SCIENTIA
Series A: *Mathematical Sciences*, Vol. 20 (2010), 83–90
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

On Centres Of h-Purity in QTAG-Modules

M. Zubair Khan* Gargi Varshney**

Abstract. Different concepts and decomposition theorems have been done for QTAG-modules by a number of authors. The purpose of this paper is essentially to study centers of h-purity and their characterizations. We have further studied subsocles and their interesting properties about range and heights establishing various facts about the same.

1. Introduction and Preliminaries

Following [8], a unital module M_R is called QTAG-module if it satisfies the following condition:
(1) Any finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

The structure theory of such modules has been developed by various authors. Analogous to centres of purity we have defined centres of n-h-purity and obtain a characterization (Theorem 4.4). For any uniform element $x \in M$, heights of x denoted as $H_M(x)$ is defined as $\sup\{d(yR/xR)/x \in yR$ and y is a uniform element in $M\}$. For any non-negative integer $n \geqslant 0, H_n(M) = \{x \in M/H_M(x) \geqslant n\}$. A submodule N of M is called h-pure in M if $H_n(N) = N \cap H_n(M)$ for all $n \geqslant 0$, and N is called h-neat if $H_1(N) = N \cap H_1(M)$. For any submodule N of M, the submodule $H^n(N) = \{x \in M/d(xR/(xR \cap N)) \leqslant n\}$ has been introduced in [1] and various related properties have been studied. For any submodule N of M, we denote $H^n_N(0)$ by $soc^n(N)$. For other basic concepts of QTAG-modules one may see [2,3,4,5,7,8].

2. Centre of h-Purity

Definition: Let M be a QTAG-module and N be a submodule of M then N is called centre of h-purity in M if every complement of N in M is h-pure submodule of M.

2000 *Mathematics Subject Classification.* Primary 16D70 Secondary 20K10.
Key words and phrases. QTAG-module, h-neat, h-pure, n-h-pure complement submodules and h-divisible modules.

Theorem 7 in [4] shows that every submodule of M^1 is centre of h-purity. Also Corollary 10 in [4] shows that for any $k \geqslant 1, H_k(M)$ is centre of h-purity in M.

Firstly we restate the following:

Proposition 2.1 [2, Lemma 1]:

(i) For any uniform elements x and $y \in M$ with $x \in yR, d(yR/xR) = m$ if and only if $H_m(yR) = xR$.

(ii) If x and y are predecessors of a uniform element z, then there is an isomorphism $\sigma : xR \to yR$ such that σ is identity on zR.

(iii) For any uniform elements x and $y \in M$, $x - y \in soc(M)$ if and only if $H_1(xR) = H_1(yR)$.

Now using the similar technique we can easily prove the following:

Proposition 2.2: If M is a QTAG-module and x, y are uniform elements in M then following hold.

(i) $x - y \in soc^n(M)$ if and only if $H_n(xR) = H_n(yR)$.

(ii) For every element $t \in soc(M), H_1((x + t)R) = H_1(xR)$.

Now we prove the following theorem which generalizes [6, Theorem 2.1]

Theorem 2.3 : If M is a QTAG-module and N is a submodule of M. Then there exists a submodule K of M such that K is maximal with respect to $K \cap N = 0$ and K is not h-pure in M if and only if the following condition is satisfied.

($\star$) there exists uniform element $u \in N$ and $v \in M$ such that $u + v$ is uniform and

