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On Semi h-Pure Submodules of QTAG-Modules

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ABSTRACT. The study of QTAG-modules was initiated by Singh [9]. The structure theory of such modules has been developed on similar lines as that of torsion abelian groups. Different concepts and decomposition theorems have been done for QTAG-modules by a number of authors. The purpose of this paper is to study the semi h-pure submodules of QTAG-modules and their characterizations. The concept of semi h-pure submodules is introduced by A. Mehdi [7]. A submodule N of a QTAG-module M is semi h-pure in M if it is not h-pure but it is contained in a h-pure submodule of M. It is well known that all submodules of M are semi h-pure if and only if M is a direct sum of a h-divisible and a bounded submodule. In [6], Khan introduced an invariant for every submodule N of M and for every non-negative integer n, denoted by $Q_n(M,N)$. Here we obtain a necessary condition on a submodule N to be a semi h-pure submodule of M. This condition turns out to be also sufficient if N is an almost h-dense submodule of M. But, in general, this condition is not sufficient. For example, if N is a subsocle of M, then $Q_n(M,N) = 0$ for all $n \geqslant 0$. However, it is known that N is not necessarily a semi h-pure submodule.

In section 2, we introduce a new invariant for every submodule N of M and every non-negative integer n, denoted by $P_n(M,N)$. This invariant gives a sufficient condition on a submodule to be a semi h-pure submodule of M. But, in general, this condition is not necessary. Here we also give some interesting properties of $P_n(M,N)$ and the relation between $Q_n(M,N)$ and $P_n(M,N)$.

In section 3, we give a new characterization of kernels of h-purity in terms of $P_n(M,N)$. In view of this characterization, it is clear that N satisfies the necessary condition that $Q_n(M,N) = 0$ for all $n \geqslant 0$.

In section 4, we establish a sufficient condition for a submodule to be a semi h-pure submodule i.e. If $P_n(M,N) = 0$ for all $n \geqslant k$, then N is semi h-pure submodule. Furthermore, we have N is h-pure in M if $k = 0$ and N is kernel of h-purity in M if $k = 1$.

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1. Introduction and Priliminaries

Following [9], a unital module M_R is called QTAG-module if it satisfies the following condition:
(1) Every finitely generated submodule of every homomorphic image of M is a direct sum of uniserial modules.
All rings considered in this paper contain unity and modules are unital QTAG-module. The structure theory of such modules has been developed by various authors. A module in which the lattice of its submodule is totally ordered with finite composition length is called a uniserial module. An element $x \in M$ is called a uniform element if xR is a nonzero uniform (hence uniserial) submodule of M. For any module M_R with a composition series, $d(M)$ denotes its length. If $x \in M$ is uniform, then $e(x) = d(xR)$ and $H_M(x) = \sup\{d(yR/xR)/y \in M$ and y is uniform with $x \in yR\}$ are called exponent of x and height of x, respectively. For any non-negative integer $n \geqslant 0, H_n(M) = \{x \in M/H_M(x) \geqslant n\}$. A submodule N of M is called h-pure in M if $H_n(N) = N \cap H_n(M)$ for all $n \geqslant 0$, and N is called h-neat if $H_1(N) = N \cap H_1(M)$. The module M is called h-divisible if $H_1(M) = M$. A submodule N of M is called h-dense if M/N is h-divisible, and N is called almost h-dense in M if for every h-pure submodule K of M containing N, M/K is h-divisible. For any module M, $Soc(M)$ denotes the socle of M. A subsocle S of a QTAG-module M is said to be h-dense in $Soc(M)$ if $S + Soc(H_n(M)) = Soc(M)$ for all $n \geqslant 0$. If N is a submodule of M, then h-neat hull of N is defined as the minimal h-neat submodule K of M, such that $N \subseteq K$ and a submodule K of M is complement of N if it is maximal with the property $K \cap N = 0$. For other basic concepts of QTAG-modules one may see [3,4,5,8,9].

We start with the following definitions.

