SCIENTIA
Series A: *Mathematical Sciences*, Vol. 19 (2010), 25–36
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

ls-Ponomarev-systems and 1-sequence-covering mappings

Nguyen Van Dung

Abstract. In this paper, we prove that f is an 1-sequence-covering (resp., 2-sequence-covering) mapping from a locally separable metric space M onto a space X if and only if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star wsn-cover (resp., double point-star so-cover) for X, where $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ is an ls-Ponomarev-system, and investigate further properties of mappings in the ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$.

1. Introduction

Characterizing images of metric spaces under covering-mappings by spaces having certain networks is one of attracted problems in general topology. Recently, some authors were interested in finding a general method to construct a covering-mapping with metric domain. By this work, characterizations of images of metric spaces were obtained systematically. In [**1**], the authors introduced the ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$, and used this notion to give necessary and sufficient conditions such that the mapping f is a compact mapping (a compact-covering mapping, a sequence-covering mapping, a pseudo-sequence-covering mapping, a sequentially-quotient mapping) from a locally separable metric space M onto a space X. As applications of these results, the authors systematically obtained internal characterizations of certain compact images of locally separable metric spaces. Among covering-mappings, 1-sequence-covering mappings and 2-sequence-covering mappings which were introduced by S. Lin in [**11**] play important roles, and cause many attentions in [**3**], [**6**], [**7**], [**9**], [**12**], [**13**]. Thus, it is interested in finding a necessary and sufficient condition such that f is an 1-sequence-covering (2-sequence-covering) mapping for an ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$.

In this paper, we prove that f is an 1-sequence-covering (resp., 2-sequence-covering) mapping from a locally separable metric space M onto a space X if and only if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star wsn-cover (resp., double point-star

2000 *Mathematics Subject Classification.* Primary 54E35, 54E40; Secondary 54D99, 54E99.
Key words and phrases. ls-Ponomarev-system, 1-sequence-covering, 2-sequence-covering, double point-star wsn-cover, double point-star so-cover.

so-cover) for X, where $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ is an ls-Ponomarev-system, and investigate further properties of mappings in the ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$. These results make the study of covering-mappings from locally separable metric spaces more completely.

Throughout this paper, all spaces are Hausdorff, all mappings are continuous and onto, a convergent sequence includes its limit point, and $\mathbb{N}$ denotes the set of all natural numbers. Let $f : X \longrightarrow Y$ be a mapping, $\mathcal{P}$ be a family of subsets of a space X, and $K \subset X$, we denote $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$, $st(K, \mathcal{P}) = \bigcup\{P \in \mathcal{P} : P \cap K \neq \emptyset\}$, and $st(x, \mathcal{P}) = st(\{x\}, \mathcal{P})$ for every $x \in X$. We say that a convergent sequence $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ converging to x is *eventually* in a subset U of a space X if $\{x_n : n \geqslant n_0\} \cup \{x\} \subset U$ for some $n_0 \in \mathbb{N}$.

For terms are not defined here, please refer to [**4**] and [**17**].

2. 1-sequence-covering mappings in Ponomarev-systems

Definition 2.1. Let P be a subset of a space X.

(1) P is a *sequential neighborhood* of x [**5**], if for every convergent sequence S converging to x in X, S is eventually in P.

(2) P is a *sequentially open* subset of X [**5**], if for every $x \in P$, P is a sequential neighborhood of x.

Definition 2.2. Let $\mathcal{P}$ be a cover for a space X.

(1) $\mathcal{P}$ is a *network* for X [**15**], if $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ with $\mathcal{P}_x \subset \{P \in \mathcal{P} : x \in P\}$, and for every $x \in U$ with U open in X, there exists $P \in \mathcal{P}_x$ such that $x \in P \subset U$, where $\mathcal{P}_x$ is a *network at* x in X.

(2) $\mathcal{P}$ is an *sn-cover* for X [**13**], if each $P \in \mathcal{P}$ is a sequential neighborhood of some point in X, and for each $x \in X$, there exists $P \in \mathcal{P}$ such that P is a sequential neighborhood of x.

(3) For each $x \in X$, $\mathcal{P}$ is a *wsn-cover at* x in X or a $wsn(x)$*-cover* for X, if there exists $P \in \mathcal{P}$ such that P is a sequential neighborhood of x.

$\mathcal{P}$ is a *wsn-cover* for X [**7**] if, for every $x \in X$, $\mathcal{P}$ is a $wsn(x)$-cover for X.

(4) For each $x \in X$, $\mathcal{P}$ is an *so-cover at* x in X or an $so(x)$*-cover* for X, if for each $P \in \mathcal{P}$ with $x \in P$, P is a sequential neighborhood of x.

$\mathcal{P}$ is an *so-cover* for X [**13**], if every element of $\mathcal{P}$ is a sequentially open subset of X. It is easy to see that $\mathcal{P}$ is an *so*-cover for X if and only if, for every $x \in X$, $\mathcal{P}$ is an $so(x)$-cover for X.

(5) $\mathcal{P}$ is a *point-star network* for X, if $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ where each $\mathcal{P}_n$ is a cover for X and, for each $x \in X$, $\{st(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x in X. Note that a point-star network is a *σ-strong network* in the sense of [**8**], and $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *point-star network* in the sense of [**14**].

Definition 2.3. Let $f : X \longrightarrow Y$ be a mapping.