(i) $e(v) > e(u) = 1$

(ii) $H(v) = H(u) < H(u + v)$

(iii) $vR \cap N = 0$

Proof : Let K be a submodule of M maximal with respect to $K \cap N = 0$ and K be not h-pure in M. Let n be the least positive integer such that $K \cap H_n(M) \neq H_n(K)$ then appealing to [4, Proposition 4] we have $n \geqslant 2$. Let x be a uniform element in $K \cap H_n(M)$, then there exists a uniform element $y \in M$ such that $y \notin K, x \in yR$ and $d(yR/xR) = n$. Let zR/xR be a submodule of yR/xR such that $d(zR/xR) = 1$, then $d(yR/zR) = n-1$. By h-neatness of K, there exists a uniform element $t \in K$ such that $x \in tR$ and $d(tR/xR) = 1$. Hence, there exists an isomorphism $\sigma : zR \to tR$ which is the identity on xR. Trivially $e(z-\sigma(z)) \leqslant 1$, so $z-\sigma(z) = u+w$ where $u \in soc(N)$ and $w \in soc(K)$. It is easy to see that u and w are uniform. Let $H(u) \geqslant n-1$ then we can find a uniform element $s \in M$ such that $d(sR/uR) = n-1$. Now $z-u = w+\sigma(z) \in K$ and $z - u \in H_{n-1}(M)$, so $z - u = w + \sigma(z) \in K \cap H_{n-1}(M) = H_{n-1}(K)$. Since $(w+\sigma(z))R$ is homomorphic image of $zR, w+\sigma(z)$ is an uniform element. Now we can find a uniform element $w' \in K$ such that $w + \sigma(z) \in w'R$ and $d(w'R/(w + \sigma(z))R) = n - 1$ Trivially $d(w + \sigma(z))R > 1$, so we can find a submodule $gR \subseteq (w + \sigma(z))R$ such that $d((w + \sigma(z))R/gR) = 1$. Now appealing to proposition (1.1) and (1.2) we get $H_1(zR) = xR, H_1((w + \sigma(z))R) = gR = H_1(\sigma(z)R) = H_1(zR) = xR$, which in turn gives $x \in H_n(K)$, a contradiction. Hence $H(u) < n - 1$. Let $v = w + \sigma(z)$ then $e(v) > e(u) = 1$ and $H(u) = H(v) < H(z) = H(u + v)$, since $v \in K, vR \cap N = 0$. Therefore the conditions of the theorem are satisfied.

Conversely, suppose that the conditions are satisfied. Let for some natural number $n, H(v) < n \leqslant H(u + v)$ and $T_n = soc(H_n(M))$. Since $e(v) > e(u) = 1, e(v) \geqslant 2$. Let $zR = soc(vR)$, then $d(vR/zR) \geqslant 1$ and we get $zR \subseteq H_1(vR)$. Also $H_1((u + v)R) = H_1(vR) \supseteq zR$, consequently $z \in T_n$. Since $vR \cap N = 0, z \notin N$. Let $T_n = S \oplus T_n \cap soc(N), z \in S$. Also (ii) gives $u \notin T_n \cap soc(N)$, so $soc(N) = T \oplus (T_n \cap soc(N)), u \in T$. Now $T_n + soc(N) = S \oplus T \oplus (T_n \cap soc(N))$. Similarly we get $soc(M) = L \oplus (T_n + soc(N))$ for some subsocle L. Let $T_0 = L \oplus S$ then $soc(M) = T_0 \oplus soc(N)$, with $z \in T_0$. Let π be the projection of $soc(M)$ onto $soc(N)$ then $\pi(T_n) = (T_n \cap soc(N))$. Let $U = T_0 + vR$ then $soc(U) = T_0 + soc(vR) = T_0 + zR = T_0$. Therefore $soc(U) \cap soc(N) = 0$ and we get $U \cap N = 0$. Now we embed U into a complement K of N. Let tR be a submodule of vR such that $d(vR/tR) = 1$. As $H_1((v + u)R) = H_1(vR) = tR$ we get $H(t) \geqslant n + 1$. Now we show that $H_K(t) \leqslant n$. Let $H_K(t) \geqslant n + 1$ then there exists a uniform element $y \in K$ such that $t \in yR$ and $d(yR/tR) = n + 1$. Let wR/tR be a submodule of yR/tR such that $d(wR/tR) = 1$ and $d(yR/wR) = n$. Hence there exists an isomorphism $\sigma : vR \to wR$ which is the identity on tR. The map $\eta : vR \to (v - \sigma(v))R$ is an R-epimorphism with $tR \leqslant Ker\eta$. Hence $e(v - \sigma(v)) \leqslant 1$ and we get $v - \sigma(v) \in soc(M)$. Since, $H(u + v) \geqslant n, u + v \in H_n(M)$. Therefore, $u + v - \sigma(v) \in H_n(M)$, consequently $u + v - \sigma(v) \in soc(M) \cap H_n(M) = T_n$. Also $v - \sigma(v) \in K$, so $v - \sigma(v) \in K \cap soc(M) = K \cap (T_0 + soc(N)) = T_0$. Therefore, $u = \pi(u + v - \sigma(v)) \in \pi(T_n) = T_n \cap soc(N)$ and we get $H(u) \geqslant n$ but $H(u) = H(v) < n$. Therefore, we reach at a contradiction. This shows that $H_K(t) \leqslant n$. Therefore, K is not h-pure in M.