Definition 1.1: A submodule N of a QTAG-module M is said to be semi h-pure submodule of M if it is not h-pure but it is contained in a h-pure submodule of M. The minimal h-pure submodule of M, containing N is said to be the h-pure hull of N in M. [7]

Definition 1.2: For every non-negative integer n, we denote by $N^n(M)$ the submodule: $N+H_{n+1}(M) \cap Soc(H_n(M))$ and by $N_n(M)$ the submodule: $N \cap Soc((H_n(M)) + Soc(H_{n+1}(M))$ and by $Q_n(M,N) = N^n(M)/N_n(M)$. [6]

2. Some Properties of $P_n(M, N)$

In this section we introduce a new invariant $P_n(M, N)$, for every submodule N of M and every non-negative integer n. We present here some interesting properties of $P_n(M, N)$ and the relation between $Q_n(M, N)$ and $P_n(M, N)$.

Notation 2.1: For every non-negative integer n, we denote by $P_N^M(n)$ the submodule: $Soc(H_n(M/N)) = Soc\Big((H_n(M)+N)/N\Big)$ and $P_M^N(n)$ the submodule: $(Soc(H_n(M))+N)/N$.

Remark 2.2: Evidently if $Soc(N)$ is h-dense in $Soc(M)$ then

$$P_M^N(n) = P_M^N(n+1)$$

for all $n \geqslant 0$.

Definition 2.3: For every non-negative integer n, we define

$$P_n(M,N) = \frac{P_N^M(n)}{P_M^N(n)}.$$

Next, we recall the definition of submodule $H^k(N)$ of M which is established in [4].

Definition 2.4: For any submodule N of M and for any $k \geqslant 0, H^k(N)$ defined by the submodule generated by those uniform elements $x \in M$ for which $d\Big(xR/(xR \cap N)\Big) \leqslant k$.

Remark 2.5: It is evident from the above definition that,

$$P_n(M,N) \simeq \Big(H^1(N) \cap (H_n(M)+N)\Big) \Big/ \Big(Soc(H_n(M)+N\Big).$$

Lemma 2.6: Let T be a proper h-pure submodule of M containing N. Then $P_N^M(n) = P_N^T(n) + P_M^N(n)$ and $P_N^T(n) \cap P_M^N(n) = P_T^N(n)$, for all $n \geqslant 0$.

Proof: Let $\bar{x} \in P_N^M(n)$, where $x \in H_n(M)$. Therefore there exists $y \in N$ such that $d(xR/yR) = 1$, then $y \in N \cap T \cap H_{n+1}(M)$. Since T is h-pure in M, $y \in H_{n+1}(T)$. Therefore there exists $z \in H_n(T)$ such that $d(zR/yR) = 1$. Appealing to Lemma 2.3 in [9], we get $e(x-z) \leqslant 1$ and $(x-z) \in Soc(H_n(M))$. Next, $\bar{x} = \bar{z} + \bar{u}$, where $u \in Soc(H_n(M))$ and $\bar{x} \in P_N^T(n) + P_M^N(n)$. Hence, $P_N^M(n) = P_N^T(n) + P_M^N(n)$. Now let $\bar{x} \in P_N^T(n) \cap P_M^N(n)$, then we have $\bar{x} = \bar{y} = \bar{z}$, where $\bar{y} \in P_N^T(n)$ and $\bar{z} \in P_M^N(n)$. As $y - z \in N$, where $y \in H_n(T)$ and $z \in Soc(H_n(M))$ we have $y - z \in T \cap H_n(M)$ and so, $y - z = u \in H_n(T)$. Next, $z = y - u \in Soc(H_n(T))$. Hence, $\bar{x} = \bar{z} = y - u + N \in P_T^N(n)$ and $P_N^T(n) \cap P_M^N(n) = P_T^N(n)$.

Theorem 2.7: Let K be a h-pure submodule of M containing N. Then $P_n(M,N) \simeq P_n(K,N)$, for all $n \geqslant 0$.

