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Some Characterizations of Submodules of *QTAG*-MODULES

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Abstract. A module M over an associative ring with unity is a $QTAG$-module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules. There are many fascinating concepts related to these modules of which h-pure submodules and N-high submodules are very significant.Here we characterize h-pure hulls of $QTAG$-module in terms of N-high submodules of M. We also characterize submodules of $QTAG$-modules which are the intersections of finitely many h-pure submodules.

1. Introduction

All rings considered here contain unity and modules are unital $QTAG$-modules. The structure of these modules is studied by various authors, but the concept of h-pure modules still fascinates. Sometimes a submodule N of M is not h- pure but it is contained in a h-pure submodule of M.The minimal h-pure submodule of M containing N is the h-pure hull of N in M. We call these submodules as semi h-pure submodules. We find that for every semi h-pure submodule N,there exists a subsocle S of M such that all h-pure hulls of N are S-high submodules of M.We also characterize the intersection of finitely many h-pure submodules containing N.

A module with totally ordered lattice of submodules with finite composition length is a uniserial module.An element $x \in M$ is uniform if xR is a nonzero uniform (hence uniserial) module and for any module M over R with a unique composition sereis,$d(M)$ denotes its composition length.For a uniform element $x \in$M, $e(x) = d(xR)$ and $H_M(x) = sup\{d(\frac{yR}{xR}) \mid y \in M, x \in yR$ and y is uniform$\}$ are the exponent and height of x in M, respectively.$H_k(M)$ is the submodule of M generated by the elements of height at least k.A submodule N of M is h-pure in M if $N \cap H_k(M) = H_k(N), \forall\ k \geqslant 0$ and M is h-divisible if $H_1(M) = M$. A submodule N of M is dense in M if $N = \bigcap_{k=0}^{\infty}(N + H_k(M))$ and N is almost dense in M if

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$soc(H_k(M)) \subset N + H_{k+1}(M), k \geqslant 0$. For other concepts we refer to [3,4,5].

2. Semi h-pure Submodules and Intersections of Pure Hulls.

First of all we should mention the following notations used by Khan [2].

$$N^k(M) = (N + H_{k+1}(M)) \cap Soc(H_k(M))$$

$$N_k(M) = (N \cap Soc(H_k(M))) + Soc(H_{k+1}(M))$$

$$Q_k(M, N) = N^k(M)/N_k(M)$$

We start by defining the following:

Definition 1. A submodule N of M is semi h-pure in M if it is not h-pure but it is contained in a h-pure submodule of M.The minimal h-pure submodule of M, containing N is said to be the h-pure hull of N in M.

Remark 2. It is important to note that h-pure hull need not be unique.

Definition 3. A submodule N of M is a kV-submodule of M if there exists an integer k such that $N^t(M) \cong N_t(M),\ \forall t \geqslant k$. If $k = 0$, then N is a V-submodule of M.

Lemma 4. Let N be a semi h-pure submodule of M and K a h-pure hull of N in M.Then for an integer k the following conditions are equivalent:
(i)$Soc(H_{k-1}(K)) \not\subset N,\ Soc(H_k(K)) \subset N$ for some $k \in Z^+$.
(ii) N is a kV-submodule of M.

Proof. Since K is the minimal h-pure submodule of M containing N, $Soc(H_k(K)) \subseteq N + H_{k+1}(K)$, therefore N is almost dense in K.
Again $Soc\left(\dfrac{H_t(K) + N}{N}\right) = \left(\dfrac{Soc(H_{t+1}(K)) + N}{N}\right)$ if and only if $N^t(M) \cong N_t(M)$.
Now (i) implies that $Soc(H_{k-1}(K) + N) \ncong N$ and $Soc(H_t(K) + N) \cong N$ for all $t \geqslant k$. Therefore $Q_{k-1}(K, N) \neq 0$ and $Q_t(K, N) = 0$ for all $t \geqslant k$. Now $N^{k-1}(M) \ncong N_{k-1}(M)$ and $N^t(M) \cong N_t(M)$ for all $t \geqslant k$ implying that N is a kV-submodule of M.

On the other hand if N is a kV-submodule of M then $N^{k-1}(K) \ncong N_{k-1}(K)$ and $N^t(K) \cong N_t(K)$ for all $t \geqslant k$.For a semi h-pure submodule N contained in a h-pure hull $K \subset M$, K may be expressed as a direct sum $K = A \oplus C$, where C is bounded.Since C is bounded $Soc(C) = Soc(N)$ and there exists a non negative integer l such that $Soc(H_l(K)) \subset N$. Since $Soc(H_t(M)) + N \cong N$ for all $t \geqslant l$, we have $l = k$.