(1) For each $y \in Y$, f is an $1(y)$-*sequence-covering* mapping or an *1-sequence-covering mapping at* y, if there exists $x_y \in f^{-1}(y)$ such that whenever $\{y_n : n \in \mathbb{N}\}$ is a sequence converging to y in Y there exists a sequence $\{x_n : n \in \mathbb{N}\}$ converging to x_y in X with each $x_n \in f^{-1}(y_n)$.

f is an *1-sequence-covering* mapping [**11**], if f is an $1(y)$-sequence-covering mapping for every $y \in Y$.

(2) For each $y \in Y$, f is a $2(y)$-*sequence-covering* mapping or a *2-sequence-covering mapping at* y, if whenever $x_y \in f^{-1}(y)$ and $\{y_n : n \in \mathbb{N}\}$ is a sequence converging to y in Y there exists a sequence $\{x_n : n \in \mathbb{N}\}$ converging to x_y in X with each $x_n \in f^{-1}(y_n)$.

f is a *2-sequence-covering* mapping [**11**], if f is a $2(y)$-sequence-covering mapping for every $y \in Y$.

(3) f is an *s-mapping* (resp., *compact* mapping) [**2**] if, for each $y \in Y$, $f^{-1}(y)$ is a separable (resp., compact) subset of X.

(4) f is an *ss-mapping* [**10**] if, for each $y \in Y$, there exists a neighborhood U of y in Y such that $f^{-1}(U)$ is a separable subset of X.

(5) f is a *cs-mapping* [**16**] if, for each compact subset K of Y, $f^{-1}(K)$ is a separable subset of X.

(6) f is a π-*mapping* [**2**] if, for each $y \in Y$ and each neighborhood U of y in Y, $d(f^{-1}(y), X - f^{-1}(U)) > 0$, where X is a metric space with a metric d.

Lemma 2.1. *Let $f : X \longrightarrow Y$ be an 1-sequence-covering mapping at y in Y, and U be a sequential neighborhood of x_y in X. Then $f(U)$ is a sequential neighborhood of y in Y.*

Proof. Let $\{y_n : n \in \mathbb{N}\}$ be a sequence converging to y in Y. Then there exists a sequence $\{x_n : n \in \mathbb{N}\}$ converging to x_y in X with each $x_n \in f^{-1}(y_n)$. Since U is a neighborhood of x_y, $\{x_n : n \in \mathbb{N}\} \cup \{x_y\}$ is eventually in U. It implies that $\{y_n : n \in \mathbb{N}\} \cup \{y\}$ is eventually in $f(U)$. This proves that $f(U)$ is a sequential neighborhood of y. □

Lemma 2.2. *Let $f : X \longrightarrow Y$ be a 2-sequence-covering mapping, and U be a sequentially open subset of X. Then the following hold.*

(1) *$f(U)$ is a sequentially open subset of Y.*
(2) *$f|_U : U \longrightarrow f(U)$ is a 2-sequence-covering mapping.*

Proof. (1). For each $y \in f(U)$, let $\{y_n : n \in \mathbb{N}\}$ be a sequence converging to y in Y. Pick some $x_y \in f^{-1}(y) \cap U$, then there exists a sequence $\{x_n : n \in \mathbb{N}\}$ converging to x_y in X with each $x_n \in f^{-1}(y_n)$. Since U is sequentially open, $\{x_n : n \in \mathbb{N}\} \cup \{x_y\}$ is eventually in U. It implies that $\{y_n : n \in \mathbb{N}\} \cup \{y\}$ is eventually in $f(U)$. This proves that $f(U)$ is a sequential neighborhood of y for every $y \in f(U)$, hence $f(U)$ is sequentially open in Y.

(2). For each $y \in f(U)$, let $\{y_n : n \in \mathbb{N}\}$ be a sequence converging to y in $f(U)$. For each $x_y \in f^{-1}(y) \cap U = f|_U^{-1}(y)$, there exists a sequence $\{x_n : n \in \mathbb{N}\}$ converging to x_y in X. Since U is sequentially open, there exists $n_0 \in \mathbb{N}$ such that $\{x_n : n \geqslant n_0\} \subset U$. For each $n < n_0$, since $y_n \in f(U)$, there exists $x'_n \in f^{-1}(y_n) \cap U$. Put $t_n = x'_n$ if $n < n_0$ and $t_n = x_n$ if $n \geqslant n_0$. Then $\{t_n : n \in \mathbb{N}\}$ is a sequence in U converging to x_y with each $t_n \in f^{-1}(y_n) \cap U = f|_U^{-1}(y_n)$. It implies that $f|_U$ is a 2-sequence-covering mapping. □

Definition 2.4. Let $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a point-star network for X. For every $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P_\alpha : \alpha \in A_n\}$, and endowed A_n with discrete topology. Put

$$M = \Big\{a = (\alpha_n) \in \prod_{n\in\mathbb{N}} A_n : \{P_{\alpha_n} : n \in \mathbb{N}\} \text{ forms a network at some point } x_a \text{ in } X\Big\}.$$

Then M, which is a subspace of the product space $\prod_{n\in\mathbb{N}} A_n$, is a metric space, x_a is unique, and $x_a = \bigcap_{n\in\mathbb{N}} P_{\alpha_n}$ for every $a \in M$. Define $f : M \to X$ by $f(a) = x_a$, then f is a mapping and $(f, M, X, \{\mathcal{P}_n\})$ is a *Ponomarev-system* [**14**].

