y^3 = z^5 = (xyxy^2)^2 = (yz^2)^2 = (xz)^2 = 1\rangle$. We choose $X, Y \in \mathrm{PSL}(2, \mathbb{C})$ with $\mathrm{tr}X = \mathrm{tr}Y = 1$, $\mathrm{tr}(XY) = \lambda = 2\cos\frac{\pi}{5}$. Now using Theorem 2.10 we construct $Z \in \mathrm{PSL}(2, \mathbb{C})$ with $\mathrm{tr}Z = \lambda$, $\mathrm{tr}(XZ) = 0$ and $\mathrm{tr}(YZ) = \lambda - 1$. Then YZ^2 has order 2, and $x \mapsto X$, $y \mapsto Y$, $z \mapsto Z$ defines an essential representation $\rho : G \to \mathrm{PSL}(2, \mathbb{C})$ with non-real $\mathrm{tr}(XYZ)$. Hence G has a free subgroup of rank 2.

(15) $l = 3$, $m = 4$, $n = 2$ and $G = \langle x, y, z \mid x^3 = y^4 = z^2 = W_1^2(x, y) = (yz)^2 = (xz)^2 = 1\rangle$ with $W_1(x, y) = xyx^2y^2$, $xyxy^3$ or $xy^2x^2y^2$. Let H be the subgroup of G generated by x and y. H has index 2 in G and a presentation $H = \langle x, y, z \mid x^3 = y^4 = W_1^2(x, y) = W_1^2(x^{-1}, y^{-1}) = 1\rangle$. If $W_1(x, y) = xyx^2y^2$ or $W_1(x, y) = xy^2x^2y^2$ we introduce the relation $y^2 = 1$ and get the factor group $\mathbb{Z}_2 * \mathbb{Z}_3 \cong \mathrm{PSL}(2, \mathbb{Z})$, a free product of a cyclic group of order 2 and a cyclic group of order 3, and hence G has a free subgroup of rank 2. If $W_1(x, y) = xyxy^3$ then $y(W_1(x^{-1}, y^{-1}))^{-1}y^{-1} = W_1(x, y)$ and $H = G_1 = \langle x, y \mid x^3 = y^4 = (xyxy^3)^2 = 1\rangle$, and hence H, and also G, has a free subgroup of rank 2 [**R**].

(16) $l = 3$, $m = 4$, $n = 3$ and $G = \langle x, y, z \mid x^3 = y^4 = z^3 = (xyxy^3)^2 = (yz)^2 = (xz)^2 = 1\rangle$. Consider the subgroup H of G generated by $a_1 = yx$, $a_2 = zx^{-1}yx^{-1}$, $a_3 = xyx^{-1}yz^{-1}$ and $a_4 = y^{-1}x^{-1}zy^{-1}zx^{-1}$. If we use GAP [**GAP**] then we get that H has index 48 in G and a presentation

$$
\begin{aligned}
H = \ & \langle a_1, a_2, a_3, a_4 \mid a_4^{-1}a_2^2a_4a_2^{-2} = a_4a_1^2a_4^{-1}a_1^{-2} = \\
& a_1^{-1}a_4^{-1}a_1a_3a_2^{-1}a_4a_2a_3^{-1} = a_3^{-1}a_1a_4a_1^{-1}a_3a_2a_4^{-1}a_2^{-1} = \\
& a_2^{-1}a_1^{-1}a_2a_3^{-1}a_1a_2^{-1}a_3a_1a_2a_1^{-1} = a_2^{-1}a_1a_2a_3a_2a_1a_2^{-1}a_1^{-1}a_3^{-1}a_1^{-1} = \\
& a_3a_1a_4a_1^{-1}a_4^{-1}a_3^{-1}a_1a_2^{-1}a_4a_1a_4^{-1}a_1^{-1}a_2a_1^{-1} = \\
& a_4^{-1}a_2^{-1}a_4a_2a_1^{-1}a_2a_3^{-1}a_2^{-1}a_4^{-1}a_2a_4a_3a_2^{-1}a_1 = \\
& a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1}a_2^{-1}a_4a_1^{-1}a_4^{-1}a_2a_1a_3a_4a_1a_4^{-1}a_2a_4 = \\
& a_2^{-1}a_1a_4^{-1}a_2^{-1}a_4a_2a_3^{-1}a_4^{-1}a_2^{-1}a_4a_1^{-1}a_4^{-1}a_2a_4a_1a_4^{-1}a_2a_4a_1^{-1}a_3 = 1\rangle
\end{aligned}
$$