Proof: Let $T = P_N^M(n),\ U = P_M^N(n)$ and $V = K/N$. We find that

$$\begin{aligned} T \cap V &= Soc\big((H_n(M)+N)/N\big) \cap K/N \\ &= Soc\big(((H_n(M)+N)\cap K)/N\big) \\ &= Soc\big(((H_n(M)\cap K)+N)/N\big) \\ &= Soc\big((H_n(K)+N)/N\big) \\ &= P_N^K(n) \end{aligned}$$

and

$$\begin{aligned} U \cap V &= \big((Soc(H_n(M))+N)/N\big) \cap K/N \\ &= \big((Soc(H_n(M))+N)\cap K\big)/N \\ &= \big((Soc(H_n(M))\cap K)+N\big)/N \\ &= \big(Soc(H_n(K))+N\big)/N \\ &= P_K^N(n) \end{aligned}$$

Therefore $P_n(M,N) = T/U$ and $P_n(K,N) = (T\cap V)/(U\cap V)$. By Dedekind short exact sequence and Theorem 3.5 in [6], we have $P_n(M,N) \simeq P_n(K,N)$, for all $n \geqslant 0$.

An immediate consequence of the above theorem is stated below.

Corollary 2.8: N is h-pure in M if and only if $P_n(M,N) = 0$, for all $n \geqslant 0$.

Remark 2.9: $P_M^N(n) \cap P_N^M(n+1) = \Big(\big((N + H_{n+1}(M)) \cap Soc(H_n(M))\big) + N\Big)\Big/N.$

Now we establish the relation between $Q_n(M,N)$ and $P_n(M,N)$.

Proposition 2.10: For some non-negative integer n, $Q_n(M,N) = 0$ if and only if $P_M^N(n) \cap P_N^M(n+1) = P_M^N(n+1)$.

Proof: Necessity is an immediate consequence of Remark 2.9. Since

$$\Big((N + H_{n+1}(M)) \cap Soc(H_n(M))\Big) \subset \Big(Soc(H_{n+1}(M)) + N\Big),$$

then by Remark 2.9, the converse is also trivial.

Next, we give a necessary condition on a submodule N to be a semi h-pure submodule of M.

Proposition 2.11: Let N be a semi h-pure submodule of M, then there exists a non-negative integer k such that $Q_n(M,N) = 0$ for all $n \geqslant k$.

Proof: Since N is semi h-pure in M, then from Theorem 3 in [2], there exists a h-pure submodule K of M and a non-negative integer k such that $Soc(H_k(K)) \subset N \subset K$. Then

$$N^n(K) = (N + H_{n+1}(K)) \cap Soc(H_n(K)) = Soc(H_n(K)) = N_n(K)$$

for $n \geqslant k$. Therefore $Q_n(K,N) = 0$ for all $n \geqslant m$, and from Theorem 4.2 in [6], $Q_n(M,N) = 0$ for all $n \geqslant k$.

Using $P_n(M,N)$ we can characterize h-neat submodules of M as follows:

Proposition 2.12: A submodule N of M is h-neat if and only if $P_0(M,N) = 0$.

Proof: Let us suppose that N is h-neat in M. Let $\bar{y} \in P_N^M(0)$, where $y \in M$. Then $\bar{y}R = (yR+N)/N \simeq yR/(yR \cap N)$. Hence $d\big(yR/(yR \cap N)\big) = 1$. Put $yR \cap N = zR$. Since N is h-neat, therefore there exists a uniform element $w \in N$ such that $y \in wR$ and $d(wR/zR) = 1$. Appealing to Lemma 2.3 in [9], we get $e(y-z) \leqslant 1$, so $y - z \in Soc(M)$ and we get $\bar{y} \in P_M^N(0)$. Thus $P_N^M(0) = P_M^N(0)$. Hence $P_0(M,N) = 0$. Conversely, let x be a uniform element in $N \cap H_1(M)$, then we can find a uniform element $y \in M$ such that $d(yR/xR) = 1$. Hence $e(\bar{y}) = 1$ and so $\bar{y} \in Soc(M/N)$. Therefore, $\bar{y} = \bar{z}$, where $z \in Soc(M)$. Now $xR = H_1(yR) = H_1((y-z)R) \subseteq H_1(N)$. Hence N is h-neat submodule of M.

