SCIENTIA
Series A: *Mathematical Sciences*, Vol. 20 (2010), 81–92
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

Boundedness for Multilinear Commutator on Hardy and Herz-Hardy Space

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ABSTRACT. In this paper, the $(H^p_{\vec{b}}, L^p)$ and $(H\dot{K}^{\alpha,p}_{q,\vec{b}}, \dot{K}^{\alpha,p}_q)$ type boundedness for the multilinear commutator of certain integral operator.

1. Introduction

Let $b \in BMO(R^n)$, and T be a Calderón-Zygmund operator. The commutator $[b, T]$ generated by b and T is defined by

$$[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x).$$

A classical result of Coifman, Rochberg and Weiss (see [4]) proved that the commutator $[b, T]$ is bounded on $L^p(R^n)$ $(1 < p < \infty)$. However, it was observed that the $[b, T]$ is not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)$. But if $H^p(R^n)$ is replaced by a suitable atomic space $H^p_b(R^n)$ and $H^{p,\infty}_b(R^n)$, then $[b, T]$ maps continuously $H^p_b(R^n)$ into $L^p(R^n)$ and $H^{p,\infty}_b(R^n)$ into $L^{p,\infty}(R^n)$ (see [1]). In recent years, the theory of Herz type Hardy spaces has been developed (see [5][12][13]). In this paper, we will introduce some multilinear commutators of certain integral operators and prove the continuity of the multilinear commutators and $BMO(R^n)$ functions on certain Hardy and Herz-Hardy spaces. The integral operators include the Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

2. Definitions and Results

Let us first introduce some definitions (see [1][2][14-16]). In this paper, Q will denote a cube of R^n with sides parallel to the axes, and for a cube Q let $f_Q = |Q|^{-1} \int_Q f(x)dx$ and the sharp function of f is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.$$

2000 *Mathematics Subject Classification.* Primary 42B20, Secondary 42B25.

Key words and phrases. Integral operator; Multilinear commutator; Littlewood-Paley operator; Marcinkiewicz operator; Bochner-Riesz operator; BMO; Hardy space; Herz-Hardy space.

It is well-known that (see [15])

$$f^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - C| dy.$$

We say that b belongs to $BMO(R^n)$ if $b^{\#}$ belongs to $L^{\infty}(R^n)$ and define $||b||_{BMO} = ||b^{\#}||_{L^{\infty}}$. It has been known that (see [15])

$$||b - b_{2^k Q}||_{BMO} \leqslant Ck||b||_{BMO}.$$

Given a positive integer m and $1 \leqslant j \leqslant m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(j)}||_{BMO}$.

Definition 1. Let b_i $(i = 1, \cdots, m)$ be a locally integrable functions and $0 < p \leqslant 1$. A bounded measurable function a on R^n is called a $(p, \vec{b})$ atom, if

(1) *supp* $a \subset B = B(x_0, r)$,

(2) $||a||_{L^\infty} \leqslant |B(x_0, r)|^{-1/p}$,

(3) $\int_B a(y)dy = \int_B a(y) \prod_{l \in \sigma} b_l(y)dy = 0$ for any $\sigma \in C_j^m$, $1 \leqslant j \leqslant m$.

A temperate distribution (see [15][16]) f is said to belong to $H_{\vec{b}}^p(R^n)$, if, in the Schwartz distribution sense, it can be written as

$$f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x),$$

where $a_j' s$ are $(p, \vec{b})$ atoms, $\lambda_j \in C$ and $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$. Moreover, $||f||_{H_{\vec{b}}^p} = \inf(\sum_{j=1}^{\infty} |\lambda_j|^p)^{1/p}$, where the infimum are taken over all the decompositions of f as above.