In [**7**, Theorem 3.10], for a Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$, Y. Ge and S. Lin proved that f is an 1-sequence-covering mapping if and only if each $\mathcal{P}_n$ is a *wsn*-cover for X. Now, we modify this result as follows.

Proposition 2.1. *Let $(f, M, X, \{\mathcal{P}_n\})$ be a Ponomarev-system. Then the following are equivalent for every point $x \in X$.*

(1) *f is an $1(x)$-sequence-covering mapping.*
(2) *Each $\mathcal{P}_n$ is a $wsn(x)$-cover for X.*

Proof. (1) ⇒ (2). Let f be an $1(x)$-sequence-covering mapping. There exists $a_x \in f^{-1}(x)$ such that whenever $\{x_n : n \in \mathbb{N}\}$ is a sequence converging to x in X there exists a sequence $\{a_n : n \in \mathbb{N}\}$ converging to a_x in M with each $a_n \in f^{-1}(x_n)$. Let $a_x = (\alpha_n)$, we shall prove that P_{α_k} is a sequential neighborhood of x for every $k \in \mathbb{N}$. Whenever $\{y_i : i \in \mathbb{N}\}$ is a sequence converging to x in X, there exists a sequence $\{b_i : i \in \mathbb{N}\}$ converging to a_x in M with each $b_i \in f^{-1}(y_i)$. Put $U_k = \{b = (\beta_n) \in M : \beta_k = \alpha_k\}$, then U_k is an open neighborhood of a_x in M. So the sequence $\{b_i : i \in \mathbb{N}\} \cup \{a_x\}$ is eventually in U_k, hence the sequence $\{y_i : i \in \mathbb{N}\} \cup \{x\}$ is eventually in $f(U_k) \subset P_{\alpha_k}$. This proves that P_{α_k} is a sequential neighborhood of x.

(2) ⇒ (1). Let each $\mathcal{P}_n$ be a $wsn(x)$-cover for X. For each $n \in \mathbb{N}$, there exists $P_{\alpha_n} \in \mathcal{P}_n$ such that P_{α_n} is a sequential neighborhood of x. Then $\{P_{\alpha_n} : n \in \mathbb{N}\}$ forms a network at x in X. Let $\{x_i : i \in \mathbb{N}\}$ be a sequence converging to x in X. Then $\{x_i : i \in \mathbb{N}\} \cup \{x\}$ is eventually in each P_{α_n}. For each $i \in \mathbb{N}$, put $\alpha_{in} = \alpha_n$ if $x_i \in P_{\alpha_n}$, otherwise, pick some $\alpha_{in} \in A_n$ such that $x_i \in P_{\alpha_{in}}$. Then $\{P_{\alpha_{in}} : n \in \mathbb{N}\}$ forms a network at x_i in X. Put $a_i = (\alpha_{in})$ and $a_x = (\alpha_n)$, we find that $a_i \in f^{-1}(x_i)$ and $a_x \in f^{-1}(x)$. Since $\{x_i : i \in \mathbb{N}\} \cup \{x\}$ is eventually in P_{α_n}, there exists $n(i)$ such that $\alpha_{in} = \alpha_n$ for every $i \geqslant n(i)$. This proves that $\{a_i : i \in \mathbb{N}\}$ is a sequence converging to a_x in M. Then f is an $1(x)$-sequence-covering mapping. □

Next, we state a necessary and sufficient condition such that f is a 2-sequence-covering mapping at $x \in X$ for a Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$.

PROPOSITION 2.2. *Let $(f, M, X, \{\mathcal{P}_n\})$ be a Ponomarev-system. Then the following are equivalent for every $x \in X$.*

(1) *f is a $2(x)$-sequence-covering mapping.*
(2) *Each $\mathcal{P}_n$ is an $so(x)$-cover for X.*

PROOF. (1) $\Rightarrow$ (2). Let f be a $2(x)$-sequence-covering mapping. For each $x \in P \in \mathcal{P}_n$, there exists $a = (\alpha_n) \in M$ such that $P_{\alpha_n} = P$. As in the proof (1) $\Rightarrow$ (2) of Proposition 2.1, we find that P is a sequential neighborhood of x in X. This proves that $\mathcal{P}_n$ is an $so(x)$-cover for X.

(2) $\Rightarrow$ (1). Let $\mathcal{P}$ be an $so(x)$-network for X. For each $a_x = (\alpha_n) \in f^{-1}(x)$, we find that each P_{α_n} is a sequential neighborhood of x in X. As in the proof (2) $\Rightarrow$ (1) of Proposition 2.1, we find that for each sequence $\{x_i : i \in \mathbb{N}\}$ converging to x in X there exists a sequence $\{a_i : i \in \mathbb{N}\}$ converging to a_x in M with each $a_i \in f^{-1}(x_i)$. This proves that f is a $2(x)$-sequence-covering mapping. □

COROLLARY 2.1. *Let $(f, M, X, \{\mathcal{P}_n\})$ be a Ponomarev-system. Then the following are equivalent.*

(1) *f is a 2-sequence-covering mapping.*
(2) *Each $\mathcal{P}_n$ is an so-cover for X.*

3. 1-sequence-covering mappings in ls-Ponomarev-systems

The notions of double point-star cover (double point-star cs-cover, double point-star cs^*-cover, double point-star cfp-cover, double point-star wcs-cover) have been introduced and investigated in [**1**]. In this section, we investigate further properties of the ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$, and introduce the notion of double point-star wsn-cover (resp., double point-star so-cover) to state a necessary and sufficient condition such that f is an 1-sequence-covering (resp., 2-sequence-covering) mapping.

