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On the Open Geodetic Number of a Graph

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Abstract. For a connected graph G of order n, a set $S \subseteq V(G)$ is a geodetic set of G if each vertex $v \in V(G)$ lies on a x-y geodesic for some elements x and y in S. The minimum cardinality of a geodetic set of G is defined as the geodetic number of G, denoted by $g(G)$. A geodetic set of cardinality $g(G)$ is called a g-set of G. A set S of vertices of a connected graph G is an open geodetic set of G if for each vertex v in G, either 1) v is an extreme vertex of G and $v \in S$ or 2) v is an internal vertex of a x-y geodesic for some $x, y \in S$. An open geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the open geodetic number, $og(G)$. The open geodetic numbers of certain standard graphs are determined. Connected graphs with open geodetic number 2 are characterized. For positive integers r, d and $l \geqslant 2$ with $r < d \leqslant 2r$, there exists a connected graph of radius r, diameter d and open geodetic number l. It is proved that for a tree T of order n and diameter d, $og(T) = n - d + 1$ if and only if T is a caterpillar. Also for integers n, d and k with $2 \leqslant d < n$, $2 \leqslant k < n$ and $n - d - k + 1 \geqslant 0$, there exists a graph G of order n, diameter d and open geodetic number k. It is also proved that $og(G) - 2 \leqslant og(G') \leqslant og(G) + 1$, where G' is the graph obtained from G by adding a pendant edge to G.

1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. For basic graph theoreotic terminology we refer to Harary [**6**]. The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. An u-v path of length $d(u, v)$ is called an *u-v geodesic*. It is known that this distance is a metric on the vertex set $V(G)$. For any vertex v of G, the *eccentricity* $e(v)$ of v is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, *rad* G and the maximum eccentricity is its *diameter, diam* G of G. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices which are adjacent with v. A vertex v is an *extreme vertex* of G if the subgraph induced by its neighbors is complete.For a cut vertex v in a connected graph

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G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v and $V(H)$ is called a *branch* of G at v. A *geodetic set* of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The *geodetic number* $g(G)$ of G is the cardinality of a minimum geodetic set. A vertex x is said to *lie on* a u-v geodesic P if x is a vertex of P and x is called an *internal vertex* of P if $x \neq u, v$. If x is an internal vertex of an $u - v$ geodesic, we also use the notation $x \in I(u, v)$. A set S of vertices of a connected graph G is an *open geodetic set* if for each vertex v in G, either (1)v is an extreme vertex of G and $v \in S$ or (2) v is an internal vertex of a x-y geodesic for some $x, y \in S$. An open geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the *open geodetic number,* $og(G)$. Certainly, every open geodetic set is a geodetic set and so $g(G) \leqslant og(G)$. The geodetic number of a graph was introduced in [**1, 4, 7**] and further studied in [**2, 3**]. The open geodetic number of a graph was introduced and studied in [**5, 8**] in the name open geodomination in graphs. Throughout the following, G denotes a connected graph with at least two vertices.

The following theorems are used in the sequel.

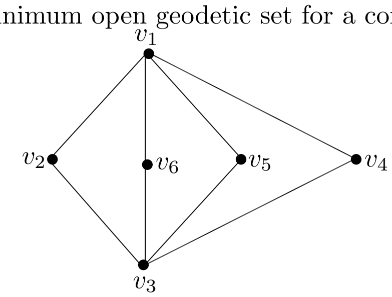
THEOREM 1.1. [**6**] A vertex v of a connected graph G is a cut vertex of G if and only if there exist vertices u and w distinct from v such that v lies on every u-w path of G.

THEOREM 1.2. [**5**] If a nontrivial connected graph G contains no extreme vertices, then $og(G) \geqslant 4$.

2. Open geodetic number of a graph

DEFINITION 2.1. [**5**] A set S of vertices in a connected graph G is an *open geodetic set* if for each vertex v in G, either (1) v is an extreme vertex of G and $v \in S$ or (2) v is an internal vertex of an x-y geodesic for some $x, y \in S$. An open geodetic set of minimum cardinality is a *minimum open geodetic set* and this cardinality is the *open geodetic number* $og(G)$ of G.

EXAMPLE 2.1. For the graph G in Figure 2.1, it is easily checked that neither a 2-element subset nor a 3-element subset is an open geodetic set. Since $S = \{v_1, v_2, v_3, v_5\}$ is a minimum open geodetic set of G, $og(G) = 4$. Also, $S = \{v_1, v_2, v_3, v_5\}$, $S' = \{v_1, v_2, v_3, v_4\}$ and $S'' = \{v_1, v_2, v_3, v_6\}$ are minimum open geodetic sets. Thus, there can be more than one minimum open geodetic set for a connected graph.