Definition 2. Let $0 < p, q < \infty$, $\alpha \in R$. For $k \in Z$, set $B_k = \{x \in R^n : |x| \leqslant 2^k\}$ and $C_k = B_k \setminus B_{k-1}$,and$\chi_k = \chi_{C_k}$ for $k \in Z$,where χ_{C_k} is the characteristic function of set C_k. Denote the characteristic function of B_0 by χ_0.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(R^n) = \left\{ f \in L_{loc}^q(R^n \setminus \{0\}) : ||f||_{\dot{K}_q^{\alpha,p}} < \infty \right\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p \right]^{1/p}.$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \left\{ f \in L_{loc}^q(R^n) : ||f||_{K_q^{\alpha,p}} < \infty \right\},$$

where

$$||f||_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p + ||f\chi_0||_{L^q}^p \right]^{1/p}.$$

Definition 3. Let $\alpha \in R^n,\ 1 < q < \infty,\ \alpha \geqslant n(1-1/q),\quad b_i \in BMO(R^n),\ 1 \leqslant i \leqslant m$. A function $a(x)$ is called a central $(\alpha, q, \vec{b})$ -atom (or a central $(\alpha, q, \vec{b})$-atom of restrict type), if

(1) $supp\ a \subset B = B(x_0, r)$(or for some $r \geqslant 1$),

(2) $||a||_{L^q} \leqslant |B(x_0, r)|^{-\alpha/n}$,

(3) $\int_B a(x)x^\beta dx = \int_B a(x)x^\beta \prod_{i\in\sigma} b_i(x)dx = 0$ for any $\sigma \in C_j^m$ $,1 \leqslant j \leqslant m$, $0 \leqslant |\beta| \leqslant \alpha$, where $\beta = (\beta_1, ..., \beta_n)$ is the multi-indices with $\beta_i \in N$ for $1 \leqslant i \leqslant n$ and $|\beta| = \sum_{i=1}^n \beta_i$.

A temperate distribution f is said to belong to $H\dot{K}_{q,\vec{b}}^{\alpha,p}(R^n)$(or $HK_{q,\vec{b}}^{\alpha,p}(R^n)$), if it can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$), in the $S'(R^n)$ sense, where a_j is a central $(\alpha, q, \vec{b})$-atom(or a central $(\alpha, q, \vec{b})$-atom of restrict type) supported on $B(0, 2^j)$ and $\sum_{-\infty}^{\infty} |\lambda_j|^p < \infty$(or $\sum_{j=0}^{\infty} |\lambda_j| < \infty$). Moreover,

$$||f||_{H\dot{K}_{q,\vec{b}}^{\alpha,p}}(\text{ or } ||f||_{HK_{q,\vec{b}}^{\alpha,p}}) = \inf(\sum_j |\lambda_j|^p)^{1/p},$$

where the infimum are taken over all the decompositions of f as above.

Definition 4. Suppose b_j $(j = 1, \cdots, m)$ are the fixed locally integrable functions on R^n. Let $F_t(x, y)$ define on $R^n \times R^n \times [0, +\infty)$. Set

$$F_t(f)(x) = \int_{R^n} F_t(x, y)f(y)dy$$

and

$$F_t^{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(y))F_t(x, y)f(y)dy,$$

for every bounded and compactly supported function f. Let H be the Banach space $H = \{h : ||h|| < \infty\}$ such that, for each fixed $x \in R^n$, $F_t(f)(x)$ and $F_t^{\vec{b}}(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H. The multilinear commutator related to F_t is defined by

$$T_{\vec{b}}(f)(x) = ||F_t^{\vec{b}}(f)(x)||,$$

where F_t satisfies: for fixed $\varepsilon > 0$

$$||F_t(x, y)|| \leqslant C|x - y|^{-n}$$

and

$$||F_t(y, x) - F_t(z, x)|| + ||F_t(x, y) - F_t(x, z)|| \leqslant C|y - z|^\varepsilon |x - z|^{-n-\varepsilon},$$

if $2|y - z| \leqslant |x - z|$. We also define that $T(f)(x) = ||F_t(f)(x)||$.

Note that when $b_1 = \cdots = b_m$, $T_{\vec{b}}$ is just the m order commutator (see [1][14]). It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1-4][6-10][15][17]).

Now we state our theorems as following.