Consider a semi h-pure submodule N of M such that $Q_t(M, N) = 0$ for all $t \geqslant k$ for some k.If K is the h-pure hull of N in M, then there exists submodules A and C

such that $K = A \oplus C, Soc(A) = Soc(N),\ H_{k-1}(C) \neq 0$ and $H_k(C) = 0$ and

$$Soc((H_t(M) + N)/N) = Soc((H_t(K) + N)/N) \oplus (Soc(H_t(M) + N)/N).$$

Now we are able to prove the following:

Theorem 5. Let N be a semi h-pure submodule of M.Then there exists a subsocle S of M such that every h-pure hull of N is S-high in M.

Proof. Since N is semi h-pure in M,there exists a non-negative integer k such that

$$(N + H_{t+1}(M)) \cap Soc(H_t(M)) \cong (N \cap Soc(H_t(M))) + Soc(H_{t+1}(M))$$

$\forall\ t \geqslant k$ i.e. N is a kV-submodule of M.Consider the h-pure hull K of N in M,then

$$Soc(H_k(K)) = Soc(N \cap H_k(M)).$$

Therefore,

$$Soc(H_k(M)) = Soc(N \cap H_k(M)) \oplus S_k$$

for some subsocle S_k of M.

Since $N^t(M) = N^t(K) + N_t(M)$ and $Soc(H_t(K)) \subset Soc(H_{t+1}(K)) + N$, $\forall t \geqslant 0, Soc(H_{k-1}(M)) = N^{k-1}(M) \oplus S_{k-1}$ for some subsocle S_{k-1} of M.

Therefore,

$$\begin{aligned} Soc(H_{k-1}(M)) &= N^{k-1}(K) + N_{k-1}(M) \oplus S_{k-1} \\ &= (Soc(H_{k-1}(K)) + Soc(H_k(M))) \oplus S_{k-1} \\ &= Soc(H_{k-1}(K)) \oplus S_k \oplus S_{k-1} \end{aligned}$$

On repeating the same process, after a finite number of steps we get

$$Soc(M) = Soc(K) \oplus S_k \oplus S_{k-1} \oplus ...... \oplus S_0.$$

where each S_i is a subsocle of M.

This implies that every h-pure hull of N in M is S-high in M,where $S = S_k \oplus S_{k-1} \oplus ...... \oplus S_0$.

An immediate consequence of the above result is stated below:

For any two h-pure hulls L, K of a submodule N of M

$$Soc(H_k(M))/Soc(H_k(L)) \cong Soc(H_k(M))/Soc(H_k(K)),\ for\ k \geqslant 0$$

But by [6] we can't say that $Soc(H_k(L)$ and $Soc(H_k(K)$ are congruent modulo M.

Now we shall try to find a relation between K/N and M/K, where K is a h-pure hull of N.If L and K are two h-pure hulls of a semi h-pure submodule N of M, then $Soc(H_k(L/N)) \cong Soc(H_k(K/N))$ for all $k \geqslant 0$. The cardinality of the minimal generating set of $Soc(H_k(K/N))$, denoted by $g(Soc(H_k(K/N)))$ plays a very important role in this study.Since $Soc(H_k(L)) \cong Soc(H_k(K))$ this cardinal doesn't depend on

the h-pure hull of N and it is a relative invariant of N in M.The α^{th}-Ulm Kaplansky invariant (and other related concepts) of a QTAG-module was defined in [4] as
$f_M(\alpha) = g\left(\dfrac{Soc(H_\alpha(M))}{Soc(H_{\alpha+1}(M))}\right)$

Proposition 6. Let N be a semi h-pure submodule of M and K, L be the h-pure hulls of N in M.Then

$$f_{M/K}(t) = f_{M/L}(t),\ \forall\ t \geqslant 0.$$

Proof. As in Theorem 5, $N^t(M) = N^t(K) + N_t(M)$ and $Soc(H_t(K)) \subset N + H_{t+1}(K)$ for all $t \geqslant 0$. Therefore the t^{th}-Ulm Kaplansky invariant of N with respect to M, $f_t(M, N)$ is

$$g(Soc(H_t(M))/(Soc(H_t(K) + Soc(H_{t+1}(M)))$$

Since a submodule K is h-pure in M if all the elements of the $Soc(K)$ have the same height in K as in M,this result with the above discussion enabled us to write t^{th} Ulm Kaplansky invariant of N with respect to M.