G
Figure 2.1

Remark 2.1. For the graph G given in Figure 2.1, $S = \{v_1, v_3\}$ is a minimum geodetic set so that $g(G) = 2$. Thus the geodetic number and the open geodetic number of a graph are different.

Theorem 2.1. For any connected graph G of order n, $2 \leqslant og(G) \leqslant n$.

Proof. An open geodetic set needs at least two vertices and so $og(G) \geqslant 2$. Also the set of all vertices of G is an open geodetic set of G so that $og(G) \leqslant n$. Thus $2 \leqslant og(G) \leqslant n$. □

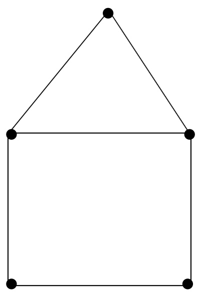
Remark 2.2. The bounds in Theorem 2.1 are sharp. For the complete graph $K_n (n \geqslant 2)$, $og(K_n) = n$. The set of two end vertices of a path $P_n (n \geqslant 2)$ is its unique minimum open geodetic set so that $og(P_n) = 2$. Thus the complete graph K_n has the largest possible open geodetic number n and that the nontrivial paths have the smallest open geodetic number 2.

The following theorem is obvious from the definition of open geodetic set.

Theorem 2.2. Every open geodetic set of a graph G contains its extreme vertices. Also, if the set S of all extreme vertices of G is an open geodetic set, then S is the unique minimum open geodetic set of G.

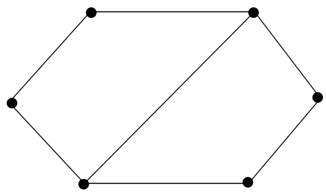
Corollary 2.1. For the complete graph $K_n (n \geqslant 2)$, $og(K_n) = n$.

Remark 2.3. If $og(G) = n$ for a connected graph G of order n, then it is not true that G is complete. It is clear that for the cycle C_4, $og(C_4) = 4$. Also for the house graph G given in Figure 2.2 and for the graph G given in Figure 2.3, $og(G) = 5$ and $og(G) = 6$ respectively. It is to be noted that for a graph G of order n, we have $g(G) = n$ if and only if $G = K_n$.



G

Figure 2.2



G

Figure 2.3

Theorem 2.3. For the complete bipartite graph $K_{m,n} (2 \leqslant m \leqslant n)$, $og(K_{m,n}) = 4$.

Proof. Let $G = K_{m,n}$. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be the partite sets of G. Since G contains no extreme vertices, by Theorem 1.2, $og(G) \geqslant 4$. Let S be any set of four vertices formed by taking two vertices from each of U and W. Then it is clear that S is an open geodetic set of G and so $og(G) = 4$. □

Theorem 2.4. For the wheel $W_n = K_1 + C_{n-1}(n \geqslant 5)$, $og(W_n) = n - 1$.

Proof. Let $W_n = K_1 + C_{n-1}(n \geqslant 5)$ with x the vertex of K_1 and $V(C_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$. It is clear that x does not belong to any minimum open geodetic set of W_n. If S is a subset of $V(C_{n-1})$ of cardinality at most $n-2$, let $v_i(1 \leqslant i \leqslant n-1)$ be such that $v_i \notin S$ and $v_{i+1} \in S$. Then v_{i+1} is not an internal vertex of any geodesic joining a pair of vertices in S. Hence S is not an open geodetic set of W_n. Since $W = \{v_1, v_2, \dots, v_{n-1}\}$ is an open geodetic set of W_n, it follows that W is the unique minimum open geodetic set of W_n and so $og(W_n) = n - 1$. □

Theorem 2.5. For the cycle $C_n(n \geqslant 4)$,

$$og(C_n) = \begin{cases} 4 & \text{if } n \text{ is even} \\ 5 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. By Theorem 1.2, $og(C_n) \geqslant 4$. First, let $n = 2k$ and the cycle be C_{2k} : $v_1, v_2, \dots, v_k, \dots, v_{2k}, v_1$. It is clear that the set $S = \{v_1, v_k, v_{k+1}, v_{2k}\}$ is a minimum open geodetic set of C_{2k} so that $og(C_{2k}) = 4$. Now, let $n = 2k + 1$ and the cycle be $C_{2k+1} : v_1, v_2, \dots, v_k, \dots, v_{2k}, v_{2k+1}, v_1$. Let $S' = \{x, y, u, v\}$ be a set of four vertices of C_{2k+1}. We consider two cases.

Case 1. S contains two antipodal vertices, say u, v. Then $u \notin I(t, v)$ and $v \notin I(t, u)$ for $t = x, y$. Also, it is clear that either $u \notin I(x, y)$ or $v \notin I(x, y)$. Hence S' is not an open geodetic set of C_{2k+1}.