Theorem 1. Let $\varepsilon > 0$, $b_i \in BMO(R^n)$, $1 \leqslant i \leqslant m$, $\vec{b} = (b_1, \cdots, b_m)$, $n/(n+\varepsilon) < p \leqslant 1$. Suppose that T is bounded on $L^q(R^n)$ for any $1 < q < \infty$. Then the multilinear commutator $T_{\vec{b}}$ is bounded from $H_{\vec{b}}^p(R^n)$ to $L^p(R^n)$.

Theorem 2. Let $0 < p < \infty$, $1 < q < \infty$, $n(1-1/q) \leqslant \alpha < n(1-1/q)+\varepsilon$, $\varepsilon > 0$ and $b_i \in BMO(R^n), 1 \leqslant i \leqslant m$, $\vec{b} = (b_1, \cdots, b_m)$. Suppose that T is bounded on $L^q(R^n)$ for any $1 < q < \infty$. Then the multilinear commutator $T_{\vec{b}}$ is bounded from $H\dot{K}^{\alpha,p}_{q,\vec{b}}(R^n)$ to $\dot{K}^{\alpha,p}_q(R^n)$.

3. Proof of Theorems

We begin with the lemma.

Lemma.([15]) Let $1 < r < \infty$, $b_j \in BMO(R^n)$ for $j = 1, \cdots, k$ and $k \in N$. Then

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q| dy \leqslant C \prod_{j=1}^k ||b_j||_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q|^r dy\right)^{1/r} \leqslant C \prod_{j=1}^k ||b_j||_{BMO}.$$

Proof of Theorem 1. It suffices to show that there exist a constant $C > 0$, such that for every $(p, \vec{b})$ atom a,

$$||T_{\vec{b}}(a)||_{L^p} \leqslant C.$$

Let a be a $(p, \vec{b})$ atom supported on a ball $B = B(x_0, r)$. We write

$$\int_{R^n} |T_{\vec{b}}(a)(x)|^p dx = \int_{|x-x_0|\leqslant 2r} |T_{\vec{b}}(a)(x)|^p dx + \int_{|x-x_0|>2r} |T_{\vec{b}}(a)(x)|^p dx = I + II.$$

For I, taking $q > 1$, by Hölder's inequality and the L^q- boundedness of $T_{\vec{b}}$(see [3]), we have

$$\begin{aligned} I &\leqslant \left(\int_{|x-x_0|\leqslant 2r} |T_{\vec{b}}(a)(x)|^q dx\right)^{p/q} \cdot |B(x_0,2r)|^{1-p/q} \\ &\leqslant C||T_{\vec{b}}(a)||^p_{L^q} \cdot |B(x_0,2r)|^{1-p/q} \\ &\leqslant C||\vec{b}||^p_{BMO}||a||^p_{L^q}|B|^{1-p/q} \\ &\leqslant C||\vec{b}||^p_{BMO}. \end{aligned}$$

For II, when $m = 1$, by the Hölder's inequality and the vanishing moment of a, we get, for $x \in (2B)^c$ and $u \in B$,

$$\begin{aligned} |T_{b_1}(a)(x)| &= ||F_t^{b_1}(a)(x)|| \\ &\leqslant \int_B ||F_t(x,y) - F_t(x,u)|||b_1(x) - b_1(y)||a(y)|dy \\ &\leqslant C\int_B \frac{|u-y|^\varepsilon}{|x-u|^{n+\varepsilon}} \int_B (|b(x) - (b_1)_B| + |(b_1)_B - b(y)|)dy||a||_{L^\infty} \\ &\leqslant C\frac{|B|^{\frac{\varepsilon}{n}}}{|x-u|^{n+\varepsilon}}(|b(x) - (b_1)_B| + ||b_1||_{BMO})|B|^{1-1/p}, \end{aligned}$$