Proposition 2.13: If $Soc(N)$ is h-dense in $Soc(M)$, then every h-neat submodule containing $Soc(N)$ is h-pure in M. In particular N has a h-pure hull whose socle is $Soc(N)$.

Proof: Since $Soc(N)$ is h-dense in $Soc(M)$, so $Soc(M) \subset Soc(N) + H_n(M)$. Let K be a h-neat submodule of M containing $Soc(N)$. Then,

$$K^n(M) = (K \cap H_{n+1}(M)) \cap Soc(H_n(M)) = Soc(H_n(M)) = K_n(M).$$

Therefore, $Q_n(M,K) = 0$ for all $n \geqslant 0$. Now appealing to Theorem 4.3 in [6], K is h-pure submodule of M. Thus every h-neat hull of N is a h-pure hull of N.

3. Kernel of h-purity

Firstly we recall the definition of kernel of h-purity.

Definition 3.1: If N is a submodule of a QTAG-module M, then N is called kernel of h-purity if all h-neat hulls of N are h-pure submodules of M. [3]

In [3], Khan has characterized kernel of h-purity. But, since kernel of h-purity are semi h-pure submodules, so here arise a case to give another characterization, which is given by using the concept of $P_n(M,N)$.

In view of the above discussion, it is trivial that if N is a kernel of h-purity, then N satisfies $Q_n(M,N)=0$ for almost all n.

First, we need the following useful Lemmas.

Lemma 3.2: If $P_M^N(0)=P_M^N(k)\neq P_M^N(k+1)$ and $P_{k+1}(M,N)\neq 0$ for some non-negative integer k, then there exists a h-neat hull of N in M which is not h-pure in M.

Proof: Since $P_M^N(0)=P_M^N(k)\neq P_M^N(k+1)$, then there exists $x\in Soc(M)$ such that $x\notin Soc(H_{k+1}(M))+Soc(N)$. Since $P_{k+1}(M,N)\neq 0$, then we get an element $\bar{y}\in P_N^M(k+1)$ such that $\bar{y}\notin P_M^N(k+1)$, where $y\in H_{k+1}(M)$ and there exists an element $z\in M$ such that $d(zR/yR)=k+1$. Let us define $K=(x+y)R+N$. Now we have to show that $Soc(K)=Soc(N)$. Suppose on contrary $Soc(K)\neq Soc(N)$. For any $(x+y)r+a\in Soc(K)$, where $r\in R$ and $a\in N$, either $xrR=xR$ or $xr=0$. If $xr=0$, then $yr+a\in Soc(M)$. Now by hypothesis, $\bar{y}\in P_M^N(0)=P_M^N(k)$. Appealing to Theorem 4.3 in [6], we know that all h-neat hulls of N are not h-pure in M if $Q_k(M,N)\neq 0$. Therefore, we assume that $Q_k(M,N)=0$. Using Proposition 2.10, we get $\bar{y}\in P_M^N(k+1)$, which shows a contradiction. Hence, $Soc(K)=Soc(N)$.
Now, let T be a h-neat hull of N in M such that $K\subseteq T$, and so $\bar{y}\in P_T^M(k+1)$. If we suppose that $\bar{y}\in P_M^T(k+1)$, then $y=u+v$, for some $u\in Soc(H_{k+1}(M))$ and $v\in T$. Therefore, $x=y+x-(u+v)$ and $y+x-v\in Soc(T)=Soc(N)$. Hence, $x\in Soc(H_{k+1}(M))+Soc(N)$, which shows a contradiction. Hence, $\bar{y}\notin P_M^T(k+1)$ and $P_{k+1}(M,T)\neq 0$. Appealing to Corollary 2.8, T is not h-pure in M.