Case 2. No two vertices of S are antipodal.

Let x', x'' be the antipodal vertices of x. Then $x', x'' \notin S'$. Let P be the x-x' geodesic and Q the x-x'' geodesic in C_{2k+1}. If $y, u, v \in V(P)$ or $y, u, v \in V(Q)$, then S' is not an open geodetic set of C_{2k+1}. Let $y \in V(P)$ and $u, v \in V(Q)$. Then $y \notin I(s, t)$ for $s, t \in S'$ and so S' is not an open geodetic set of C_{2k+1}. Thus $og(C_{2k+1}) \geqslant 5$. It is clear that $S = \{v_1, v_2, v_{k+1}, v_{k+2}, v_{k+3}\}$ is a minimum open geodetic set of C_{2k+1} and so $og(C_{2k+1}) = 5$. □

Theorem 2.6. Let G be a connected graph with cut vertices. Then every open geodetic set of G contains at least one vertex from each component of G.

Proof. Let v be a cut vertex of G. Let $G_1, G_2, \dots, G_k$ $(k \geqslant 2)$ be the components of $G - v$. Let S be an open geodetic set of G. Suppose that S contains no vertex from a component say $G_i(1 \leqslant i \leqslant k)$. Let u be a vertex of G_i. Then by Theorem 2.2, u is not an extreme vertex of G. Since S is an open geodetic set of G, there exist vertices $x, y \in S$ such that u lies on a x-y geodesic $P : x = u_0, u_1, u_2, \dots, u, \dots, u_l = y$ such that $u \neq x, y$. By Theorem 1.1, the x-u subpath of P and the u-y subpath of P both contain v. Hence it follows that P is not a path, contrary to assumption. □

Corollary 2.2. Let G be a connected graph with cut vertices and let S be an open geodetic set of G. Then every branch of G contains an element of S.

Theorem 2.7. Let G be a connected graph with cut vertices and S a minimum open geodetic set of G. Then no cut vertex of G belongs to S.

Proof. Let S be any minimum open geodetic set of G. Let $v \in S$. We prove that v is not a cut vertex of G. Suppose that v is a cut vertex of G. Let $G_1, G_2, \ldots, G_k$ ($k \geqslant 2$) be the components of $G - v$. Then v is adjacent to at least one vertex of each G_i for $1 \leqslant i \leqslant k$. Let $S' = S - \{v\}$. We show that S' is an open geodetic set of G. Let x be a vertex of G. If x is an extreme vertex of G, then $x \neq v$ and so by Theorem 2.2, $x \in S'$. If x is not an extreme vertex, then, since S is an open geodetic set of G, $x \in I(u, w)$ for some $u, w \in S$. If $v \neq u, w$, then $u, w \in S'$. If $v = u$, then $v \neq w$. Assume without loss of generality that $w \in G_1$. By Theorem 2.6, S contains a vertex w' from G_i ($2 \leqslant i \leqslant k$). Then $w' \neq v$. Since v is a cut vertex of G, we have $I(w, u) \subseteq I(w, w')$. Hence $x \in I(w, w')$, where $w, w' \in S'$. Thus S' is an open geodetic set of G. This contradicts that S is a minimum open geodetic set of G. □

Remark 2.4. If $og(G) = n$ for a connected graph of order n, it follows from Theorem 2.7 that G is a block.

We leave the following problem as an open question.

Problem 2.8. Characterize the class of graphs of order n for which $og(G) = n$.

Theorem 2.9. *For any tree T, the open geodetic number $og(T)$ equals the number of end vertices of T. In fact, the set of all end vertices of T is the unique minimum open geodetic set of T.*

Proof. This follows from Theorems 2.2 and 2.7. □

Theorem 2.10. *For every pair, k, n of integers with $2 \leqslant k \leqslant n$, there exists a connected graph G of order n such that $og(G) = k$.*

Proof. For $k = n$, let $G = K_n$. Then the result follows from Corollary 2.1. For $2 \leqslant k < n$, let G be a tree of order n with k end vertices. Then the result follows from the Theorem 2.9. □

Theorem 2.11. *For a connected graph $G, og(G) = 2$ if and only if there exist extreme peripheral vertices u and v such that every vertex of G is on a diametral path joining u and v.*