so

$$
\begin{aligned}
II &\leqslant C\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}\frac{|B|^{\frac{\varepsilon p}{n}}}{|x-u|^{(n+\varepsilon)p}}(|b(x)-(b_1)_B|+||b_1||_{BOM})^p dx|B|^{p-1}\\
&\leqslant C\sum_{k=1}^{\infty}\frac{|B|^{\frac{\varepsilon p}{n}}}{|2^kr|^{(n+\varepsilon)p}}\int_{2^{k+1}B}(|b(x)-(b_1)_B|^p+||b_1||_{BMO}^p)dx|B|^{p-1}\\
&\leqslant C\sum_{k=1}^{\infty}\frac{|B|^{\frac{\varepsilon p}{n}+p-1}}{(2^kr)^{(n+\varepsilon)p}}\cdot(2^{k+1}r)^n\cdot(|2^{k+1}B|^{-1}\int_{2^{k+1}B}|b(x)-(b_1)_B|^p dx+||b_1||_{BMO}^p)\\
&\leqslant C\sum_{k=1}^{\infty}\frac{|B|^{\frac{\varepsilon p}{n}+p-1}}{(2^kr)^{(n+\varepsilon)p}}\cdot(2^kr)^n\cdot[(|2^{k+1}B|^{-1}\int_{2^{k+1}B}|b(x)-(b_1)_B|dx)^p+||b_1||_{BMO}^p]\\
&\leqslant C\sum_{k=1}^{\infty}\frac{|B|^{\frac{\varepsilon p}{n}+p-1}}{(2^kr)^{(n+\varepsilon)p}}\cdot(2^kr)^n\cdot[(k||b_1||_{BOM})^p+||b_1||_{BMO}^p]\\
&\leqslant C||b_1||_{BMO}^p\sum_{k=1}^{\infty}k^p2^{k[n-(n+\varepsilon)]p}\\
&\leqslant C||b_1||_{BMO}^p.
\end{aligned}
$$

This finishes the proof of the case of $m=1$.

When $m>1$, denoting $\lambda=(\lambda_1,\cdots,\lambda_m)$ with $\lambda_i=(b_i)_B,\ 1\leqslant i\leqslant m$, where $(b_i)_B=|B(x_0,r)|^{-1}\int_{B(x_0,r)}b_i(x)dx$, by Hölder's inequality and the vanishing moment of a, we get

$$
\begin{aligned}
II &= \sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}|T_{\vec b}(a)(x)|^p dx\\
&\leqslant C\sum_{k=1}^{\infty}|2^{k+1}B|^{1-p}\left(\int_{2^{k+1}B\setminus 2^kB}|T_{\vec b}(a)(x)|dx\right)^p\\
&\leqslant C\sum_{k=1}^{\infty}|2^{k+1}B|^{1-p}\\
&\quad\times\left(\int_{2^{k+1}B\setminus 2^kB}\left(\int_B||F_t(x,y)-F_t(x,u)||\prod_{j=1}^m|b_j(x)-b_j(y)||a(y)|dy\right)dx\right)^p,
\end{aligned}
$$

noting that $u\in B$ and $x\in 2^{k+1}B\setminus 2^kB$, then

$$
\begin{aligned}
&\int_B||F_t(x,y)-F_t(x,u)||\prod_{j=1}^m|b_j(x)-b_j(y)||a(y)|dy\\
\leqslant\ & C\int_B\frac{|u-y|^{\varepsilon}}{|x-u|^{n+\varepsilon}}\prod_{j=1}^m|b_j(x)-b_j(y)||a(y)|dy\\
\leqslant\ & C\frac{|B|^{\frac{\varepsilon}{n}}}{|x-u|^{n+\varepsilon}}\int_B\prod_{j=1}^m|b_j(x)-b_j(y)||a(y)|dy.
\end{aligned}
$$