Again we have

$$f_{M/K}(t) = g(Soc(H_t(M/K))/(Soc(H_{t+1}(M/K)))$$

As $Soc(H_t(M/K))/Soc(H_{t+1}(M/K)) \cong (K+Soc(H_t(M)))/(K+Soc(H_{t+1}(M)))$

$$= (Soc(H_t(M)) + Soc(H_t(K)) + K)/(Soc(H_{t+1}(M)) + Soc(H_t(K)) + K)$$

and $(Soc(H_t(M)) + Soc(H_t(K))) \cap K \subseteq Soc(H_{t+1}(M)) + Soc(H_t(K))$,

we have

$$\begin{aligned} f_{M/K}(t) &= g(Soc(H_t(M))/(Soc(H_t(K)) + Soc(H_{t+1}(M)))) \\ &= f_t(M, N). \end{aligned}$$

Similarly $f_{M/L}(t) = f_t(M/N), \forall\ t \geqslant 0$ and the result follows.

Lemma 7. Let K be a h-pure submodule of M containing the submodule N.Then

$$Soc(H_k(M/N))/Soc(H_k(K/N)) \cong Soc(H_k(M))/Soc(H_k(K))$$

for every integer $k \geqslant 0$.

Proof. We shall prove this lemma by using the famous Dedekind short exact sequence.

Since $Soc((H_k(M)+N)/N)/Soc((H_k(K)+N)/N)$

$$= (Soc((H_k(K)+N)/N) + Soc((H_k(M)+N)/N))/Soc((H_k(K)+N)/N)$$

$$\cong (Soc((H_k(M))+N)/N))/(Soc(H_k(K)+N)/N)) \cap (Soc(H_k(M))+N)/N))$$

$$\cong (Soc(H_k(M))+N)/N)/(Soc(H_k(K))+N)/N))$$

$$\cong (Soc(H_k(M))+N))/(Soc(H_k(K))+N)).$$

Again

$$Soc(H_k(M)) \cap N \subset K \cap Soc(H_k(M)) = Soc(H_k(K))$$

Therefore we have

$$(Soc(H_k(M)+N))/(Soc(H_k(K)+N)) \cong Soc(H_k(M))/Soc(H_k(K)).$$

By Dedekind short exact sequence we have

$$Soc\left(\frac{H_k(M)+N}{N}\right) / \left(\frac{SocH_k(K)+N}{N}\right) \cong Soc(H_k(M))/Soc(H_k(K))$$

for every integer $k \geqslant 0$.

To study the intersections of finitely many h-pure submodules of M containing N,we need some notations and lemmas:

Lemma 8. Let N be a semi h-pure and kV-submodule of M contained in a h-pure submodule A of M.Then

$$g(((Soc(H_t(M))+N)/N)/(Soc(H_t(A))+N)/N)) \leqslant g(Soc(H_t(M))/Soc(H_t(K)))$$

for every integer $t \geqslant 0$ and h-pure hull K of N in M.

Proof. Since K is a h-pure hull of N in M,$Soc(H_k(K)) = Soc(N \cap H_k(M))$ and $Soc(H_k(M)) = Soc(N \cap H_k(M)) \oplus S_k$ for some subsocle S_k of M.For every $t \geqslant k$, we have $Soc(H_t(A)) = Soc(N \cap H_t(M)) \oplus (A \cap S_t)$ and $Soc(H_t(M)) = Soc(H_t(A)) \oplus S'_t$ for some submodule S'_t of the subsocle S_t.
Now by Lemma 7,

$$Soc((H_t(M)+N)/N)/Soc((H_t(A)+N)/N) \cong Soc(H_t(M))/Soc(H_t(A))$$
$$\cong S'_t \subset S_t \cong Soc(H_t(M))/Soc(H_t(K))$$

So we may assume $t < k$. On the similar lines of Theorem 5,we can say that

$$\begin{aligned} Soc(H_{k-1}(M) &= ((N + H_k(M)) \cap Soc(H_{k-1}(M))) \oplus S_{k-1} \\ &= ((N + H_k(A)) \cap Soc(H_{k-1}(A))) + Soc(H_k(M)) \oplus S_{k-1} \\ &= ((N + H_k(A)) \cap Soc(H_{k-1}(A))) + Soc(N \cap H_k(M)) \oplus S_k \oplus S_{k-1} \end{aligned}$$

where S_{k-1} is a subsocle of M.