Proof. Let u and v be extreme peripheral vertices of G such that each vertex of G is on a diametral path P joining u and v. Then $S = \{u, v\}$ is an open geodetic set of G and so $og(G) = 2$. Conversely, let $og(G) = 2$ and let $S = \{u, v\}$ be a minimum open geodetic set of G. Necessarily, both u and v are extreme vertices of G. We claim that $d(u, v) = d(G)$, where $d(G)$ dentoes the diameter of G. If $d(u, v) < d(G)$, then let x and y be two vertices of G such that $d(x, y) = d(G)$. Now, it follows that x and y lie on distinct geodesics joining u and v. Hence

$$d(u, v) = d(u, x) + d(x, v) \tag{2.1}$$

$$\text{and } d(u, v) = d(u, y) + d(y, v). \tag{2.2}$$

By the triangle inequality,

$$d(x, y) \leqslant d(x, u) + d(u, y). \tag{2.3}$$

Since $d(u,v) < d(x,y)$, (3) becomes

$$d(u,v) < d(x,u) + d(u,y). \tag{2.4}$$

Using (4) in (1), we get $d(x,v) < d(x,u) + d(u,y) - d(u,x) = d(u,y)$. Thus,

$$d(x,v) < d(u,y). \tag{2.5}$$

Also by triangle inequality, we have

$$d(x,y) \leqslant d(x,v) + d(v,y). \tag{2.6}$$

Now, using (2) and (5),(6) becomes $d(x,y) < d(u,y) + d(v,y) = d(u,v)$. Thus, $d(G) < d(u,v)$, which is a contradiction. Hence $d(u,v) = d(G)$ and since $S = \{u,v\}$ is a minimum open geodetic set of G, it follows that each vertex of G is on a diametral path joining u and v. □

THEOREM 2.12. Let G be a non complete connected graph of order n. If G contains a vertex of degree $n-1$, then $og(G) \leqslant n-1$.

PROOF. Let x be a vertex of degree $n-1$. Since G is not complete, x is not an extreme vertex. Let $S = V(G) - \{x\}$. We show that S is an open geodetic set of G. Since x is not extreme, there exist nonadjacent neighbors y and z of x. Hence it follows that $x \in I(y,z)$, where $y, z \in S$. Now, let $u \in S$. Suppose that u is not an extreme vertex of G. If $\langle N(u)\rangle$ is complete in $\langle S\rangle$, then $\langle N(u) \cup \{x\}\rangle$ is complete in G and so u is an extreme vertex of G, which is not so. Hence $\langle N(u)\rangle$ is not complete in $\langle S\rangle$. This means that there exist nonadjacent neighbors v, w of u such that $v, w \in S$. This, in turn, shows that $u \in I(v,w)$ and hence S is an open geodetic set of G. Thus $og(G) \leqslant |S| = n-1$. □

REMARK 2.5. The bound in Theorem 2.12 can be strict. For the graph G in Figure 2.4, $S = \{v_2, v_4, v_5\}$ is a minimum open geodetic set of G so that $og(G) = 3 < 4$. Also, the bound in Theorem 2.12 is sharp. For the wheel $W_n = K_1 + C_{n-1} (n \geqslant 5)$, $og(W_n) = n-1$.

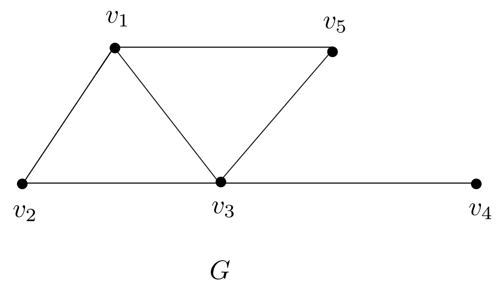


Figure 2.4

THEOREM 2.13. For any tree T of order $n \geqslant 3$, $og(T) = n-1$ if and only if T is the star $K_{1,n-1}$.

PROOF. This follows from Theorem 2.9. □

In the following theorem, we construct a class of graphs G of order n for which $og(G) = n-1$.

Theorem 2.14. Let $G_i (1 \leqslant i \leqslant k)$ be vertex disjoint connected graphs of order n_i, where $k \geqslant 2$. If $og(G_i) = n_i$, then $og(K_1 + \cup G_i) = \sum n_i - 1$.

Proof. Let $G = K_1 + \cup G_i$. Let $K_1 = \{v\}$. By Theorem 2.12, $og(G) \leqslant \sum n_i - 1$. Suppose that $og(G) < \sum n_i - 1$. Let S be a minimum open geodetic set of G. Then $|S| \leqslant \sum n_i - 2$. Since v is a cut vertex of G, $v \notin S$. Also, there exists a $v_i \in V(G_i)$ such that $v_i \notin S$. Let $S_i = S \cap V(G_i)(1 \leqslant i \leqslant k)$. Then $|S_i| \leqslant n_i - 1$ for each i. We show that S_i is an open geodetic set of G_i. Let $x \in V(G_i)$. Then $x \in V(G)$. It is clear that a vertex is extreme in G_i if and only if it is extreme in G. Hence, if x is extreme in G_i, then $x \in S$ and so $x \in S_i$. If x is non-extreme in G_i, then, since S is an open geodetic set of G, we have $x \in I_G(y, z)$ for some $y, z \in S$. Since $d(y, z) = 2$, it follows that $y, z \in V(G_i)$. Since $x, y, z \in V(G_i)$, $x \in I_{G_i}(y, z)$ with $y, z \in S_i$. Hence S_i is an open geodetic set of G_i, which is a contradiction to $og(G_i) = n_i$. □

Now, we leave the following problem as an open question.