So

$$\begin{aligned}
II &\leqslant C\sum_{k=1}^{\infty}|2^{k+1}B|^{1-p}\\
&\times \left(\int_{2^{k+1}B\setminus 2^kB}|x-u|^{-(n+\varepsilon)}|B|^{\frac{\varepsilon}{n}}(\int_B\prod_{j=1}^{m}|b_j(x)-b_j(y)||a(y)|d\mu(y)\Big)dx\right)^p\\
&\leqslant C\sum_{k=1}^{\infty}|2^{k+1}B|^{1-p}\\
&\times \left(\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}\int_{2^{k+1}B\setminus 2^kB}|x-u|^{-(n+\varepsilon)}|B|^{\frac{\varepsilon}{n}}|(\vec{b}(x)-\lambda)_\sigma|dx\int_B|(\vec{b}(y)-\lambda)_{\sigma^c}||a(y)|dy\right)^p\\
&\leqslant C\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}\left(\int_B|(\vec{b}(y)-\lambda)_{\sigma^c}||a(y)|dy\right)^p\\
&\times \sum_{k=1}^{\infty}|2^{k+1}B|^{1-p}\left(\int_{2^{k+1}B\setminus 2^kB}|x-u|^{-(n+\varepsilon)}|B|^{\frac{\varepsilon}{n}}|(\vec{b}(x)-\lambda)_\sigma|dx\right)^p\\
&\leqslant C\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}|B|^{p(1-\varepsilon/n)+1}||\vec{b}_{\sigma^c}||_{BMO}^p||\vec{b}_\sigma||_{BMO}^p\sum_{k=1}^{\infty}|2^{k+1}B|k^p|2^kB|^{\frac{-(n+\varepsilon)p}{n}}\\
&\leqslant C||\vec{b}||_{BMO}^p\sum_{k=1}^{\infty}k^p2^{k[n-(n+\varepsilon)]p}\\
&\leqslant C||\vec{b}||_{BMO}^p.
\end{aligned}$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Let $f\in H\dot{K}_{q,\vec{b}}^{\alpha,p}(R^n)$ and $f(x)=\sum_{j=-\infty}^{\infty}\lambda_ja_j(x)$ be the atomic decomposition for f as in Definition 3, we write

$$\begin{aligned}
||T_{\vec{b}}(f)(x)||_{\dot{K}_q^{\alpha,p}} &= \left(\sum_{k=-\infty}^{\infty}2^{k\alpha p}||T_{\vec{b}}(f)\chi_k||_{L_q}^p\right)^{1/p}\\
&\leqslant C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}(\sum_{j=-\infty}^{\infty}|\lambda_j|||T_{\vec{b}}(a_j)\chi_k||_{L^q})^p\right]^{1/p}\\
&\leqslant C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}(\sum_{j=-\infty}^{k-3}|\lambda_j|||T_{\vec{b}}(a_j)\chi_k||_{L^q})^p\right]^{1/p}\\
&\quad +C\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}(\sum_{j=k-2}^{\infty}|\lambda_j|||T_{\vec{b}}(a_j)\chi_k||_{L^q})^p\right]^{1/p}\\
&= J+JJ.
\end{aligned}$$

For JJ, by the boundedness of $T_{\vec{b}}$ on $L^q(R^n)$ and the Hölder's inequality, we have

$$\begin{aligned} JJ &\leqslant C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j| ||T_{\vec{b}}(a_j)\chi_j||_{L^q}\right)^p\right]^{1/p} \\ &\leqslant C\left[\sum_{-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j| ||a_j||_{L^q}\right)^p\right]^{1/p} \\ &\leqslant C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j|\cdot 2^{-j\alpha}\right)^p\right]^{1/p} \\ &\leqslant C\begin{cases} \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p}\right]^{1/p}, & 0<p\leqslant 1 \\ \left[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=-\infty}^{j+2} 2^{(k-j)p/2}\right]^{1/p}, & 1<p<\infty \end{cases} \\ &\leqslant C\left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\right)^{1/p} \\ &\leqslant C||f||_{H\dot{K}^{\alpha,p}_{q,\vec{b}}}. \end{aligned}$$