Lemma 3.3: For non-negative integer k and t, we have

$$P_{k+t}(M,N)\simeq P_t\Big(H_k(M),H_k(M)\cap N\Big).$$

Proof: We know that

$$P_{k+t}(M,N) \simeq \frac{\Big(I_M(N) \cap (H_{k+t}(M)+N)\Big)}{\Big(Soc(H_{k+t}(M))+N\Big)}$$

and

$$P_t\Big(H_k(M),\ H_k(M)\ \cap\ N\Big) \simeq \frac{\Big(I_M(N) \cap (H_{k+t}(M)+N) \cap H_k(M)\Big)}{\Big((Soc(H_{k+t}(M))+N) \cap H_k(M)\Big)}$$

Put $T = I_M(N) \cap (H_{k+t}(M)+N),\ U = Soc(H_{k+t}(M))+N$ and $V = H_k(M)$. Then

$$\begin{aligned}(T \cap V)+U &= \Big(\big(I_M(N) \cap (H_{k+t}(M)+N)\big) \cap H_k(M)\Big) + \Big(Soc(H_{k+t}(M))+N\Big)\\ &= I_M(N) \cap \Big(H_{k+t}(M) + (H_k(M) \cap N) + Soc(H_{k+t}(M)) + N\Big)\\ &= I_M(N) \cap \Big(H_{k+t}(M) + H_k(N)) + Soc(H_{k+t}(M)) + N\Big)\\ &= I_M(N) \cap (H_{k+t}(M)+N)\\ &= T\end{aligned}$$

By Dedekind short exact sequence, we get $P_{k+t}(M,N) \simeq P_t\Big(H_k(M), H_k(M) \cap N\Big)$.

Using Corollary 2.8 and Lemma 3.3, we establish a theorem which gives the relation between $P_n(M,N)$ and $P_n(M,T)$, where T is h-neat hull of N in M.

Lemma 3.4: Let T be a h-neat hull of N in M. If $P_{k+n}(M,N) = 0$ for all $n \geqslant 1$, then $P_{k+n}(M,T) = 0$ for all $n \geqslant 1$.

Proof: By Corollary 2.8 and Lemma 3.3, $H_k(M) \cap N$ is h-pure in $H_k(M)$. Since $Soc(H_k(M) \cap N) = Soc(H_k(M) \cap T)$, therefore $H_k(M) \cap N = H_k(M) \cap T$. Hence $P_{k+n}(M,T) = 0$ for all $n \geqslant 1$.

Now we give a new characterization for kernel of h-purity.

Theorem 3.5: N is a kernel of h-purity in M if and only if $Soc(N)$ is h-dense in M or there exists a non-negative integer k such that $P_M^N(0) = P_M^N(k) \neq P_M^N(k+1)$ and $P_n(M,N) = 0$ for all $n > k$.

Proof: Let us suppose that N be a kernel of h-purity in M. If $Soc(N)$ is not h-dense in $Soc(M)$, then there exists a smallest non-negative integer k such that $Soc(M) = Soc(N) + Soc(H_k(M))$ and $Soc(M) \neq Soc(N) + Soc(H_t(M))$ for $t \geqslant k$. Therefore, $P_M^N(0) = P_M^N(k) \neq P_M^N(k+1)$. By Lemma 3.2, we get $P_{k+1}(M,N) = 0$. Furthermore, by Theorem 4.3 in [6], $Q_n(M,N) = 0$ for all $n \geqslant 0$, and by Proposition 2.10, $P_n(M,N) = 0$ for all $n > k$.
Conversely, let T be a h-neat hull of N in M. Then by Lemma 3.4, $P_n(M,T) = 0$ for all $n > k$. Hence, by Proposition 2.10, $Q_n(M,T) = 0$ for all $n \geqslant k$. Since $Soc(N) = Soc(T)$, $P_M^T(0) = P_M^T(k) \neq P_M^T(k+1)$. By Remark 2.2, $Q_n(M,T) = 0$ for all $n \geqslant 0$. Hence by Theorem 4.3 in [6], T is h-pure in M. If $Soc(N)$ is h-dense, then N is a kernel of h-purity.