Problem 2.15. Characterize the class of graphs G of order n for which $og(G) = n - 1$.

For every connected graph, $rad\ G \leqslant diam\ G \leqslant 2\ rad\ G$. Ostrand [**9**] showed that every two positive integers a and b with $a \leqslant b \leqslant 2a$ are realizable as the radius and diameter, respectively, of some connected graph. Now, Ostrand's theorem can be extended so that the open geodetic number can also be prescribed, when $a < b \leqslant 2a$.

Theorem 2.16. For positive integers r, d and $l \geqslant 2$ with $r < d \leqslant 2r$, there exists a connected graph G with $rad\ G = r$, $diam\ G = d$ and $og(G) = l$.

Proof. When $r = 1$, let $G = K_{1,l}$. Then $d = 2$ and by Theorem 2.9, $og(G) = l$. For $r \geqslant 2$, we construct a graph G with the desired properties as follows:

Let $C_{2r} : v_1, v_2, \ldots, v_{2r}, v_1$ be a cycle of order $2r$ and let $P_{d-r+1} : u_0, u_1, u_2, \ldots, u_{d-r}$ be a path of order $d - r + 1$. Let H be a graph obtained from C_{2r} and P_{d-r+1} by identifying v_1 in C_{2r} and u_0 in P_{d-r+1}. Let G be the graph obtained from H by adding $l - 2$ new vertices $w_1, w_2, \ldots, w_{l-2}$ to H and joining each vertex w_i $(1 \leqslant i \leqslant l - 2)$ to the vertex u_{d-r-1} and also joining the edge $v_r v_{r+2}$. The graph G is shown in Figure 2.5. Then $rad\ G = r$ and $diam\ G = d$. The graph G has $l - 1$ end vertices. Let $S = \{w_1, w_2, \ldots, w_{l-2}, u_{d-r}, v_{r+1}\}$. Then S is the set of all extreme vertices of G and it is clear that S is an open geodetic set of G and so by Theorem 2.2, $og(G) = l$.

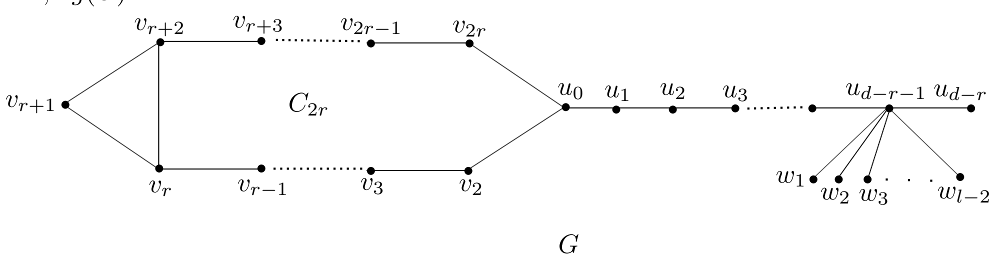
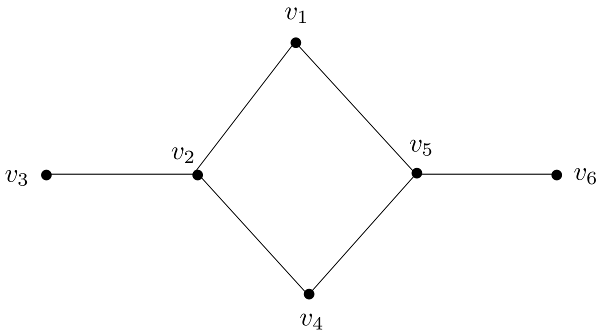