DEFINITION 3.1. Let $\{X_\lambda : \lambda \in \Lambda\}$ be a cover for a space X such that each X_λ has a sequence of covers $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$.

(1) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a *double point-star cover* for X [**1**], if, for every $\lambda \in \Lambda$, $\bigcup\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a point-star network for X_λ consisting of countable covers $\mathcal{P}_{\lambda,n}$.

(2) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a *point-countable* (resp., *point-finite, locally countable, compact-countable*) *double point-star cover* for X, if $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star cover for X such that $\{X_\lambda : \lambda \in \Lambda\}$ and each $\mathcal{P}_{\lambda,n}$ is point-countable (resp., point-finite, locally countable, compact-countable). Note that each $\mathcal{P}_{\lambda,n}$ is countable, then it is obviously point-countable (locally countable, compact-countable).

Definition 3.2. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double point-star cover for X.

(1) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a *double point-star wsn-cover* for X if, for each $x \in X$, there exists $\lambda \in \Lambda$ such that X_λ is a sequential neighborhood of x and each $\mathcal{P}_{\lambda,n}$ is a *wsn*-cover at x in X_λ.

(2) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a *point-countable* (resp., *point-finite*, *locally countable*, *compact-countable*) *double point-star wsn-cover* for X, if it is a double point-star *wsn*-cover for X and a point-countable (resp., point-finite, locally countable, compact-countable) double point-star cover for X.

(3) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a *double point-star so-cover* for X, if $\{X_\lambda : \lambda \in \Lambda\}$ is an *so*-cover for X, and each $\mathcal{P}_{\lambda,n}$ is an *so*-cover for X_λ.

(4) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a *point-countable* (resp., *point-finite*, *locally countable*, *compact-countable*) *double point-star so-cover* for X, if it is a double point-star *so*-cover for X and a point-countable (resp., point-finite, locally countable, compact-countable) double point-star cover for X.

Definition 3.3. Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double point-star cover for a space X, and $(f_\lambda, M_\lambda, X_\lambda, \{\mathcal{P}_{\lambda,n}\})$ be the Ponomarev-system for each $\lambda \in \Lambda$. Since each $\mathcal{P}_{\lambda,n}$ is countable, M_λ is a separable metric space. Put $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, and $f = \bigoplus_{\lambda\in\Lambda} f_\lambda$. Then M is a locally separable metric space, and f is a mapping from a locally separable metric space M onto X. The system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ is an *ls-Ponomarev-system* [**1**].

In [**1**], a necessary and sufficient condition such that f is a compact mapping for an ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ has been obtained as follows.

Lemma 3.1 ([**1**], Theorem 2.15). *Let $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ be an ls-Ponomarev-system. Then the following are equivalent.*

(1) *f is a compact mapping.*
(2) *$\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a point-finite double point-star cover for X.*

Proof. (1) $\Rightarrow$ (2) Let f be a compact mapping. For each $x \in X$, since $f^{-1}(x)$ is compact, $\{\lambda \in \Lambda : f^{-1}(x) \cap M_\lambda \neq \emptyset\}$ is finite. Then $\{\lambda \in \Lambda : x \in X_\lambda\}$ is finite, i.e., $\{X_\lambda : \lambda \in \Lambda\}$ is point-finite. On the other hand, for each $\lambda \in \Lambda$, since $f_\lambda^{-1}(x) = f^{-1}(x) \cap M_\lambda$ is compact, f_λ is a compact mapping. Then each $\mathcal{P}_{\lambda,n}$ is point-finite by [**7**, Theorem 3.7.(1)]. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a point-finite double point-star cover for X.

(2) $\Rightarrow$ (1). Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a point-finite double point-star cover for X. For each $x \in X$, $\Lambda_x = \{\lambda \in \Lambda : x \in X_\lambda\}$ is finite by point-finiteness of $\{X_\lambda : \lambda \in \Lambda\}$. Since each $\mathcal{P}_{\lambda,n}$ is point-finite, $f_\lambda^{-1}(x)$ is compact by [**7**, Theorem 3.7.(1)]. It implies that $f^{-1}(x) = \bigcup\{f_\lambda^{-1}(x) : \lambda \in \Lambda_x\}$ is compact, then f is a compact mapping. □

In the next, we state necessary and sufficient conditions such that f is an s-mapping (ss-mapping, cs-mapping) for an ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$.

PROPOSITION 3.1. *Let $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ be an ls-Ponomarev-system. Then the following are equivalent.*

(1) *f is an s-mapping (resp., ss-mapping, cs-mapping).*
(2) *$\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a point-countable (resp., locally countable, compact-countable) double point-star cover for X.*

PROOF. (1) $\Rightarrow$ (2). Let $f : X \longrightarrow Y$ be an s-mapping. For each $x \in X$, we find that $\{\lambda \in \Lambda : f^{-1}(x) \cap M_\lambda \neq \emptyset\}$ is countable. Then $\{\lambda \in \Lambda : x \in X_\lambda\}$ is countable, hence $\{X_\lambda : \lambda \in \Lambda\}$ is point-countable. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a point-countable double point-star cover for X.

For the parenthetic part, let f be an ss-mapping. For each $x \in X$, there exists an open neighborhood U of x such that $f^{-1}(U)$ is separable. Then $\{\lambda \in \Lambda : f^{-1}(U) \cap M_\lambda \neq \emptyset\}$ is countable, hence $\{\lambda \in \Lambda : U \cap X_\lambda \neq \emptyset\}$ is countable. Thus, $\{X_\lambda : \lambda \in \Lambda\}$ is locally countable. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is locally countable double point-star cover for X.