For J, let $C_k = B_k\backslash B_{k-1}$, $\chi_k = \chi_{C_k}$, $b_j^i = |B_j|^{-1}\int_{B_j} b_i(x)dx$, $1 \leqslant i \leqslant m, \vec{b'} = (b_j^1, \cdots, b_j^m)$. When m=1, similar to the proof of II in Theorem 1, we have

$$\begin{aligned} T_{\vec{b_1}}(a_j)(x) &= ||\int_{B_j} (b_1(x)-b_1(y))F_t(x,y)a_j(y)dy|| \\ &\leqslant C\int_{B_j} ||F_t(x,y)-F_t(x,u)|| |(b_1(x)-b_1(y))||a_j(y)|dy \\ &\leqslant C\int_{B_j} \frac{|u-y|^\varepsilon}{|x-u|^{n+\varepsilon}}|(b_1(x)-b_1(y))||a_j(y)|dy \\ &\leqslant C\frac{|B_j|^{\frac{\varepsilon}{n}}}{|x-u|^{n+\varepsilon}}\left(\int_{B_j}|a_j(y)||b_1(x)-b_j^1|dy+\int_{B_j}|a_j(y)||b_1(y)-b_j^1|dy\right) \\ &\leqslant C\frac{|B_j|^{\frac{\varepsilon}{n}}}{|x-u|^{n+\varepsilon}}\left(|b_1(x)-b_j^1||B_j|^{1-1/q-\frac{\alpha}{n}}+|B_j|^{1-1/q-\frac{\alpha}{n}}||b_1||_{BMO}\right). \\ &\leqslant C|x-u|^{-(n+\varepsilon)}\left(|b_1(x)-b_j^1|2^{j(\varepsilon+n(1-1/q)-\alpha)}+2^{j(\varepsilon+n(1-1/q)-\alpha)}||b_1||_{BMO}\right), \end{aligned}$$

then

$$\begin{aligned} &||T_{\vec{b_1}}(a_j)(x)\chi_k||_{L^q} \leqslant C2^{j(\varepsilon+n(1-1/q)-\alpha)} \\ &\times\left[\left(\int_{B_k}|b_1(x)-b_j^1|^q|x-u|^{-q(n+\varepsilon)dx}\right)^{1/q}+\left(\int_{B_k}|x-u|^{-q(n+\varepsilon)}dx\right)^{1/q}||b_1||_{BMO}\right] \end{aligned}$$

$$\begin{aligned}
&\leqslant C2^{j(\varepsilon+n(1-1/q)-\alpha)}\left[2^{-k(n+\varepsilon)}\cdot|B_k|^{1/q}||b_1||_{BMO}+2^{-k(n+\varepsilon)}\cdot|B_k|^{1/q}||b_1||_{BMO}\right]\\
&\leqslant C||b_1||_{BMO}2^{[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]},
\end{aligned}$$

thus

$$\begin{aligned}
J &\leqslant C||b_1||_{BMO}\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|2^{[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]}\right)^p\right]^{1/p}\\
&\leqslant C||b_1||_{BMO}\begin{cases}\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\sum_{j=-\infty}^{k-3}|\lambda_j|^p2^{[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]p}\right]^{1/p}, & 0<p\leqslant 1\\ \left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|^p2^{p[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]/2}\right)\right.\\ \quad\left.\times\left(\sum_{j=-\infty}^{k-3}2^{p'[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]/2}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty\end{cases}\\
&\leqslant C||b_1||_{BMO}\begin{cases}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}2^{(j-k)(\varepsilon+n(1-1/q)-\alpha)p}\right]^{1/p}, & 0<p\leqslant 1\\ \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}2^{(j-k)(\varepsilon+n(1-1/q)-\alpha)p/2}\right]^{1/p}, & 1<p<\infty\end{cases}\\
&\leqslant C||b_1||_{BMO}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p}\\
&\leqslant C||f||_{H\dot{K}^{\alpha,p}_{q,\vec{b}}}.
\end{aligned}$$