Figure 2.5

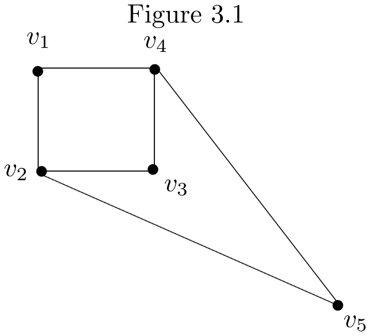
□

3. The open geodetic number and diameter of a graph

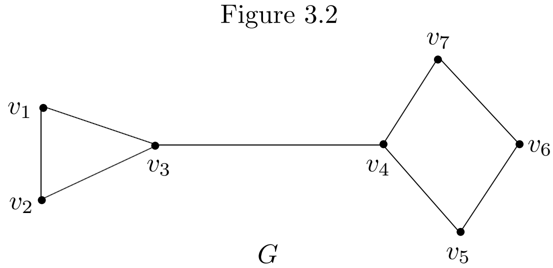
For a graph G of order n and diameter d, it is proved in [**3**] that $g(G) \leqslant n-d+1$. However, in the case of $og(G)$, it happens that $og(G) < n-d+1$, $og(G) = n-d+1$ and $og(G) > n-d+1$. For the graph G given in Figure 3.1, it is clear that $\{v_3, v_6\}$ is a minimum open geodetic set of G and so $og(G) = 2$. Since $n = 6$ and $d = 4$, we have $n - d + 1 = 3$ and so $og(G) < n - d + 1$. For the graph G given in Figure 3.2, it is clear that $\{v_1, v_2, v_3, v_4\}$ is a minimum open geodetic set of G and so $og(G) = 4$. Since $n = 5$ and $d = 2$, we have $n - d + 1 = 4$ and so $og(G) = n - d + 1$. Also, for the graph G given in Figure 3.3, it is clear that $\{v_1, v_2, v_5, v_6, v_7\}$ is a minimum open geodetic set of G and so $og(G) = 5$. Since $n = 7$ and $d = 4$, we have $n-d+1 = 4$ and so $og(G) > n - d + 1$.



G
Figure 3.1



G
Figure 3.2



G
Figure 3.3

THEOREM 3.1. For every nontrivial tree T of order n, $og(T) = n - d + 1$ if and only if T is a caterpillar.

PROOF. Let T be a nontrivial tree. Let $d(u, v) = d$ and $P : u = v_0, v_1, v_2, \dots, v_{d-1}, v_d = v$ be a diametral path. Let k be the number of end vertices of T and l the number of internal vertices of T other than $v_1, v_2, \dots, v_{d-1}$. Then $n = d - 1 + k + l$. By Theorem 2.2, $og(T) = k$ and so $og(T) = n - d - l + 1$. Hence $og(T) = n - d + 1$ if and only if $l = 0$, if and only if all the internal vertices of T lie on the diametral path P, if and only if T is a caterpillar. □

Now, we prove the following realization result.

THEOREM 3.2. If n, d and k are integers such that $2 \leqslant d < n$, $2 \leqslant k < n$ and $n - d - k + 1 \geqslant 0$, then there exists a graph G of order n, diameter d and $og(G) = k$.

PROOF. Let $P_d : u_0, u_1, u_2, \dots, u_d$ be a path of length d. First, let $n-d-k+1 \geqslant 1$. Let $K_{n-d-k+1}$ be the complete graph with vertex set $\{w_1, w_2, \dots, w_{n-d-k+1}\}$. Let H be the graph obtained from P_d and $K_{n-d-k+1}$ by joining each vertex of $K_{n-d-k+1}$ to u_i for $i = 0, 1, 2$. Then we add $k - 2$ new vertices $v_1, v_2, \dots, v_{k-2}$ to H by joining each vertex $v_i (1 \leqslant i \leqslant k - 2)$ to the vertex u_1 of P_d and obtain the graph G of Figure 3.4. Then G has order n and diameter d. Let $S = \{u_0, u_d, v_1, v_2, \dots, v_{k-2}\}$ be the set of extreme vertices of G. Then it is clear that S is an open geodetic set of G and so by Theorem 2.2, $og(G) = k$.

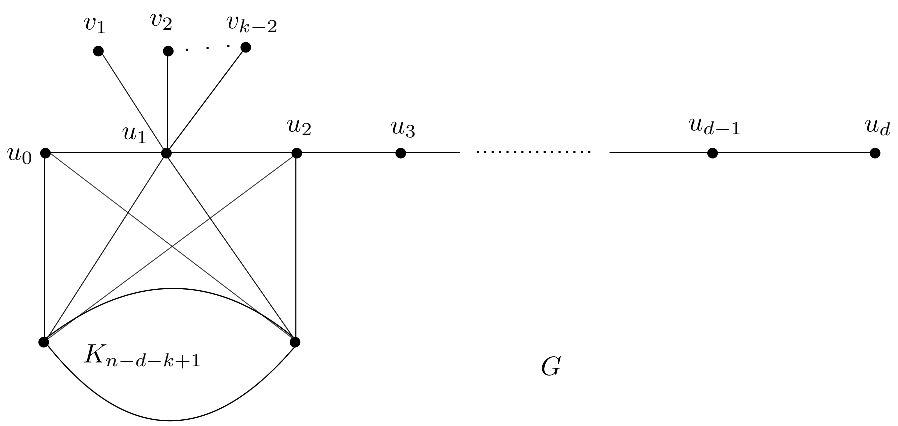


Figure 3.4

For $n - d - k + 1 = 0$, let G be the tree given in Figure 3.5. Then it is clear that G has diameter d, order $d + k - 1 = n$ and $og(G) = k$.

