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The integrals in Gradshteyn and Ryzhik. Part 15: Frullani integrals

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ABSTRACT. The table of Gradshteyn and Ryzhik contains some integrals that can be reduced to the Frullani type. We present a selection of them.

1. Introduction

The table of integrals [**3**] contains many evaluations of the form

$$\int_0^\infty \frac{f(ax)-f(bx)}{x}\,dx = [f(0)-f(\infty)]\,\ln\left(\frac{b}{a}\right). \tag{1.1}$$

Expressions of this type are called *Frullani integrals*. Conditions that guarantee the validity of this formula are given in [**1**] and [**4**]. In particular, the continuity of f' and the convergence of the integral are sufficient for (1.1) to hold.

2. A list of examples

Many of the entries in [**3**] are simply particular cases of (1.1).

EXAMPLE 2.1. The evaluation of 3.434.2 in [**3**]:

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx = \ln\left(\frac{b}{a}\right) \tag{2.1}$$

corresponds to the function $f(x)=e^{-x}$.

EXAMPLE 2.2. The change of variables $t=e^{-x}$ in Example 2.1 yields

$$\int_0^1 \frac{t^{b-1}-t^{a-1}}{\ln t}\,dt = \ln\left(\frac{a}{b}\right). \tag{2.2}$$

This is 4.267.8 in [**3**].

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EXAMPLE 2.3. A generalization of the previous example appears as entry 3.476.1 in [**3**]:

$$\int_0^\infty \left(e^{-vx^p} - e^{-ux^p} \right) \frac{dx}{x} = \frac{1}{p} \ln \left(\frac{u}{v} \right). \tag{2.3}$$

This comes from Frullani's result with a simple additional scaling.

EXAMPLE 2.4. The choice

$$f(x) = \frac{e^{-qx} - e^{-px}}{x}, \tag{2.4}$$

with p, $q > 0$ satisfies $f(\infty) = 0$ and

$$f(0) = \lim_{x \to 0} \frac{e^{-qx} - e^{-px}}{x} = p - q. \tag{2.5}$$

Then Frullani's theorem yields

$$\int_0^\infty \left(\frac{e^{-aqx} - e^{-apx}}{ax} - \frac{e^{-bqx} - e^{-bpx}}{bx} \right) \frac{dx}{x} = (p-q) \ln \left(\frac{b}{a} \right),$$

that can be written as

$$\int_0^\infty \left(\frac{e^{-aqx} - e^{-apx}}{a} - \frac{e^{-bqx} - e^{-bpx}}{b} \right) \frac{dx}{x^2} = (p-q) \ln \left(\frac{b}{a} \right).$$

This is entry 3.436 in [**3**].

EXAMPLE 2.5. Now choose

$$f(x) = \frac{x}{1 - e^{-x}} \exp(-ce^x). \tag{2.6}$$

Then Frullani's theorem yields entry 3.329 of [**3**], in view of $f(0) = e^{-c}$ and $f(\infty) = 0$:

$$\int_0^\infty \left(\frac{a \exp(-ce^{ax})}{1 - e^{-ax}} - \frac{b \exp(-ce^{bx})}{1 - e^{-bx}} \right) dx = e^{-c} \ln \left(\frac{b}{a} \right).$$

EXAMPLE 2.6. The next example uses

$$f(x) = (x + c)^{-\mu}, \tag{2.7}$$

with c, $\mu > 0$, to produce

$$\int_0^\infty \frac{(ax + c)^{-\mu} - (bx + c)^{-\mu}}{x} dx = c^{-\mu} \ln \left(\frac{b}{a} \right). \tag{2.8}$$

This is 3.232 in [**3**].

EXAMPLE 2.7. Entry 4.536.2 in [**3**] is

$$\int_0^\infty \frac{\tan^{-1}(px) - \tan^{-1}(qx)}{x} dx = \frac{\pi}{2} \ln \left(\frac{p}{q} \right). \tag{2.9}$$

This follows directly from (1.1) by choosing $f(x) = \tan^{-1} x$.