Using the above theorem we prove the following, a generalization of [5, Theorem 1]. It may be noticed that the proof given below has similarity with the corresponding proof in [5, Theorem 1].

Theorem 2.4: Let M be a QTAG-module and $T_n = soc(H_n(M)), T_\infty = soc(M^1)$ and $T_{\infty+1} = T_{\infty+2} = 0$. Let N be a submodule of M then N is center of h-purity in M, if and only if there exists k with $0 \leqslant k \leqslant \infty$ such that $T_k \supseteq soc(N) \supseteq T_{k+2}$.

Proof: Let for some $n, T_n \supseteq soc(N) \supseteq T_{n+2}$. Suppose N is not center of h-purity in M. Now if $n = \infty$ then there does not exist any uniform element in $soc(N)$ satisfying condition (ii) of Theorem 2.3. Suppose n is finite. Let $u \in soc(N), v \in M$ be uniform elements satisfying conditions of Theorem 2.3. Let $H(u) = k$ then as $u \in T_n, n \leqslant k < H(u + v)$. Since $e(v) > e(u) = 1$ we can find a submodule tR of vR such that $d(vR/tR) = 1$. Let $w = u + v$ then $H_1((u + v)R) = H_1(vR) = tR$. Let $zR = soc(vR)$ then as vR is totally ordered $zR \leqslant tR$. Hence $H(z) \geqslant n + 2$. This shows that $z \in T_{n+2} \supseteq soc(N)$ and we get a contradiction to the fact that $vR \cap N = 0$. Therefore, N is centre of h-purity in M.

Conversely, suppose $T_n \supseteq soc(N) \supseteq T_{n+2}$ is not true for any n. Then $soc(N) \nsubseteq M^1$, so $soc(N) \nsubseteq T_m$ for some m. Let k be the greatest natural number such that $soc(N) \subset T_k$. Then the maximality of k and the assumption yield $soc(N) \nsubseteq T_{k+1}$ and $T_{k+2} \nsubseteq soc(N)$. Hence there exist uniform elements $u \in soc(N)$ and $s \in T_{k+2}$ such that $H(u) = k$ and $s \notin soc(N)$. Now we can find a uniform element $y \in M$ such that $s \in yR$ and $d(yR/sR) = k+2$. Let xR/sR be a submodule of yR/sR such that $d(xR/sR) = 1$, then $d(yR/xR) = k+1, e(x) = 2$ and we get $H(x) \geqslant k+1$. Let $v = x - u$, then $H_1((x-u)R) = H_1(vR) = H_1(xR) = sR$, consequently $s \in (x-u)R$. Hence $s = (x-u)r$ for some $r \in R$. If $xr = 0$ then $ur = 0$ otherwise $s \in soc(N)$. Define $\eta : xR \to (x-u)R$ given as $xr \to (x-u)r$ then η is a well defined onto homomorphism, consequently $v = x - u$ is a uniform element. Trivially $H(v) = k$ and $H(u+v) = H(x) \geqslant k+1$. Since $e(x) = 2$ and $e(u) = 1, e(v) = 2 > e(u)$. Now suppose $vR \cap N \neq 0$ then there exists a uniform element $x' \in vR \cap N$ and $x' = vr$ for some $r \in R$. Now $x' = vr = xr - ur$. Trivially $xr \neq 0$, so either $xrR = xR$ or $xrR = sR$ and in each case we get $s \in N$ which is a contradiction. Therefore, $vR \cap N = 0$. Hence, by Theorem 2.3, N is not a center of h-purity in M. This completes the proof of the theorem.