Let f be a cs-mapping. For each compact subset K of X, we find that $f^{-1}(K)$ is separable. Then $\{\lambda \in \Lambda : f^{-1}(K) \cap M_\lambda \neq \emptyset\}$ is countable, hence $\{\lambda \in \Lambda : K \cap X_\lambda \neq \emptyset\}$ is countable. Thus, $\{X_\lambda : \lambda \in \Lambda\}$ is compact-countable. It implies that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is compact-countable double point-star cover for X.

(2) $\Rightarrow$ (1). Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a point-countable double point-star cover for X. For each $x \in X$, we find that $\Lambda_x = \{\lambda \in \Lambda : x \in X_\lambda\}$ is countable. Note that each $f_\lambda^{-1}(x)$ is separable. Then $f^{-1}(x) = \bigcup\{f_\lambda^{-1}(x) : \lambda \in \Lambda\}$ is separable. This proves that f is an s-mapping.

For the parenthetic part, let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a locally countable double point-star cover for X. For each $x \in X$, there exists a neighborhood U of x such that $\{\lambda \in \Lambda : U \cap X_\lambda \neq \emptyset\}$ is countable. Then $\{\lambda \in \Lambda : f^{-1}(U) \cap M_\lambda \neq \emptyset\}$ is countable. Since each $f^{-1}(U) \cap M_\lambda$ is separable, $f^{-1}(U)$ is separable. This proves that f is an ss-mapping.

Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a compact-countable double point-star cover for X. For each compact subset K of X, we find that $\{\lambda \in \Lambda : K \cap X_\lambda \neq \emptyset\}$ is countable. Then $\{\lambda \in \Lambda : f^{-1}(K) \cap M_\lambda \neq \emptyset\}$ is countable. Since each $f^{-1}(K) \cap M_\lambda$ is separable, $f^{-1}(K)$ is separable. This proves that f is a cs-mapping. □

For an ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$, necessary and sufficient conditions such that f is a compact-covering (sequence-covering, pseudo-sequence-covering, sequentially-quotient) mapping have been obtained by means of certain double point-star covers in [**1**]. In the next, we state a necessary and sufficient condition such that f is an 1-sequence-covering (resp., 2-sequence-covering) mapping by means of double point-star wsn-covers (resp., double point-star so-covers).

THEOREM 3.1. *Let $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ be an ls-Ponomarev-system. Then the following are equivalent.*

(1) *f is an 1-sequence-covering (resp., 2-sequence-covering) mapping.*

(2) $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ *is a double point-star wsn-cover (resp., double point-star so-cover) for* X.

Proof. (1) $\Rightarrow$ (2). Let f be an 1-sequence-covering mapping. For each $x \in X$, there exists $a_x \in f^{-1}(x)$ such that whenever $\{x_n : n \in \mathbb{N}\}$ is a sequence converging to x in X there exists a sequence $\{a_n : n \in \mathbb{N}\}$ converging to a_x in M with each $a_n \in f^{-1}(x_n)$. Let $a_x \in M_\lambda$ for some $\lambda \in \Lambda$. If $\{y_n : n \in \mathbb{N}\}$ is a sequence converging to x in X, then there exists a sequence $\{b_n : n \in \mathbb{N}\}$ converging to a_x in M with each $b_n \in f^{-1}(y_n)$. Since M_λ is open in M, $\{b_n : n \in \mathbb{N}\} \cup \{a_x\}$ is eventually in M_λ. It implies that $\{y_n : n \in \mathbb{N}\} \cup \{x\}$ is eventually in X_λ. Then X_λ is a sequential neighborhood of x.

For each sequence $\{x_n : n \in \mathbb{N}\}$ converging to x in X_λ, there exists a sequence $\{a_n : n \in \mathbb{N}\}$ converging to a_x in M with each $a_n \in f^{-1}(x_n)$. Since M_λ is open, there exists $n_0 \in \mathbb{N}$ such that $\{a_n : n \geqslant n_0\} \subset M_\lambda$. For each $n < n_0$, since $x_n \in X_\lambda$, there exists some $a'_n \in M_\lambda \cap f^{-1}(x_n)$. Put $b_n = a'_n$ if $n < n_0$, and $b_n = a_n$ if $n \geqslant n_0$. Then $\{b_n : n \in \mathbb{N}\}$ is a sequence in M_λ converging to a_x with each $b_n \in f_\lambda^{-1}(x_n)$. It implies that f_λ is an $1(x)$-sequence-covering mapping in X_λ. Then each $\mathcal{P}_{\lambda,n}$ is a *wsn*-cover at x in X_λ by Proposition 2.1.

By the above, $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star *wsn*-cover for X.

For the parenthetic part, let f be a 2-sequence-covering mapping. By Lemma 2.2, $\{X_\lambda : \lambda \in \Lambda\}$ is an *so*-cover for X, and each $f_\lambda = f|_{M_\lambda}$ is a 2-sequence-covering mapping. It follows from Corollary 2.1 that each $\mathcal{P}_{\lambda,n}$ is an *so*-cover for X_λ. Then $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star *so*-cover for X.