EXAMPLE 2.8. The function $f(x) = \ln(a + be^{-x})$ gives the evaluation of entry 4.319.3 of [**3**]:

$$\int_0^\infty \frac{\ln(a+be^{-px}) - \ln(a+be^{-qx})}{x}\, dx = \ln\left(\frac{a}{a+b}\right) \ln\left(\frac{p}{q}\right). \tag{2.10}$$

EXAMPLE 2.9. The function $f(x) = ab\ln(1+x)/x$ produces entry 4.297.7 of [**3**]:

$$\int_0^\infty \frac{b\ln(1+ax) - a\ln(1+bx)}{x^2}\, dx = ab\ln\left(\frac{b}{a}\right). \tag{2.11}$$

EXAMPLE 2.10. Entry 3.484 of [**3**]:

$$\int_0^\infty \left[\left(1+\frac{a}{qx}\right)^{qx} - \left(1+\frac{a}{px}\right)^{px}\right] \frac{dx}{x} = (e^a - 1)\ln\left(\frac{q}{p}\right), \tag{2.12}$$

is obtained by choosing $f(x) = (1 + a/x)^x$ in (1.1).

EXAMPLE 2.11. The final example in this section corresponds to the function

$$f(x) = \frac{a+be^{-x}}{ce^x + g + he^{-x}} \tag{2.13}$$

that produces entry 3.412.1 of [**3**]:

$$\int_0^\infty \left[\frac{a+be^{-px}}{ce^{px}+g+he^{-px}} - \frac{a+be^{-qx}}{ce^{qx}+g+he^{-qx}}\right] \frac{dx}{x} = \frac{a+b}{c+g+h}\ln\left(\frac{q}{p}\right). \tag{2.14}$$

3. A separate source of examples

The list presented in this section contains integrals of Frullani type that were found in volume 1 of Ramanujan's Notebooks [**2**].

EXAMPLE 3.1.

$$\int_0^\infty \frac{\tan^{-1} ax - \tan^{-1} bx}{x}\, dx \;=\; \frac{\pi}{2}\ln\frac{a}{b}$$

EXAMPLE 3.2.

$$\int_0^\infty \ln\frac{p+qe^{-ax}}{p+qe^{-bx}}\frac{dx}{x} \;=\; \ln\left(1+\frac{q}{p}\right)\ln\frac{b}{a}$$

EXAMPLE 3.3.

$$\int_0^\infty \left[\left(\frac{ax+p}{ax+q}\right)^n - \left(\frac{bx+p}{bx+q}\right)^n\right]\frac{dx}{x} \;=\; \left(1-\frac{p^n}{q^n}\right)\ln\frac{a}{b}$$

where a, b, p, q are all positive.

EXAMPLE 3.4.

$$\int_0^\infty \frac{\cos ax - \cos bx}{x}\, dx \;=\; \ln\frac{b}{a}$$

EXAMPLE 3.5.

$$\int_0^\infty \sin\left(\frac{(b-a)x}{2}\right)\sin\left(\frac{(b+a)x}{2}\right)\frac{dx}{x} \;=\; \int_0^\infty \frac{\cos ax - \cos bx}{2x}\, dx \;=\; \frac{1}{2}\ln\frac{b}{a}$$

EXAMPLE 3.6.

$$\int_0^\infty \sin px \, \sin qx \, \frac{dx}{x} \; = \; \int_0^\infty \frac{\cos[(p-q)x] - \cos[(p+q)x]}{2x} \, dx \; = \; \frac{1}{2} \ln \frac{p+q}{p-q}$$

EXAMPLE 3.7. The evaluation of

$$\int_0^\infty \ln \left(\frac{1 + 2n \cos ax + n^2}{1 + 2n \cos bx + n^2} \right) \frac{dx}{x} \; = \; \begin{cases} \ln \frac{b}{a} \ln (1+n)^2 & n^2 < 1 \\ \ln \frac{b}{a} \ln \left(1 + \frac{1}{n}\right)^2 & n^2 > 1 \end{cases}$$

is more delicate and is given in detail in the next section.