3. Height of Subsocles

Firstly we give the following definitions:

Definition: Let S be a subsocle of a QTAG-module M, then height of S is defined as a non-negative integer k such that $S \subseteq H_k(M)$ but $S \nsubseteq H_{k+1}(M)$ and we write $h(S) = k$.

If no such k is possible then we write $h(S) = \infty$, so $S \subseteq M^1$.

Definition: A subsocle S of a QTAG-module M is called open if $soc(H_n(M)) \subseteq S$ for some non-negative integer n.

Definition: If S is open subsocle of a QTAG-module M with $h(S) = k$ then the range of S is the least non-negative integer n such that $soc(H_{k+n}(M)) \subseteq S$ and we write range$(S) = n$.

Now from Theorem 2.4, it is evident that a subsocle S of finite height is center of h-purity if and only if range$(S) \leqslant 2$.

Proposition 3.1: Let S be a subsocle of a QTAG-module M and n be any non-negative integer then

(1) $S \cap H_{n+1}(M) = 0$ if and only if $soc(H_n(M/S)) \subseteq soc(M)/S$.
(2) $S + soc(H_n(M)) = soc(M)$ if and only if $soc(M)/S \subseteq H_n(M/S)$.

Proof: (1) Let $S \cap H_{n+1}(M) = 0$ and $\bar{x} \in soc(H_n(M/S)) = soc((H_n(M)+S)/S)$, then $x \in H_n(M)$ and $H_1(\bar{x}R) = 0$ which in turn implies $H_1(xR) \subseteq S$, so

$H_1(xR) \subseteq S \cap H_{n+1}(M) = 0$. Therefore, $x \in soc(M)$ and we get $soc(H_n(M/S)) \subseteq soc(M)/S$.

Conversely, suppose $S \cap H_{n+1}(M) \neq 0$. Let x be a uniform element in $S \cap H_{n+1}(M)$, then there is a uniform element $y \in M$ such that $d(yR/xR) = n+1$. Let $zR/xR = soc(yR/xR)$, then $d(yR/zR) = n$ and $d(zR/xR) = 1$, so $z \in H_n(M)$ and $H_1(zR) = xR \subseteq S$. Now $H_1(\bar{z}R) = \bar{0}$, so we get $\bar{z} \in soc(H_n(M/S)) \subseteq soc(M)/S$, which gives $z \in soc(M)$ but this is not possible. Therefore, $S \cap H_{n+1}(M) = 0$.

(2) Let $soc(M) = S + soc(H_n(M))$ and $\bar{x} \in soc(M)/S$, then $\bar{x} = y + S, y \in soc(H_n(M))$, consequently $\bar{x} \in H_n(M/S)$.

Conversely if we take $x \in soc(M)$ then $x + S = z + S$ where $z \in H_n(M)$. Hence, $x = z + s, s \in S$ and we get $soc(M) = S + soc(H_n(M))$.

Proposition 3.2: Let S be a subsocle of a QTAG-module M such that $h(S) = k$ and $soc(H_{k+n+1}(M)) \nsubseteq S$ for some integer $n \geqslant 0$. Then there exists a complementary subsocle T of S in $soc(M)$ such that $h(soc(M)/T) = k$ and $soc(H_{k+n}(M/T)) \nsubseteq soc(M)/T$.