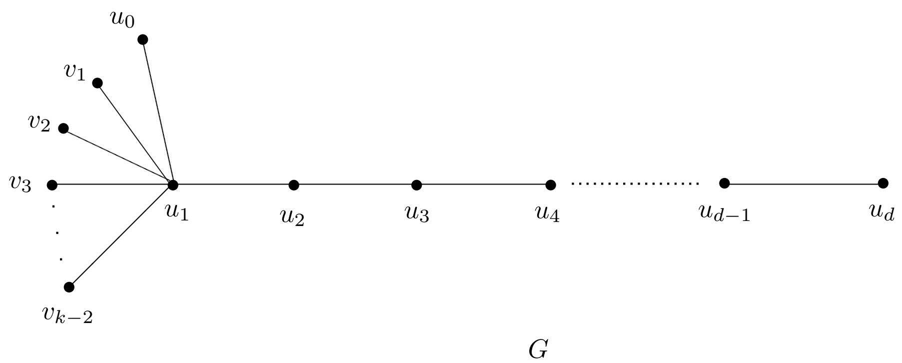


Figure 3.5

□

4. Open geodetic number and addition of a pendant edge

A fundamental question in graph theory concerns how the value of a parameter is affected by making a small change in the graph. In this section, we study how the open geodetic number of a graph is affected by the addition of a pendant edge.

THEOREM 4.1. If G' is a graph obtained by adding a pendant edge to a connected graph G, then $og(G) - 2 \leqslant og(G') \leqslant og(G) + 1$.

PROOF. Let G' be the graph obtained from G by adding a pendant edge uv, where u is not a vertex of G and v is a vertex of G. Let S' be a minimum open geodetic set of G'. Then $og(G') = |S'|$. By Theorem 2.2 $u \in S'$ and by Theorem 2.7 $v \notin S'$. We consider two cases.

Case 1. v is an extreme vertex of G.

Let $S = (S' - \{u\}) \cup \{v\}$. Then $|S| = |S'| = og(G')$. We show that S is an open geodetic set of G. Let x be a vertex of G. Suppose that x is an extreme vertex of G. If $x = v$, then $x \in S$. if $x \neq v$, then x is also an extreme vertex of G' and so $x \in S'$. Since $x \neq u, v$, we have $x \in S$. So, assume that x is not an extreme vertex of G. Then $x \neq v$. Since S' is an open geodetic set of G', $x \in I(y, z)$, where $y, z \in S'$. If $u \neq y, z$, then $x \in I(y, z)$ with $y, z \in S$. If $u = y$ or $u = z$, say $y = u$, then, since $x \neq v$ it follows that $x \in I(v, z)$, where $v, z \in S$. Thus S is an open geodetic set of G and so $og(G) \leqslant |S| = |S'| = og(G')$.

Case 2. v is not an extreme vertex of G.

Then there exist nonadjacent neighbors v', v'' of v in G and it follows that $v \in I(v', v'')$. Let $S = (S' - \{u\}) \cup \{v, v', v''\}$. Then $|S| \leqslant |S'| + 2$. We show that S is an open geodetic set of G. Let x be a vertex of G such that $x \neq v$. Suppose that x is an extreme vertex of G. Then x is also an extreme vertex of G' and so $x \in S'$. Since $x \neq u$, we have $x \in S$. So, assume that x is not an extreme vertex of G. Since $x \neq u$, it is clear that x is also not an extreme vertex of G' and so $x \in I(y, z)$ with $y, z \in S'$. Then, as in Case 1, S is an open geodetic set of G so that $og(G) \leqslant |S| \leqslant |S'| + 2 = og(G') + 2$. Thus in both cases, $og(G) - 2 \leqslant og(G')$.

For the upper bound, let S be a minimum open geodetic set of G. Since u is an extreme vertex of G', $S \cup \{u\}$ is an open geodetic set of G'. Hence $og(G') \leqslant |S \cup \{u\}| = og(G) + 1$. $\square$

Remark 4.1. The bounds in Theorem 4.1 are sharp. For the graph G given in Figure 4.1, it is easily seen that $S = \{v_1, v_3, v_4, v_5\}$ is a minimum open geodetic set of G so that $og(G) = 4$. Let G' be the graph in Figure 4.2 obtained from G by adding the pendant edge v_5v_6. Then $S' = \{v_3, v_6\}$ is a minimum open geodetic set of G' so that $og(G') = 3$. Thus $og(G) - 2 = og(G')$. For any path G of length atleast 2, we have $og(G) = 2$. Let G' be the tree obtained from G by adding the pendant edge at a cut vertex of G. Then $og(G') = 3$. Thus $og(G') = og(G) + 1$.