EXAMPLE 3.8. The value

$$\int_0^\infty \frac{e^{-ax} \sin ax - e^{-bx} \sin bx}{x} \, dx \; = \; 0$$

follows directly from (1.1) since, in this case $f(x) = e^{-x} \sin x$ satisfies $f(\infty) = f(0) = 0$.

EXAMPLE 3.9.

$$\int_0^\infty \frac{e^{-ax} \cos ax - e^{-bx} \cos bx}{x} \, dx \; = \; \ln \frac{b}{a}.$$

4. A more delicate example

Entry 4.324.2 of [**3**] states that

$$\int_0^\infty \left[\ln(1 + 2a \cos px + a^2) - \ln(1 + 2a \cos qx + a^2) \right] \frac{dx}{x} = \begin{cases} 2 \ln \left(\frac{q}{p} \right) \ln(1+a) & -1 < a \leqslant 1 \\ 2 \ln \left(\frac{q}{p} \right) \ln(1 + 1/a) & a < -1 \text{ or } a \geqslant 1. \end{cases} \tag{4.1}$$

This requires a different approach since the obvious candidate for a direct application of Frullani's theorem, namely $f(x) = \ln(1 + 2a \cos x + a^2)$, does not have a limit at infinity.

In order to evaluate this entry, start with

$$\int_0^1 x^y dx = \frac{1}{y+1}, \tag{4.2}$$

so

$$\int_0^1 dy \int_0^1 x^y dx = \int_0^1 dx \int_0^1 x^y dy = \int_0^1 \frac{x-1}{\ln x} dx = \int_0^1 \frac{dy}{y+1} = \ln 2. \tag{4.3}$$

This is now generalized for arbitrary symbols α and β as

$$\int_0^\infty \frac{e^{\alpha t} - e^{\beta t}}{t} \, dt = \ln \left(\frac{\beta}{\alpha} \right). \tag{4.4}$$

To prove (4.4), make the substitution $u = e^{-t}$ that turns the integral into

$$\begin{aligned}\int_0^1 \frac{u^{-1-\beta} - u^{-1-\alpha}}{\ln u} du &= \int_0^1 du \int_{-1-\alpha}^{-1-\beta} u^w dw \\ &= \int_{-1-\alpha}^{-1-\beta} dw \int_0^1 u^w du \\ &= \int_{-1-\alpha}^{-1-\beta} \frac{dw}{w+1} \\ &= \ln\left(\frac{\beta}{\alpha}\right).\end{aligned}$$

Now observe that $|\frac{2a\cos(rx)}{1+a^2}| \leqslant 1$; therefore it is legitimate to expand the logarithmic terms as infinite series using $\ln(1+z) = \sum_k (-1)^{k-1}\frac{z^k}{k}$. The outcome reads

$$\int_0^\infty \frac{dx}{x} \sum_{k\geqslant 1} \frac{(-1)^{k-1}A^k(\cos^k px - \cos^k qx)}{k} = \\ \sum_{k\geqslant 1} \frac{(-1)^{k-1}A^k}{2^k k} \int_0^\infty \frac{(e^{ipx}+e^{-ipx})^k - (e^{iqx}+e^{-iqx})^k}{x} dx;$$

where $A = 2a/(1+a^2)$. The inner integral is evaluated using some binomial expansions. That is,

(4.5)
$$\int_0^\infty \frac{(e^{ipx}+e^{-ipx})^k - (e^{iqx}+e^{-iqx})^k}{x} dx = \sum_{r=0}^k \binom{k}{r} \int_0^\infty \frac{e^{(2r-k)ipx} - e^{(2r-k)iqx}}{x} dx.$$