Proof: Trivially $S \cap soc(H_{k+n+1}(M)) \subsetneq soc(H_{k+n+1}(M))$. Since $soc(H_{k+n+1}(M))$ is bounded, we shall have $soc(H_{k+n+1}(M)) = T_0 \oplus S \cap soc(H_{k+n+1}(M))$. It is easy to see that $T_0 \cap S = 0$ and $T_0 \subseteq H_{k+1}(M)$. As $S \cap H_{k+1}(M) \oplus T_0 \subseteq soc(H_{k+1}(M))$, we can find a subsocle T_1 such that $soc(H_{k+1}(M)) = S \cap H_{k+1}(M) \oplus T_0 \oplus T_1$. Now using the definition of height of S, we will have $S \cap H_{k+1}(M) \subsetneq S$. Hence, $S = S \cap H_{k+1}(M) \oplus S'$ for some subsocle S'. Trivially $S' \subseteq H_k(M)$ and $S' \cap H_{k+1}(M) = 0$, since $soc(H_{k+1}(M)) \oplus S' \subseteq soc(H_k(M))$, we get a subsocle T_2 such that $soc(H_k(M)) = soc(H_{k+1}(M)) \oplus S' \oplus T_2$. Trivially $S \cap (T_0 \oplus T_1 \oplus T_2) = 0$. Let $soc(M) = soc(H_k(M)) \oplus T_3$ and $T = T_0 \oplus T_1 \oplus T_2 \oplus T_3$ then $soc(M) = soc(H_k(M)) \oplus T_3 = soc(H_{k+1}(M)) + S' + T_2 + T_3 = S \cap soc(H_{k+1}(M)) \oplus T_0 \oplus T_1 \oplus S' \oplus T_2 \oplus T_3 = S \oplus T$. Hence, $(S+T)/T = soc(M)/T \subseteq H_k(M)/T$. Now, since $T_0 \neq 0$, $T \cap H_{k+n+1}(M) \neq 0$ and consequently by Proposition 3.1, $soc(H_{k+n}(M/T)) \nsubseteq soc(M)/T$. Also as $soc(M) \neq T + soc(H_{k+1}(M))$, appealing to Proposition 3.1, we get $soc(M)/T \nsubseteq H_{k+1}(M/T)$. Hence $h(soc(M)/T) = k$.

Theorem 3.3: Let S be a open subsocle of a QTAG-module M such that $h(S) = k$ and n be a non-negative integer. Then range$(S) \leqslant n+1$ if and only if range$(soc(M)/T) \leqslant n$, for every subsocle T of M such that $soc(M) = T \oplus S$.

Proof: Let range$(S) \leqslant n+1$ then $soc(H_{k+n+1}(M)) \subseteq S \subseteq (H_k(M))$. Trivially $T \cap H_{k+n+1}(M) = 0$. Hence, by Proposition 3.1, $soc(H_{k+n}(M/T)) \subseteq soc(M)/T$. It is trivial to see that $soc(M) = soc(H_k(M)) + T$, so by Proposition 3.1, we get $soc(M)/T \subseteq H_k(M/T)$. Therefore, range$(soc(M)/T) \leqslant n$.

Conversely, let range$(soc(M)/T) \leqslant n$. Now we show that $soc(H_{k+n+1}(M)) \subseteq S$. Let $soc(H_{k+n+1}(M)) \nsubseteq S$, then by Proposition 3.2, we find a subsocle T such that

$soc(M) = T \oplus S$ such that $h(soc(M)/T) = k$ and $soc(H_{k+n}(M/T)) \nsubseteq soc(M)/T$ and hence range$(soc(M)/T) \nleqslant n$. Which is a contradiction. Therefore, $soc(H_{k+n+1}(M)) \subseteq S$ and we get range $(S) \leqslant n + 1$.

4. Centre of n-h-Purity

In this section we define a new concept of n-h-purity which generalizes the concept of h-purity and obtain a characterization of center of n-h-purity.

Definition: A submodule N of a QTAG-module M is called n-h-pure in M if $N/soc^n(N)$ is h-pure in $M/soc^n(N)$, where n is a non-negative integer.
It is evident that if $n = 0$ then n-h-purity is simply h-purity.