(2) $\Rightarrow$ (1). Let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double point-star *wsn*-cover for X. For each $x \in X$, there exists $\lambda \in \Lambda$ such that X_λ is a sequential neighborhood of x and each $\mathcal{P}_{\lambda,n}$ is a *wsn*-cover at x in X_λ. It follows from Proposition 2.1 that f_λ is an $1(x)$-sequence-covering mapping. Then there exists $a_x \in f_\lambda^{-1}(x)$ such that whenever $\{x_n : n \in \mathbb{N}\}$ is a sequence converging to x in X_λ there exists a sequence $\{a_n : n \in \mathbb{N}\}$ converging to a_x in M_λ with each $a_n \in f_\lambda^{-1}(x_n)$. For each sequence $\{y_n : n \in \mathbb{N}\}$ converging to x in X, there exists $n_0 \in \mathbb{N}$ such that $\{y_n : n \geqslant n_0\}$ is a sequence in X_λ. Then there exists a sequence $\{b_n : n \geqslant n_0\}$ converging to a_x in M_λ with $b_n \in f_\lambda^{-1}(y_n)$ for every $n \geqslant n_0$. For each $n < n_0$, pick some $b'_n \in f^{-1}(y_n)$, and put $c_n = b'_n$ if $n < n_0$ and $c_n = b_n$ if $n \geqslant n_0$. Then $\{c_n : n \in \mathbb{N}\}$ is a sequence converging to a_x in M with each $c_n \in f^{-1}(y_n)$. It implies that f is an 1-sequence-covering mapping.

For the parenthetic part, let $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ be a double point-star *so*-cover for X. For each $x \in X$, put $\Lambda_x = \{\lambda \in \Lambda : x \in X_\lambda\}$. We find that $f^{-1}(x) = \bigcup\{f_\lambda^{-1}(x) : \lambda \in \Lambda_x\}$. For each $a_x \in f^{-1}(x)$, there exists $\lambda \in \Lambda_x$ such that $a_x \in f_\lambda^{-1}(x)$. If $\{x_n : n \in \mathbb{N}\}$ is a sequence converging to x in X, there exists $n_0 \in \mathbb{N}$ such that $\{x_n : n \geqslant n_0\} \subset X_\lambda$. Since each $\mathcal{P}_{\lambda,n}$ is an *so*-cover for X_λ, f_λ is a 2-sequence-covering mapping by Corollary 2.1. Then there exists a sequence $\{a_n : n \geqslant n_0\}$ converging to a_x in M_λ with each $a_n \in f_\lambda^{-1}(x_n)$. For each $n < n_0$, pick some $a'_n \in f^{-1}(x_n)$. Put $b_n = a'_n$ if $n < n_0$ and $b_n = a_n$ if $n \geqslant n_0$. Then $\{b_n : n \in \mathbb{N}\}$ is a sequence in M

converging to a_x with each $b_n \in f^{-1}(x_n)$. It implies that f is a 2-sequence-covering mapping. □

By Lemma 3.1, Proposition 3.1, and Theorem 3.1, we get the following.

Corollary 3.1. *Let $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ be an ls-Ponomarev-system. Then the following are equivalent, where "1-sequence-covering" and "double point-star wsn-cover" can be replaced by "2-sequence-covering" and "double point-star so-cover", respectively.*

(1) *f is an 1-sequence-covering s-mapping (resp., ss-mapping, cs-mapping, compact mapping).*
(2) *$\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a point-countable (resp., locally countable, compact-countable, point-finite) double point-star wsn-cover for X.*

Next, we get a new characterization for 1-sequence-covering (2-sequence-covering) compact images of locally separable metric spaces.

Corollary 3.2. *The following are equivalent for a space X.*

(1) *X is an 1-sequence-covering (resp., 2-sequence-covering) compact image of a locally separable metric space.*
(2) *X has a point-finite double point-star wsn-cover (resp., point-finite double point-star so-cover).*

Proof. (1) ⇒ (2). Let $f : M \longrightarrow X$ be an 1-sequence-covering compact mapping from a locally separable metric space M onto X. Since M is a locally separable metric space, $M = \oplus_{\lambda\in\Lambda} M_\lambda$ where each M_λ is a separable metric space by [**4**, 4.4.F]. Since each M_λ is a separable metric space, M_λ has a sequence of open countable covers $\{\mathcal{B}_{\lambda,n} : n \in \mathbb{N}\}$ such that, for each compact subset K of M_λ and each open subset U of M_λ with $K \subset U$, there exists $n \in \mathbb{N}$ satisfying $st(K, \mathcal{B}_{\lambda,n}) \subset U$ by [**4**, 5.4.E]. Let $\mathcal{C}_{\lambda,n}$ be a locally finite open refinement of each $\mathcal{B}_{\lambda,n}$. Then, for each $\lambda \in \Lambda$, $\{\mathcal{C}_{\lambda,n} : n \in \mathbb{N}\}$ is a sequence of locally finite open countable covers for M_λ such that, for each compact subset K of M_λ and each open subset U of M_λ with $K \subset U$, there exists $n \in \mathbb{N}$ satisfying $st(K, \mathcal{C}_{\lambda,n}) \subset U$. For each $\lambda \in \Lambda$ and $n \in \mathbb{N}$, put$X_\lambda = f(M_\lambda)$ and $\mathcal{P}_{\lambda,n} = f(\mathcal{C}_{\lambda,n})$. We get following facts (a)-(e).

(a) $\{X_\lambda : \lambda \in \Lambda\}$ is a cover for X.