4. Some Sufficiency Conditions for semi h-pure Submodules

In this section we proved a necessary condition that a submodule N of M is semi h-pure submodule of M if there exists a non-negative integer k such that $Q_N(M,N) = 0$ for all $n \geqslant k$. Furthermore, we show that this condition becomes sufficient if N is almost h-dense in M.

Theorem 4.1: Let N be almost h-dense in M. Then N is semi h-pure in M if and only if there exists a non-negative integer k such that $Q_n(M,N) = 0$ for all $n \geqslant k$.

Proof: Suppose that $Q_n(M,N) = 0$ for all $n \geqslant k$. Since N is almost h-dense in M, then from Theorem 5 in [1], $Soc(H_n(M)) \subset N + H_{n+1}(M)$. Hence, we have

$$Soc(H_n(M)) = N^n(M) = N_n(M) = Soc(N \cap H_n(M)) + Soc(H_{n+1}(M))$$

for all $n \geqslant k$. Therefore, $Soc(N \cap H_k(M))$ is a h-dense in $H_k(M)$. Let K be a h-neat submodule of $H_k(M)$ containing $Soc(N \cap H_k(M)$. Now from Proposition 2.12, K is h-pure in $H_k(M)$ such that $Soc(K) \subset N \cap H_k(M) \subset K$. Furthermore, $H_k(M)/K$ is h-divisible. We can easily see that $(N+K)/K$ is disjoint from $H_k(M)/K$. Thus $(N+K)/K$ can be extended to T/K such that $(T/K) \oplus (H_k(M)/K) = (M/K)$. Since K is h-pure in $H_k(M)$, then appealing to Proposition 2.5 in [5], T is h-pure in M. Now clearly $H_k(T) = K$. Therefore, $Soc(H_k(T)) \subset Soc(K) \subset N$. Hence, N is semi h-pure.

Now using the proof of above theorem and our notation, we obtain the following result. This is a sufficient condition for semi h-pure submodules.

Theorem 4.2: Let k be a non-negative integer. If $P_M^N(n) = P_M^N(n+1)$ for all $n \geqslant k$, then N is a semi h-pure submodule of M.

Proof: By Remark 2.9 and Proposition 2.10, $Q_n(M,N)=0$ and $(H_{n+1}(M)+N)\cap Soc(H_n(M)) = Soc(H_n(M))$ for all $n \geqslant k$. From the above Theorem, N is a semi h-pure submodule of M.

Let k be a non-negative integer. Suppose that $P_n(M,N)=0$ for all $n \geqslant k$. Then by Proposition 2.10, N satisfies the necessary condition for semi h-pure submodule, namely $Q_n(M,N)=0$ for all $n \geqslant k-1$. Furthermore, if $k=0$, then by Corollary 2.8, N is h-pure in M, and if $k=1$, then by Theorem 3.5, N is a kernel of h-purity in M.

With these statements, we assume that, if k is an integer and $P_n(M,N)=0$ for all $n \geqslant k$, then N is a semi h-pure submodule. Now we prove a theorem which shows that this assumption is true. This is also a sufficient condition for semi h-pure submodules.

Theorem 4.3: Let k be a non-negative integer. If $P_n(M,N)=0$ for all $n \geqslant k$, then N is a semi h-pure submodule of M.

Proof: Let us consider the family $F = \Big\{K \Big/ K \supseteq N \text{ and } K\cap H_k(M) = N \cap H_k(M)\Big\}$ of submodules of M. By Zorn's Lemma, F will contain a maximal element, say T, then $T/(N\cap H_k(M))$ is a complement of $(H_k(M))\big/(N\cap H_k(M))$ in $M/(N\cap H_k(M))$. Therefore,

$$Soc\Big(T/(N\cap H_k(M))\Big) \oplus Soc\Big((H_k(M))/(N\cap H_k(M))\Big) = Soc\Big(M/(N\cap H_k(M))\Big).$$