Definition: A subsocle S of a QTAG-module M is centre of n-h-purity if all complements of S in M are n-h-pure submodules of M.

Firstly we prove the following:

Theorem 4.1: If N is a submodule of a QTAG-module M, then there is a complement of N which is h-pure in M.

Proof: It is sufficient to consider $soc(N) \neq soc(M)$. Suppose every uniform element of $soc(M)$ is of infinite height then trivially $N \subseteq M^1$. Now appealing to [3, Corollary 8] we get a complement K of N, which is h-pure in M. Now on the other hand if there is a uniform element $x \in soc(M)$ such that $x \notin soc(N)$ and $H(x) < \infty$. As if $y \in soc(M)$ such that $y \notin soc(N)$ and $H(y) = \infty$, then $H(x+y) = H(x) < \infty$. Hence, appealing to [7, Lemma 1] we shall get a summand K such that $soc(K) = (x + y)R$ and $K \cap N = 0$. Hence, K is h-pure in M.

Theorem 4.2: $S \subseteq soc(M)$ then there exists a h-neat submodule K of M which is 1-h-pure with $soc(K) = S$.

Proof: Applying Theorem 4.1 for M/S, we get a h-pure submodule K/S in M/S, which is a complement of $soc(M)/S$. Since $(K/S) \cap (soc(M)/S) = 0$, for every uniform element $x \in soc(K), x + S = S$, so $x \in S$ and hence, $soc(K) = S$. Therefore, K is 1-h-pure in M. Now we show that K is h-neat. Let x be a uniform element in $K \cap H_1(M)$, then we get a uniform element $y \in M$ such that $d(yR/xR) = 1$. Now if $y \in K$ we get K to be h-neat submodule. Let $y \notin K$ then $((K+yR)/S) \cap (soc(M)/S) \neq 0$ implies $k + y + S = z + S$ for some $z \in soc(M), k \in K$. Hence, $0 = H_1(zR) = H_1((k + y)R = 0$, so $k + y \in soc(M)$. Therefore, $H_1(kR) = H_1(yR) = xR$ and $x \in H_1(K)$. Hence, K is h-neat.

Proposition 4.3: Let S be a subsocle of a QTAG-module M such that S is centre of n-h-purity for $n \geqslant 1$. Then $soc(M)/T$ is centre of $(n - 1)$-h-purity in M/T for every complementary subsocle T of S in $soc(M)$.

Proof: Let K/T be a complement of $soc(M)/T$ in M/T. Then trivially $K \cap S = 0$. Now we show that $N \cap S \neq 0$ for $K \subsetneq N$. Let $N \cap S = 0$ then we show that $N/T \cap (S \oplus T)/T = 0$. Let on contrary $N/T \cap (S \oplus T)/T \neq 0$, then $x + T = s + T$ where $x \in N, s \in S$ and we get $x - s \in T \subseteq K \subseteq N$, consequently $s \in N \cap S = 0$ and $x + T = T$, which is a contradiction. Therefore, K is a complement of S. Hence, $K/soc^n(K)$ is h-pure in $M/soc^n(K)$. Now we show that $K/T/soc^{(n-1)}(K/T)$ is h-pure in $M/T/soc^{(n-1)}(K/T)$. It is easy to see that $soc(K) = T$ and $soc^{(n-1)}(K/T) \subseteq soc^n(K)/T$. Now for any uniform element $x \in soc^n(K)$, let $yR = soc(xR)$ then $H_{n-1}(xR) = yR$. Hence,

$$H_{n-1}(\bar{x}R) = H_{n-1}((xR + T)/T) = (H_{n-1}(xR) + T)/T = \bar{0}.$$

Therefore, $soc^{(n-1)}(K/T) = soc^n(K)/T$. Further, under the canonical isomorphism $M/T/soc^{(n-1)}(K/T) = M/T/soc^n(K)/T \cong M/soc^n(K)$, $K/T/soc^{(n-1)}(K/T)$ is mapped onto $K/soc^n(K)$. Hence K/T is $(n-1)$-h-pure in M/T and we get the result.