(b) Each $\mathcal{P}_{\lambda,n}$ is countable.

(c) For each $\lambda \in \Lambda$, $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a point-star network for X_λ.

Let $x \in U$ with U open in X_λ, then $x \in V$ with V open in X and $V \cap X_\lambda = U$. Since f is compact, $f_\lambda^{-1}(x) = f^{-1}(x) \cap M_\lambda$ is a compact subset of M_λ and $f_\lambda^{-1}(x) \subset V_\lambda$ with $V_\lambda = f^{-1}(V) \cap M_\lambda$ open in M_λ. Then there exists $n \in \mathbb{N}$ such that $st(f_\lambda^{-1}(x), \mathcal{C}_{\lambda,n}) \subset V_\lambda$. It implies that $st(x, \mathcal{P}_{\lambda,n}) \subset f(f^{-1}(V) \cap M_\lambda) \subset V \cap X_\lambda = U$. Then $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a point-star network for X_λ.

(d) $\{X_\lambda : \lambda \in \Lambda\}$ is point-finite.

For each $x \in X$, since $f^{-1}(x)$ is compact, $f^{-1}(x)$ meets only finitely many M_λ's. It implies that $\{X_\lambda : \lambda \in \Lambda\}$ is point-finite.

(e) Each $\mathcal{P}_{\lambda,n}$ is point-finite.

For each $x \in X_\lambda$, since f is compact, $f_\lambda^{-1}(x) = f^{-1}(x) \cap M_\lambda$ is a compact subset of M_λ. Then $f_\lambda^{-1}(x)$ meets only finitely many members of $\mathcal{C}_{\lambda,n}$ by locally finiteness of $\mathcal{C}_{\lambda,n}$ for every $n \in \mathbb{N}$. It implies that x meets only finitely many members of each $\mathcal{C}_{\lambda,n}$. Therefore, $\mathcal{P}_{\lambda,n}$ is point-finite.

From (a)-(e) we find that $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a point-finite double point-star cover for X. Let $x \in X$, there exists $a_x \in M$ such that whenever $\{x_n : n \in \mathbb{N}\}$ is a sequence converging to x in X there exists a sequence $\{a_n : n \in \mathbb{N}\}$ converging to a_x in M with $a_n \in f^{-1}(x_n)$. We find that $a_x \in M_\lambda$ for some $\lambda \in \Lambda$. Since M_λ is open and $\mathcal{C}_{\lambda,n}$ is an open cover for M_λ for every $n \in \mathbb{N}$, X_λ is a sequential neighborhood of x and $\mathcal{P}_{\lambda,n}$ is a *wsn*-cover for X_λ by Lemma 2.1.

For the parenthetic part, let f be a 2-sequence-covering compact mapping. Then the point-finite double point-star cover $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a double point-star *so*-cover for X by Lemma 2.2.

By the above, $\{(X_\lambda, \{\mathcal{P}_{\lambda,n}\}) : \lambda \in \Lambda\}$ is a point-finite double point-star *wsn*-cover (resp., point-finite double point-star *so*-cover) for X.

(2) $\Rightarrow$ (1). By Lemma 3.1 and Theorem 3.1. □

For a Ponomarev-system $(f, M, X, \{\mathcal{P}_n\})$, the following is well-known.

LEMMA 3.2 ([**18**], Lemma 2.2). *Let $(f, M, X, \{\mathcal{P}_n\})$ be a Ponomarev-system. Then f is a π-mapping.*

Next, we prove a sufficient condition such that f is a π-mapping for an *ls*-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$.

PROPOSITION 3.2. *Let $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ be an ls-Ponomarev-system. If $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a point-star network for X, where $\mathcal{P}_n = \bigcup\{\mathcal{P}_{\lambda,n} : \lambda \in \Lambda\}$ for every $n \in \mathbb{N}$, then f is a π-mapping.*

PROOF. Let $x \in U$ with U open in X. Since $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a point-star network for X, $st(x, \mathcal{P}_n) \subset U$ for some $n \in \mathbb{N}$. For each $\lambda \in \Lambda$ with $x \in X_\lambda$ we find that $st(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$ where $U_\lambda = U \cap X_\lambda$. If $a = (\alpha_i) \in M_\lambda$ such that $d(f^{-1}(x), a) < \frac{1}{2^n}$, there exists $b = (\beta_i) \in f_\lambda^{-1}(x)$ such that $d_\lambda(a, b) < \frac{1}{2^n}$, where d and d_λ are metrics on M and M_λ, respectively. Therefore, $\alpha_i = \beta_i$ if $i \leqslant n$. It implies that $x \in P_{\alpha_n} = P_{\beta_n} \subset st(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$, hence $a \in f^{-1}(P_{\alpha_n}) \subset f_\lambda^{-1}(U_\lambda)$. This proves that $d_\lambda(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) \geqslant \frac{1}{2^n}$. Then

$$\begin{aligned} d(f^{-1}(x), M - f^{-1}(U)) &= \inf\{d(a, b) : a \in f^{-1}(x), b \in M - f^{-1}(U)\} \\ &= \min\left\{1, \inf\{d_\lambda(a, b) : a \in f_\lambda^{-1}(x), b \in M_\lambda - f_\lambda^{-1}(U_\lambda), \lambda \in \Lambda\}\right\} \geqslant \frac{1}{2^n} > 0. \end{aligned}$$

It implies that f is a π-mapping. □

Finally, we give an example to prove that the inverse implication of Proposition 3.2 does not hold.