Since $N\cap H_k(M)$ is h-pure in $H_k(M)$, by Corollary 2.8 and Lemma 3.3, we get

$$Soc\Big((H_k(M))/(N\cap H_k(M))\Big) = \Big(Soc(H_k(M)) + (N\cap H_k(M))\Big)\Big/\Big(N\cap H_k(M)\Big).$$

Now we prove that T is h-neat in M. Let x be a uniform element in $T\cap H_1(M)$, then there exists a uniform element $y\in M$ such that $d(yR/xR)=1$. Due to h-neatness of $T/(N\cap H_k(M))$ in $M/(N\cap H_k(M))$ there exists an element $\bar{z}\in T/(N\cap H_k(M))$ such that $\bar{z}\in \bar{y}R$ and $d(\bar{y}R/\bar{z}R)=1$. Appealing to Lemma 2.3 in [9], we get $e(\bar{y}-\bar{z})\leqslant 1$, so $\bar{y}-\bar{z}\in Soc\Big(M/(N\cap H_k(M))\Big)$. Hence, $y-z=u+v+a$, where $u\in T,\ v\in Soc(H_k(M))$ and $a\in (N\cap H_k(M))$. This implies that $xR = H_1(yR) = H_1((z+u+v+a)R)\subseteq H_1(T)$. Therefore, T is h-neat in M. Now for $x'\in Soc(M)\setminus (N\cap H_k(M)),\ x'=u'+v'+a'$ where $u'\in T, v'\in Soc(H_k(M))$ and $a'\in N\cap H_k(M)$. Since $T\supset N$ and $Soc(M) = Soc(T)+Soc(H_k(M))$. Hence, $P^T_M(0)=P^T_M(k)$. Now by Lemma 3.3, we have

$$P_{k+n}(M,T)\simeq P_n\Big(H_k(M), H_k(M)\cap T\Big) = P_n\Big(H_k(M), H_k(M)\cap N\Big) = P_{k+n}(M,N)=0,$$

for all $n\geqslant 0$. By Theorem 3.5, T is a kernel of h-purity in M. Since T is h-neat in M, T is h-pure in M. Also $Soc(H_k(T)) = T\cap H_k(M) = N\cap H_k(M)\subset N,\ N$ is semi h-pure in M.

Corollary 4.4: Let k be a non-negative integer. Suppose that $P_n(M,N) = 0$ for all $n \geqslant k$. Then N is a semi h-pure submodule of M.
Moreover,
(1) If $k = 0$, then N is h-pure in M, and
(2) If $k = 1$, then N is a kernel of h-purity in M.

References

[1] Ahmad Mofeed, Ansari, A. Halim and Khan, M.Z., *Some decomposition theorems on S_2 modules*, Tamkang Journal of Mathematics, 11(2) (1980), 203-208.

[2] Khan, M.Z., *Modules behaving like torsion abelian groups*, Math. Japonica, 22(5) (1978), 513-518.

[3] Khan, M.Z., *On h-purity in QTAG-modules*, Communications in Algebra, 16 (1988), 2649-2660.

[4] Khan, M.Z., *Some generalizations in abelian groups*, Tamkang Journal of Mathematics, 19(2) (1988), 7-11.

[5] Khan, M.Z. and Bano, R., *Some decomposition theorems in abelian groups like modules*, Soochow Journal of mathematics, 18(1) (1992), 1-7.

[6] Khan, M.Z. and Zubair, A., *On Quasi h-pure submodules of QTAG-modules*, International J. Math. and Math. Sci., 24(7) (2000), 493-499.

[7] Mehdi, Alveera and Sikander, Fahad, *Some Characterizations of submodules of QTAG-modules*, Scientia, Series A, 18 (2009), 39-46.

[8] Singh, S., *Some decomposition theorems in abelian groups and their generalizations*, Proc. Ohio. Univ. Conf. Marcel Dekker N. Y., (1976), 183-189.

[9] Sing, S., *Abelian groups like modules*, Act. Math. Hung., 50 (1987), 85-95.

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