If we introduce the relations $a_1 = a_2$ and $a_4 = 1$ then the factor group is free of rank 2. Hence H, and also G has a free subgroup of rank 2. Especially, the group G is SQ-universal, that is, every countable group can be embedded isomorphically as a subgroup of a quotient of G.

□

Theorems 3.4, 3.5 and 3.6 together with the results in Section 2 and 3 provide a proof of Theorem 1.1.

Additional Remark 3.7. *Unfortunately there is a technical error in one of the arguments in* [**FHHgRRS**] *which probably is a consequence of a typo in the computer calculations. Let* $X, Y \in \mathrm{PSL}(2,\mathbb{C})$ *with* $\mathrm{tr}(X) = \mathrm{tr}(Y) = \mathrm{tr}(XY) = \lambda = 2\cos\frac{\pi}{5}$. *We may construct* $Z \in \mathrm{PSL}(2,\mathbb{C})$ *with* $\mathrm{tr}(Z) = \mathrm{tr}(XZ) = \mathrm{tr}(YZ) = \lambda$. *We claimed that* $\mathrm{tr}(XYZ)$ *is non-real. This is not the case. In fact, we have* $\mathrm{tr}(XYZ) = 2$ *or* λ. *This has consequences only for the two groups*

$G_1 = \langle x, y, z \mid x^3 = y^5 = z^2 = (x^{-1}yxyx^{-1}y^2xy^{-1})^2 = (yz)^2 = (xz)^2 = 1\rangle$

and $G_2 = \langle x, y, z \mid x^3 = y^5 = z^2 = (x^{-1}yx^{-1}y^2xyxy^{-1})^2 = (yz)^2 = (xz)^2 = 1\rangle$.

Here we now give a correct proof that G_1 *and* G_2 *are infinite, as stated in* [**FHHgRRS**]. *For both groups we use GAP* [**GAP**]. *We first consider* G_1. *The subgroup* H *generated by* $y, xy^{-1}x$ *and* $xyx^{-1}yxy^{-1}x^{-1}$ *has index* 60 *in* G_1. *If we abelianize* H, *then we get* $\mathbb{Z}$ *as an epimorphic image of* H *(we remark that* y *is in the derived subgroup of* H*). Hence,* G_1 *is infinite. We now consider* G_2. G_2 *has as an epimorphic image the direct product* $A_5 \times A_5$, *given by* $x \mapsto (2,3,4)(7,10,8)$, $y \mapsto (1,5,2,3,4)(6,10,7,8,9)$ *and* $z \mapsto (1,5)(2,4)(6,9)(8,10)$. *Let* U *be the preimage in* G *of the diagonal subgroup of* $A_5 \times A_5$ *generated by* $(2,3,4)(7,8,10)$ *and* $(1,4)(3,5)(6,7)(8,9)$. *Let* U' *be the derived subgroup of* U. *Then* U/U' *is elementary abelian of order* 8, *and* U' *has abelianization of type* $[0,3,5]$. *Hence,* U' *also has* $\mathbb{Z}$ *as an epimorphic image, and therefore* G_2 *also is infinite.*

This corrects the technical error in [**FHHgRRS**] *and leaves the result there as stated.*

The technical error may have consequences for some of the few groups in [**FHHgRRS**] *where we used Theorem* 2.10 *(Theorem* 2.7 *in* [**FHHgRRS**]*). Nevertheless, in each of the remaining cases, not covered by our paper, using GAP* [**GAP**], *we found analogously as above a subgroup* H *of finite index with infinite abelianization.*

References

[BMS] G. Baumslag, J. Morgan, P. Shalen, Generalized Triangle groups, *Math. Proc. Comb. Phil. Soc.* **102**, (1987), 25-31.

