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The Schwarzian derivative and univalent functions theory

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Abstract. Let $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ be regular in the unit disk $E = \{z : |z| < 1\}$. In this paper we establish a condition under which $Ref(z)/z > 0$. We also apply our result to discuss the univalency of the function $w = f(z)$.

1. Introduction

Let A denote the class of functions of the form $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ which are regular in the unit disk $|z| < 1$. We denote the subclass of A consisting of regular and univalent functions $f(z)$ in the unit disk and satisfying $Re\frac{f(z)}{z} > 0$ by S_0.
Definition 1. A function $f(z)$ is said to be convex in the unit disk $|z| < 1$ if and only if

$$Re\left[1 + \frac{zf''(z)}{f'(z)}\right] > 0$$

Definition 2. A function $f(z)$ is said to be starlike in the unit disk $|z| < 1$ if and only if

$$Re\left[\frac{zf'(z)}{f(z)}\right] > 0$$

It is known [2] that if $Re\frac{f(z)}{z} > \frac{1}{2}$ for $|z| < 1$, then $f(z)$ is starlike for $|z| < 2^{-1/2}$. So it is natural to be interested in the condition under which $Re\frac{f(z)}{z} > \frac{1}{2}$ for $|z| < 1$. Many authors have worked in this direction, for example, in [3,8] it was shown that if $f(z)$ is convex in $|z| < 1$ then $Re\frac{f(z)}{z} > \frac{1}{2}$. Also in [8] it was proved that if $f(z)$ satisfies $Ref'(z) > \beta$ for $0 \leqslant \beta < 1$ and $z \in E$, then $Re\frac{f(z)}{z} > \frac{1+2\beta}{3}$. This result was improved in [9]. In this paper we also establish a condition under which $Re\frac{f(z)}{z} > \frac{1}{2}$.

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2. Preliminary and statement of results

Let $f(z) = z + a_2z^2 + a_3z^3 + \dots$, $|z| < 1$ and let $(w, z) = (\frac{w''}{w'})' - \frac{1}{2}(\frac{w''}{w'})^2$ denote the Schwarzian derivative of $f(z)$. Also, let

$$g(z) = \frac{f'(x)\left(1-|x|^2\right)}{f\left((x+z)/(1+\bar{x}z)\right) - f(x)}$$

It is known that $f(z)$ is univalent for $|z| < 1$ if $g(z)$ is univalent for $|z| < 1$.
By expansion we have

$$\frac{f'(x)\left(1-|x|^2\right)}{f\left((x+z)/(1+\bar{x}z)\right) - f(x)} = \frac{1}{z} + \bar{x} - \frac{1}{2}\frac{f''(x)}{f'(x)}\left(1-|x|^2\right) -$$

$$-\frac{1}{6}\left(1-|x|^2\right)^2\left[\left(\frac{f''(x)}{f'(x)}\right)' - \frac{1}{2}\left(\frac{f''(x)}{f'(x)}\right)^2\right] z + \dots = \frac{1}{z} + h(z,x),\ |x| < 1,\ |z| < 1$$

It was shown in [5] that $f(z)$ is univalent for $|z| < 1$ if and only if $|h'(z, x)| < 1$, for $|x| < 1, |z| < 1$
We shall make use of the following lemmas to prove our main results:
Lemma 1. ([9]) If

$$f(z) = z + a_2z^2 + a_3z^3 + \dots \in S_0$$

then the partial sums

$$s_n(z) = z + a_2z^2 + a_3z^3 + a_nz^n \ \ (n = 2, 3, \dots)$$

are univalent in $|z| < \frac{1}{4}$
Lemma 2. ([6]) If $f(z) \in S_0$ then $f(z)$ is univalent in $|z| < 2^{1/2} - 1$.
Lemma 3. ([6]) Wolff-Noshiro's lemma.
If $f(z)$ is analytic in $|z| < R$ and $Ref'(z) > 0$ $(|z| < R)$ then $f(z)$ is univalent in $|z| < R$.

Our main result in this paper is the following theorem.
Theorem 2.1. *If $f(z) = z + a_2z^2 + a_3z^3 + \dots$ is regular for $|z| < 1$ and $|h(z, x)| < 1$ for $|z| < 1, |x| < 1$, then $Re\frac{f(z)}{z} > \frac{1}{2}$*

3. Proof of the main results and aplications

In this section, we provide the proof of our main result in this paper. We also provide application of our result.

Proof of the Theorem 2.1. Let $f(z) = z + a_2z^2 + a_3z^3 + \dots$ be regular for $|z| < 1$. Consider the function

$$\frac{f'(x)\left(1-|x|^2\right)}{f\left((x+z)/(1+\bar{x}z)\right) - f(x)} = \frac{1}{z} + h(z,x),$$

since $f(0) = 0$ and $f'(0) = 1$ we have that $h(z, 0) = \frac{1}{f(z)} - \frac{1}{z}$
By the condition of the theorem and since $|z| < 1$ we have that

$$|\frac{z}{f(z)} - 1| < 1$$

which implies that

$$Re\frac{f(z)}{z} > \frac{1}{2}.$$

As applications we prove the following theorems.

Theorem 3.1. *If* $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ *is regular for* $|z| < 1$ *and* $|h(z, x)| < 1$ *for* $|z| < 1, |x| < 1$*, then the partial sums*

$$S_k(z) = z + a_2z^2 + \ldots + a_kz^k, (k = 2, 3, \ldots)$$

are univalent in the disk $|z| < \frac{1}{4}$.

Proof. By applying Theorem 2.1 we have that

$$Re\frac{f(z)}{z} > \frac{1}{2}.$$

Hence, by applying a theorem of Yamagushi [9] it follows that the partial sums $S_k(z) = z + a_2z^2 + \ldots + a_kz^k, (k = 2, 3, \ldots)$ are univalent in the disk $|z| < \frac{1}{4}$.

Theorem 3.2. *If* $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ *is regular for* $|z| < 1$ *and* $|h(z, x)| < 1$ *for* $|z| < 1, |x| < 1$*, then*

$$Ref'(re^{i\theta}) \geqslant \frac{1 - 2r - r^2}{(1 + r)^2}$$

for $0 \leqslant r < 2^{1/2} - 1$

Proof. Let $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ be regular for $|z| < 1$ and if $|h(z, x)| < 1$, then $Re\frac{f(z)}{z} > 1/2$ by Theorem 2.1. Hence, by a theorem of Yamagushi [9] we have that

$$Ref'(re^{i\theta}) \geqslant \frac{1 - 2r - r^2}{(1 + r)^2}$$

for $0 \leqslant r < 2^{1/2} - 1$. The bound is sharp.

Theorem 3.3. *If* $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ *is regular for* $|z| < 1$ *and* $|h(z, x)| < 1$*, then* $f(z)$ *is univalent for* $|z| < 2^{1/2} - 1$

Proof. Let $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ be regular for $|z| < 1$ and let $|h(z, x)| < 1$, then by Theorem 2.1 $Ref'(re^{i\theta}) \geqslant \frac{1-2r-r^2}{(1+r)^2}$ for $0 \leqslant r < 2^{1/2} - 1$.

which implies that
$Ref'(re^{i\theta}) > 0$ for $0 \leqslant r < 2^{1/2} - 1$
The result follows by a well known Wolff-Noshiro's lemma.

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