Now we have $Soc(H_{k-1}(M)) = Soc(H_{k-1}(A)) \oplus C_{k-1}$, where C_{k-1} is a submodule of $S_k \oplus S_{k-1}$. Because $Soc(H_{k-1}(M)) = Soc(H_{k-1}(K)) \oplus S_k \oplus S_{k-1}$, the result holds for $t = k-1$. On the lines of Lemma 4, if we repeat the steps we have

$$g((Soc(H_t(M)+N)/N)/(Soc((H_t(A)+N)/N) \leqslant g(Soc(H_t(M)/Soc(H_t(N))).$$

Lemma 9. Let N be a semi h-pure submodule of M which is an intersection of finitely many h-pure submodules in M.Then there exist a positive integer l such that

$$g(Soc(K/N)) \leqslant l\ g(Soc\ M/Soc\ K)$$

where K is a h-pure hull of N.

Proof. Let $N = \bigcap\limits_{i=1}^{l} K_i$, where each K_i is h-pure submodule of M containing N. We may define $K_0 = K$ and $L_m = \bigcap\limits_{i=0}^{m} Soc\left(\frac{K_i+N}{N}\right)$ and we have

$$g\left(\frac{L_0}{L_1}\right) + g\left(\frac{L_1}{L_2}\right) + \text{..........} + g\left(\frac{L_{l-1}}{L_l}\right) = g\left(Soc\left(\frac{K}{N}\right)\right)$$

As

$$\frac{L_{m-1}}{L_m} = \frac{L_{m-1}}{L_m \cap Soc(K_m/N)} \cong \frac{L_m + Soc(K_m/N)}{Soc(K_m/N)} \subseteq Soc(M/N)/Soc(K_m/N).$$

By the previous lemma

$$g\left(\frac{L_{m-1}}{L_m}\right) \leqslant g(Soc(M/N)/Soc(K/N)) = g(Soc(M)/Soc(K)).$$

This implies that

$$g(Soc(K/N)) \leqslant l\ g\left(Soc(M)/Soc(N)\right).$$

Lemma 10. Let N be a semi h-pure submodule of M.Then for every integer $k \geqslant 0$,

$$g(Soc(H_k(K/N))) = g\left(Soc\left(\frac{H_k(M)}{H_k(M)\cap N}\right)\right) \text{ and}$$

$$g(Soc(H_k(M))/Soc(H_k(K))) = g(Soc(H_k(M))/Soc(H_k(M)) \cap N)$$

Proof. Since K is a h-pure hull of N in M, $H_k(K)$ is a h-pure hull of $N \cap H_k(M)$ in $H_k(M)$ and $N \cap H_k(M) = N \cap H_k(K)$.

Therefore,

$g(Soc(H_k(K))/(H_k(K) \cap N)) = g(Soc(H_k(K/N))$

$= g(Soc(H_k(M))/(Soc(H_k(M)) \cap N))$ and

$$g(Soc(H_k(M))/(Soc(H_k(M) \cap N)) = g\left(\frac{Soc(H_k(M))}{Soc(H_k(K))}\right).$$

Now we can prove the following:

Theorem 11. Let N be a semi h-pure submodule of M.If N is an intersection of finitely many h-pure submodules in M,then for all integer $k \geqslant 0$, there exist a positive integer l_k such that

$$g(Soc(H_k(K/N))) \leqslant l_k \ g\left(\frac{Soc(H_k(M))}{Soc(H_k(K))}\right)$$

where K is a h-pure hull of N in M.

Proof. Let $N = \bigcap_{i=1}^{l} K_i$, where each K_i is a h-pure submodule of M containing N.Now $N \cap H_k(M)$ is semi h-pure in $H_k(M)$ and we also have that

$$N \cap H_k(M) = \bigcap_{i=1}^{l} (K_i \cap H_k(M)) = \bigcap_{i=1}^{l} H_k(Ki)$$

and $H_k(K_i)$ is a h-pure submodule of $H_k(M)$ containing $N \cap H_k(M)$. Therefore by Lemma 9 and 10 there exists a positive integer l_k

$$g(Soc(H_k(K/N))) \leqslant l_k \ g\left(\frac{Soc(H_k(M))}{Soc(H_k(K))}\right).$$

The above theorem does not throw any light on the sufficiency of the conditions.In the end we would like to state some open problems:

Problem 1. To find out the cases when the conditions of theorem 11 become sufficient for a semi h-pure submodules to be the intersection of h-pure submodules.

Problem 2. If $(Soc(H_n(M)) + N)/N = (N \cap Soc(H_n(M))) + Soc(H_{t+1}(M)) = 0$ implies that $Soc\left(H_k(\frac{M}{N})\right) = 0$ for every non negative integer n then is it possible to express N as the intersection of h-pure submodules in M?

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