It is time to employ equation (4.4). A closer look at (4.5) shows that care must be exercised. The integrals are sensitive to the *parity* of k. More precisely, the quantity $2r-k$ vanishes if and only if k is even and $r = k/2$, in which case there is a zero contribution to summation. Otherwise, the second integral in (4.5) is always equal to $\ln(q/p)$. Therefore,

$$\sum_{r=0}^k \binom{k}{r} \int_0^\infty \left[e^{(2r-k)ipx} - e^{(2r-k)iqx}\right] \frac{dx}{x} = \begin{cases} 2^k \ln\left(\frac{q}{p}\right) & \text{if } k \text{ is odd,} \\ \left(2^k - \binom{k}{k/2}\right) \ln\left(\frac{q}{p}\right) & \text{if } k \text{ is even.}\end{cases}$$

Combining the results obtained thus far yields

(4.6)

$$
\begin{aligned}
I &= \int_0^\infty \frac{\ln(1+2a\cos(px)+a^2) - \ln(1+2a\cos(qx)+a^2)}{x} dx \\
&= \int_0^\infty \frac{dx}{x} \sum_{k \geqslant 1} \frac{(-1)^{k-1} A^k (\cos^k px - \cos^k qx)}{k} \\
&= \sum_{k \geqslant 1} \frac{(-1)^{k-1} A^k}{k 2^k} \sum_{r=0}^{k} \binom{k}{r} \int_0^\infty \frac{e^{(2r-k)ipx} - e^{(2r-k)iqx}}{x} dx \\
&= \ln\left(\frac{q}{p}\right) \sum_{k \text{ odd}} \frac{(-1)^{k-1} A^k}{k} + \ln\left(\frac{q}{p}\right) \sum_{k \text{ even}} \frac{(-1)^{k-1} A^k}{k} \left(1 - \frac{1}{2^k}\binom{k}{k/2}\right) \\
&= \ln\left(\frac{q}{p}\right) \sum_{k \geqslant 1} \frac{(-1)^{k-1} A^k}{k} + \ln\left(\frac{q}{p}\right) \sum_{k \geqslant 1} \frac{1}{2k} \left(\frac{A}{2}\right)^{2k} \binom{2k}{k} \\
&= \ln\left(\frac{q}{p}\right) \ln(1+A) + \frac{1}{2} \ln\left(\frac{q}{p}\right) \sum_{k \geqslant 1} \binom{2k}{k} \frac{1}{k} \left(\frac{A^2}{2^2}\right)^k .
\end{aligned}
$$

The last step utilizes the Taylor series

$$
\sum_{k \geqslant 1} \binom{2k}{k} \frac{Q^k}{k} = -2 \ln\left(\tfrac{1}{2}\left[1+\sqrt{1-4Q}\right]\right) \tag{4.7}
$$

This follows from the binomial series $\sum_{k \geqslant 0} \binom{2k}{k} R^k = 1/\sqrt{1-4R}$ after rearranging in the manner

$$
\sum_{k \geqslant 1} \binom{2k}{k} R^{k-1} = \frac{1}{R\sqrt{1-4R}} - \frac{1}{R} = \frac{4}{\sqrt{1-4R}(1+\sqrt{1-4R})},
$$

and then integrating by parts (from 0 to Q)

$$
\sum_{k \geqslant 1} \binom{2k}{k} \frac{Q^k}{k} = \int_0^Q \frac{4 \cdot dR}{\sqrt{1-4R}(1+\sqrt{1-4R})} = \int_1^{\sqrt{1-4Q}} \frac{-2 \cdot du}{1+u} = -2 \ln\left(\tfrac{1}{2}\left[1+\sqrt{1-4Q}\right]\right).
$$

Formula (4.7) applied to equation (4.6) leads to

$$
I = \ln\left(\frac{q}{p}\right) \ln(1+A) - \ln\left(\frac{q}{p}\right) \ln\left(\tfrac{1}{2}\left[1+\sqrt{1-4Q}\right]\right).
$$

It remains to replace $Q = A^2/2^2 = a^2/(1+a^2)^2$ and use the identity

$$
1 - 4Q = \frac{(a^2-1)^2}{(a^2+1)^2}.
$$

Observe that the expression for $\sqrt{1-4Q}$ depends on whether $|a| > 1$ or not. The proof is complete.

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