[Co] H.S.M. Coxeter, The complete enumeration of finite groups of the form $R_i^2 = (R_i, R_j)^{k_{ij}}$, *J. London Math. Soc.* **10**, (1935), 21-25.
[EHRT] M. Edjvet, J. Howie, G. Rosenberger and R. Thomas, Finite generalized tetrahedron groups with a high-power relator, *Geometriae Dedicata* **94**, (2002), 111-139.
[ERST] M. Edjvet, G. Rosenberger, M. Stille and R. Thomas, On certain finite generalized tetrahedron groups, *in* M. D. Atkinson, N. D. Gilbert, J. Howie, S. A. Linton and E. F. Robertson (eds.), *Computational and Geometric Aspects of Modern Algebra*, London Math. Soc. Lecture Note Series **275**, Cambridge University Press (2000), 54-65.
[FHgRR] B. Fine, A. Hulpke, V. große Rebel and G. Rosenberger, The Tits Alternative for Spherical Generalized Tetrahedron Groups, *Algebra Colloquium* **15**:4, (2008), 541-554.
[FHHgRRS] B. Fine, M. Hahn, A. Hulpke, V. große Rebel, G. Rosenberger and M. Scheer, All Finite Generalized Tetrahedron Groups, *Algebra Colloquium* **15**:4, (2008), 555-580.
[FLRR] B. Fine, F. Levin, F. Roehl and G. Rosenberger, The generalized tetrahedron groups, *in* R. Charney, M. Davis and M. Shapiro (eds.), *Geometric Group Theory*, Ohio State University, Mathematical Research Institute Publications **3**, de Gruyter (1995), 99-119.
[FR] B. Fine and G. Rosenberger, Algebraic Generalizations Of Discrete Groups, Marcel Dekker Inc., (1999).
[GAP] GAP-Groups, Algorithms, Programming - a System for Computational Discrete Algebra, *University of St. Andrews*, `www.gap-system.org`, Version **4.4.9**, (2006)
[HgRR] M. Hahn, V. große Rebel and G. Rosenberger, The Tits alternative for Tsaranov generalized tetrahedron groups, *Groups-Complexity-Cryptology*, (2009), 207-216.
[HK] J. Howie and N. Kopteva, The Tits alternative for generalised tetrahedron groups, *J. Group Theory* **9**, (2006), 173-189.
[gR] V. große Rebel, Die Tits-Alternative für verallgemeinerte Tetraedergruppen, Dissertation, Universität Dortmund, `http://hdl.handle.net/2003/23165`, (2006).
[R] G. Rosenberger, On free subgroups of generalized triangle groups, *Algebra i Logika* **28 (2)** (1989), 227-240.
[Ti] J. Tits, Free subgroups in linear groups, *J. Algebra* **20**, (1972), 250-270.
[T1] S.V. Tsaranov, On a generalization of Coxeter groups, *Algebras, Groups and Geometries* **6**, (1989), 281-318.
[T2] S.V. Tsaranov, Finite generalized Coxeter groups, *Algebras, groups and Geometries* **6**, (1989), 421-452.

Received 21 12 2010, revised 25 06 2011

[a]Department of Mathematics,
Fairfield University,
Fairfield, Connecticut 06430,
USA
E-mail address: `fine@fair1.fairfield.edu`

[b]Department of Mathematics,
Colorado State University,
Colorado 80523-1874,
USA.
E-mail address: `hulpke@math.colostate.edu`

[c]TU Dortmund,
Fakultät Mathematik, 44221 Dortmund,
Germany.
E-mail address: `volkmar.rebel@tecis.de`
E-mail address: `stefanie-schauerte@gmx.de`

[d]Universität Hamburg,
Bundestrasse 55, 20146 Hamburg,
Germany.
E-mail address: gerhard.rosenberger@math.uni-hamburg.de