When $m>1$, Let $b_j^i=|B_j|^{-1}\int_{B_j}b_i(x)dx$, $1\leqslant i\leqslant m, \vec{b'}=(b_j^1,\cdots,b_j^m)$. We have

$$\begin{aligned}
T_{\vec{b}}(a_j)(x) &= ||\int_{B_j}\prod_{i=1}^m(b_i(x)-b_i(y))F_t(x,y)a_j(y)dy||\\
&\leqslant C\int_{B_j}||F_t(x,y)-F_t(x,u)||\prod_{i=1}^m|(b_i(x)-b_i(y))||a_j(y)|dy\\
&\leqslant C\int_{B_j}\frac{|u-y|^\varepsilon}{|x-u|^{n+\varepsilon}}\prod_{i=1}^m|(b_i(x)-b_i(y))||a_j(y)|dy\\
&\leqslant C\frac{|B_j|^{\frac{\varepsilon}{n}}}{|x-u|^{n+\varepsilon}}\int_{B_j}\prod_{i=1}^m|(b_i(x)-b_i(y))||a_j(y)|dy\\
&\leqslant C\frac{|B_j|^{\frac{\varepsilon}{n}}}{|x-u|^{n+\varepsilon}}\sum_{i=0}^m\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b'})_\sigma|\int_{B_j}|a_j(y)||(\vec{b}(y)-\vec{b'})_{\sigma^c}|dy\\
&\leqslant C\frac{|B_j|^{\frac{\varepsilon}{n}}}{|x-u|^{n+\varepsilon}}\sum_{i=0}^m\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b'})_\sigma||B_j|^{1-1/q-\frac{\alpha}{n}}||\vec{b}_{\sigma^c}||_{BMO}\\
&\leqslant C|x-u|^{-(n+\varepsilon)}\cdot 2^{j(\varepsilon+n(1-1/q)-\alpha)}\sum_{i=0}^m\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b'})_\sigma|||\vec{b}_{\sigma^c}||_{BMO},
\end{aligned}$$

so

$$\begin{aligned}
&||T_{\vec{b}}(a_j)\chi_k||_{L^q} \\
\leqslant\ & C2^{j(\varepsilon+n(1-1/q)-\alpha)}||\vec{b}_{\sigma^c}||_{BMO}\left(\int_{B_k}\frac{1}{|x-u|^{(n+\varepsilon)q}}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}|(\vec{b}(x)-\vec{b}')_\sigma|^q dx\right)^{1/q} \\
\leqslant\ & C||\vec{b}_{\sigma^c}||_{BMO}2^{j(\varepsilon+n(1-1/q)-\alpha)}\cdot 2^{-k(n+\varepsilon)+kn/q} \\
\leqslant\ & C||\vec{b}||_{BMO}
\end{aligned}$$

and

$$\begin{aligned}
J \leqslant\ & C||\vec{b}||_{BMO}\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|2^{[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]}\right)^p\right]^{1/p} \\
\leqslant\ & C||\vec{b}||_{BMO}\begin{cases}\left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\sum_{j=-\infty}^{k-3}|\lambda_j|^p 2^{[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]p}\right]^{1/p}, & 0<p\leqslant 1\\ \left[\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|^p 2^{p[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]/2}\right)\right. & \\ \quad\left.\times\left(\sum_{j=-\infty}^{k-3}2^{p'[j(\varepsilon+n(1-1/q)-\alpha)-k(n+\varepsilon)+kn/q]}\right)^{p/p'}\right]^{1/p}, & 1<p<\infty\end{cases} \\
\leqslant\ & C||\vec{b}||_{BMO}\begin{cases}\left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}2^{(j-k)(\varepsilon+n(1-1/q)-\alpha)p}\right]^{1/p}, & 0<p\leqslant 1\\ \left[\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+3}^{\infty}2^{(j-k)(\varepsilon+n(1-1/q)-\alpha)p/2}\right]^{1/p}, & 1<p<\infty\end{cases} \\
\leqslant\ & C||\vec{b}||_{BMO}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p} \\
\leqslant\ & C||f||_{H\dot{K}^{\alpha,p}_{q,\vec{b}}}.
\end{aligned}$$

This completes the proof of Theorem 2.

Remark. Theorem 2 also holds for nonhomogeneous Herz-type spaces, we omit the details.