Now we prove the main result of this section:

Theorem 4.4: A subsocle S of a QTAG-module M is centre of n-h-purity for some $n \geqslant 0$ if and only if either $h(S) = \infty$, or S is open subsocle of M such that range$(S) \leqslant n + 2$.

Proof: Let S be a centre of n-h-purity and $h(S) < \infty$. Suppose $h(S) = k$, then we show that $soc(H_{k+n+2}(M)) \subseteq S$, which in turn will imply range$(S) \leqslant n + 2$. Let $soc(H_{k+n+2}(M)) \nsubseteq S$, then appealing to Proposition 3.2, we will find a subsocle T such that $soc(M) = S \oplus T$, $h(soc(M)/T) = k$ and $soc(H_{k+n+1}(M/T) \nsubseteq soc(M)/T$. As remarked in section 3, for $n = 0$, range$(S) \leqslant 2$, so we use induction. However, appealing to Proposition 4.3, we get $soc(M)/T$ as centre of $(n-1)$-h-purity. Therefore, range$(soc(M)/T) \leqslant n - 1 + 2 = n + 1$, consequently, $soc(H_{k+n+1}(M/T) \subseteq soc(M)/T$, which is a contradiction. Hence range$(S) \leqslant n + 2$.

Conversely, if $h(S) = \infty$, then by [3, Corollary 8], S is centre of h-purity and hence for $n = 0, S$ is centre of n-h-purity. Suppose range$(S) \leqslant n+2$ and $soc(H_{k+n+2}(M)) \subseteq S \subseteq H_k(M)$. Let K be a complement of S in M. Now we prove that

$$soc(H_{k+2}(M/soc^n(K)) \subseteq (soc(M) + soc^n(K))/soc^n(K) \subseteq H_k(M/soc^n(K)).$$

For any uniform element $x \in H_{k+2}(M)$, Let $\bar{x} \in soc(H_{k+2}(M/soc^n(K))$. Then $H_1(\bar{x}R) = 0$, hence, $H_1(xR) \subseteq K$, but due to [4, Proposition 4], K is h-neat and so there is a uniform element $t \in K$ such that $H_1(xR) = H_1(tR) = zR$. Now as $x \in H_{k+2}(M)$, there is a uniform element $y \in M$ such that $d(yR/xR) = k + 2$, consequently $H_{k+3}(yR) = H_1(tR) = zR$ and we get $H_{k+3+n-1}(yR) = H_n(tR) = H_{n-1}(zR)$, but $H_{k+n+2}(yR) = H_n(tR) \subseteq K \cap H_{k+n+2}(M) = 0$. Hence, $t \in soc^n(K)$. Further, as $H_1(xR) = H_1(tR)$, we get $x - t \in soc(M)$. Therefore,

$$x - t + soc^n(K) = x + soc^n(K) = \bar{x} \in (soc(M) + soc^n(K))/soc^n(K)$$

and we get the first inclusion. Trivially $H_k(M/soc^n(K)) = (H_k(M)+soc^n(K))/soc^n(K)$ and as K is complement of $S, soc(M) = S + soc(K)$. Therefore, the second inclusion also follows. Hence, range$((soc(M) + soc^n(K))/soc^n(K)) \leqslant 2$ and we get $(soc(M) + soc^n(K))/soc^n(K)$ as centre of h-purity in $M/soc^n(K)$. Further it is easy to see that $K/soc^n(K)$ is complement of $(soc(M)+soc^n(K))/soc^n(K)$ in $M/soc^n(K)$ and hence $K/soc^n(K)$ is h-pure submodule of $M/soc^n(K)$. Therefore, S is centre of n-h-purity.

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Received 31 08 2009, revised 10 10 2010

*Current address: Department of Mathematics
Aligarh Muslim University,
Aligarh-202 002
India
E-mail address: mz_alig@yahoo.com

**Current address: Department of Mathematics,
Aligarh Muslim University,
Aligarh-202 002
India
E-mail address: gargi2110@gmail.com