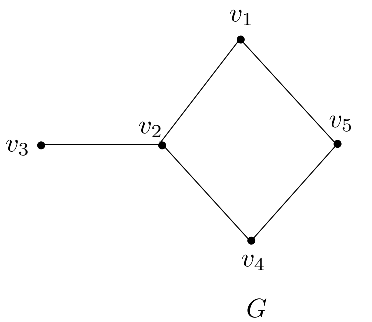


Figure 4.1

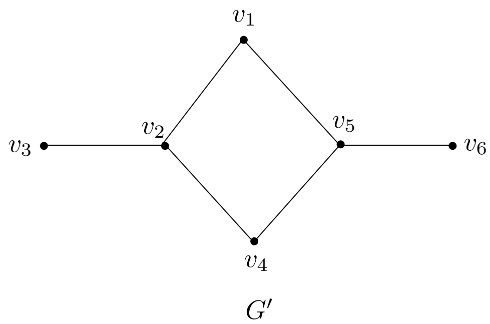


Figure 4.2

Theorem 4.2. If G' is a graph obtained from a connected graph G by adding a pendant edge uv, where u is not a vertex of G and v is a vertex of G and if $og(G') = og(G) + 1$, then v does not belong to any minimum open geodetic set of G.

Proof. Assume that v belongs to some minimum open geodetic set S of G. Let $S' = (S - \{v\}) \cup \{u\}$. Then $|S| = |S'|$. We show that S' is an open geodetic set of G'. Let $x \in V(G')$. Suppose that x is an extreme vertex of G'. Then $x \neq v$. If $x = u$, then by the definition of S', we have $x \in S'$. If $x \neq u$, then x is an extreme vertex of G and so $x \in S$. Hence it follows that $x \in S'$. If x is not an extreme vertex of G', then $x \neq u$. Hence $x \in V(G)$. If $x = v$, then $x \in I(y, u)$ for any $y \in S$ with $y \neq x$. If $x \neq v$, then, since S is an open geodetic set of G, $x \in I(y, z)$, where $y, z \in S$. If $v \neq y, z$, then $y, z \in S'$. If $v = y$ or $v = z$, say $y = v$, then $x \in I(v, z)$, where $v, z \in S$. Since v is a cut vertex of G', it follows that $x \in I(u, z)$ with $u, z \in S'$. Thus S' is an open geodetic set of G'. Hence $og(G') \leqslant |S'| = |S| = og(G)$, which is a contradiction. $\square$

Remark 4.2. The converse of Theorem 4.2 is false. For the graph G given in Figure 4.3, it is easily seen that $S = \{v_2, v_4, v_6, v_8\}$ is the unique minimum open geodetic set so that $og(G) = 4$. Let G' be the graph given in Figure 4.4, obtained from G by adding the pendent edge v_7v_9. Then $S' = \{v_2, v_4, v_9\}$ is the unique minimum open geodetic set of G' so that $og(G') = 3$. Thus $og(G') \neq og(G) + 1$ and v_7 does not belong to S.

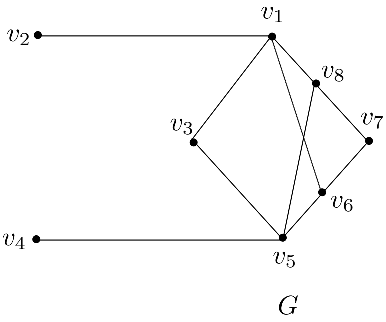


Figure 4.3

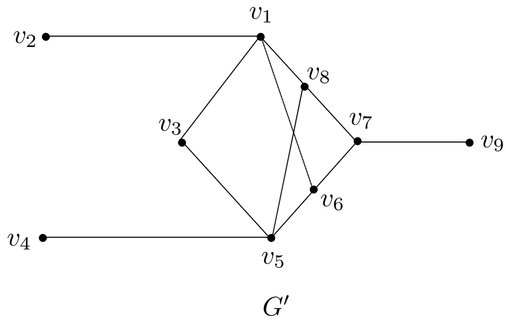


Figure 4.4

We leave the following problem as an open question.

PROBLEM 4.3. Characterize the class of graphs G for which $og(G') = og(G) + 1$, where G' is the graph obtained from G by adding a pendant edge to G.

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