Example 3.1. There exists an ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ such that the following hold.

(1) f is a compact mapping.
(2) $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is not a point-star network for X.

Proof. Let $X = \{x, y\}$ be a discrete space. Put $X_1 = X_2 = X$, and put $\mathcal{P}_{1,1} = \mathcal{P}_{2,2} = \{\{x\}, \{y\}\}$ and $\mathcal{P}_{1,n} = \{X\}$ if $n \geqslant 2$, $\mathcal{P}_{2,n} = \{X\}$ if $n \neq 2$. We find that $\bigcup\{\mathcal{P}_{1,n} : n \in \mathbb{N}\}$ is a point-star network for X_1, and $\bigcup\{\mathcal{P}_{2,n} : n \in \mathbb{N}\}$ is a point-star network for X_2. Then the ls-Ponomarev-system $(f, M, X, \{\mathcal{P}_{\lambda,n}\})$ exists, where $\{X_\lambda : \lambda \in \Lambda\} = \{X_1, X_2\}$.

(1). f is a compact mapping.

For each $x \in X$, we find that $f_1^{-1}(x) = (\alpha_n)$ where $\alpha_1 = \alpha$ and $\alpha_n = \gamma$ for $n \geqslant 2$, and $f_2^{-1}(x) = (\alpha_n)$ where $\alpha_2 = \alpha$ and $\alpha_n = \gamma$ for $n \neq 2$. Then $f^{-1}(x) = f_1^{-1}(x) \cup f_2^{-1}(x)$ is a compact subset of M. On the other hand, $f_1^{-1}(y) = (\beta_n)$ where $\beta_1 = \beta$ and $\beta_n = \gamma$ for $n \geqslant 2$, and $f_2^{-1}(y) = (\beta_n)$ where $\beta_2 = \beta$ and $\beta_n = \gamma$ for $n \neq 2$. Then $f^{-1}(y) = f_1^{-1}(y) \cup f_2^{-1}(y)$ is a compact subset of M. It implies that f is a compact mapping.

(2). $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is not a point-star network for X.

We find that $\mathcal{P}_1 = \mathcal{P}_2 = \{\{x\}, \{y\}, X\}$, and $\mathcal{P}_n = \{X\}$ if $n \geqslant 2$. Then $st(x, \mathcal{P}_n) = X$ for every $n \in \mathbb{N}$. This proves that $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is not a point-star network for X. □

Remark 3.1. Since every compact mapping from a metric space is a π-mapping, Example 3.1 shows that the inverse implication of Proposition 3.2 does not hold.

Acknowledgment. The author would like to thank Prof. Y. Ge for giving him useful references to do this paper.

References

[1] T. V. An and N. V. Dung, *ls-Ponomarev-systems and compact images of locally separable metric spaces*, Methods Funct. Anal. Topology **15** (2009), no. 4, 391 – 400.
[2] A. V. Arhangel'skii, *Mappings and spaces*, Russian Math. Surveys **21** (1966), 115 – 162.
[3] N. V. Dung, *On sequence-covering msss-images of locally separable metric spaces*, Lobachevskii J. Math. **30** (2009), no. 1, 67 – 75.
[4] R. Engelking, *General topology*, Sigma series in pure mathematics, vol. 6, Heldermann Verlag, Berlin, 1988.
[5] S. P. Franklin, *Spaces in which sequences suffice*, Fund. Math. **57** (1965), 107 – 115.
[6] Y. Ge, *Weak forms of open mappings and strong forms of sequence-covering mappings*, Mat. Vesnik **59** (2007), 1 – 8.
[7] Y. Ge and S. Lin, *On Ponomarev-systems*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **10** (2007), 455 – 467.
[8] Y. Ikeda, C. Liu, and Y. Tanaka, *Quotient compact images of metric spaces, and related matters*, Topology Appl. **122** (2002), 237 – 252.

[9] Z. Li, Q. Li, and X. Zhou, *On sequence-covering msss-maps*, Mat. Vesnik **59** (2007), 15 – 21.
[10] S. Lin, *On a generation of Michael's theorem*, Northeastern Math. J. **4** (1988), 162 – 168.
[11] S. Lin, *On sequence-covering s-mappings*, Adv. Math. (China) **25** (1996), 548 – 551.
[12] S. Lin, *Point-countable covers and sequence-covering mappings*, Chinese Science Press, Beijing, 2002.
[13] S. Lin and P. Yan, *Sequence-covering maps of metric spaces*, Topology Appl. **109** (2001), 301 – 314.
[14] S. Lin and P. Yan, *Notes on cfp-covers*, Comment. Math. Univ. Carolin. **44** (2003), no. 2, 295 – 306.
[15] E. Michael, $\aleph_0$*-spaces*, J. Math. Mech. **15** (1966), 983 – 1002.
[16] Z. Qu and Z. Gao, *Spaces with compact-countable k-networks*, Math. Japonica **49** (1999), 199 –205.
[17] Y. Tanaka, *Theory of k-networks II*, Questions Answers in Gen. Topology **19** (2001), 27 – 46.
[18] Y. Tanaka and Y. Ge, *Around quotient compact images of metric spaces, and symmetric spaces*, Houston J. Math. **32** (2006), no. 1, 99 – 117.

Received 16 02 2009, revised 16 10 2009

Mathematics Faculty,
Dongthap University,
Caolanh City,
Dongthap Province, Vietnam

E-mail address: nvdung@dthu.edu.vn