4. Applications

Now we give some applications of Theorems in this paper.

Application 1. Littlewood-Paley operator.

Fixed $\varepsilon>0$. Let ψ be a fixedfunction which satisfies the following properties:

(1) $\int_{R^n}\psi(x)dx=0$,

(2) $|\psi(x)|\leqslant C(1+|x|)^{-(n+1)}$,

(3) $|\psi(x+y)-\psi(x)|\leqslant C|y|^\varepsilon(1+|x|)^{-(n+1+\varepsilon)}$ when $2|y|<|x|$.

The Littlewood-Paley multilinear commutator are defined by

$$g_\psi^{\vec{b}}(f)(x)=\left(\int_0^\infty|F_t^{\vec{b}}(f)(x)|^2\frac{dt}{t}\right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^{m} (b_j(x) - b_j(y))\psi_t(x-y)f(y)dy$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define that

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

which are the Littlewood-Paley operator(see [16]). Let H be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t\right)^{1/2} < \infty\right\},$$

then, for each fixed $x \in R^n$, $F_t^{\vec{b}}(f)(x)$ may be viewed as the mapping from $[0, +\infty)$ to H, and it is clear that

$$g_\psi^{\vec{b}}(f)(x) = ||F_t^{\vec{b}}(f)(x)||, \quad g_\psi(f)(x) = ||F_t(f)(x)||.$$

It is easily to see that g_ψ satisfies the conditions of Theorem 1 and Theorem 2 (see [6-8]), thus Theorem 1 and Theorem 2 hold for $g_\psi^{\vec{b}}$.

Application 2. Marcinkiewicz operator.

Fixed $0 < \gamma \leqslant 1$. Let Ω be homogeneous of degree zero on R^n with $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$. The Marcinkiewicz multilinear commutator are defined by

$$\mu_\Omega^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_t^{\vec{b}}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{|x-y|\leqslant t} \prod_{j=1}^{m} (b_j(x) - b_j(y)) \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

Set

$$F_t(f)(x) = \int_{|x-y|\leqslant t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy;$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

which are the Marcinkiewicz operator(see [17]). Let H be the space

$$H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3\right)^{1/2} < \infty\right\}.$$

Then, it is clear that

$$\mu_\Omega^{\vec{b}}(f)(x) = ||F_t^{\vec{b}}(f)(x)||, \quad \mu_\Omega(f)(x) = ||F_t(f)(x)||.$$

It is easily to see that μ_Ω satisfies the conditions of Theorem 1 and Theorem 2 (see [9][17]), thus Theorem 1 and Theorem 2 hold for $\mu_\Omega^{\vec{b}}$.

Application 3. Bochner-Riesz operator.

Let $\delta > (n-1)/2$, $B_t^\delta(f)\hat{}(\xi) = (1-t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n}B^\delta(z/t)$ for $t > 0$. Set

$$F_{\delta,t}^{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^{m}(b_j(x) - b_j(y))B_t^\delta(x-y)f(y)dy.$$

The maximal Bochner-Riesz multilinear commutator are defined by

$$B_{\delta,*}^{\vec{b}}(f)(x) = \sup_{t>0}|B_{\delta,t}^{\vec{b}}(f)(x)|.$$

We also define that

$$B_{\delta,*}(f)(x) = \sup_{t>0}|B_t^\delta(f)(x)|$$

which is the maximal Bochner-Riesz operator(see [11]). Let H be the space $H = \{h : ||h|| = \sup_{t>0}|h(t)| < \infty\}$, then

$$B_{\delta,*}^{\vec{b}}(f)(x) = ||B_{\delta,t}^{\vec{b}}(f)(x)||, \quad B_*^\delta(f)(x) = ||B_t^\delta(f)(x)||.$$

It is easily to see that $B_{\delta,*}^{\vec{b}}$ satisfies the conditions of Theorem 1 and Theorem 2 (see [8]), thus Theorem 1 and Theorem 2 hold for $B_{\delta,*}^{\vec{b}}$.

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Received 25 05 2009, revised 10 08 2010

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