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A more important Galois connection between distance functions and inequality relations

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Abstract. In a former paper, by using the ideas of R. DeMarr, the second author established a useful Galois connection between distance functions on X and inequality relations on $X \times \mathbb{R}$.

Now, by using the ideas of A. Brøndsted, M. Altman and the second author, we establish a more important Galois connection between distance functions and inequality relations on the same set X.

Introduction

Let X be a set and $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. Denote by $\mathcal{D}_X$ and $\mathcal{E}_X$ the function space $\bar{\mathbb{R}}^{X^2}$ and the power set $\mathcal{P}(X^2)$, respectively. Moreover, let Φ be a fixed member of $\mathcal{D}_X$.

Now, following the ideas of Brøndsted [7] and Altman [1], for any $d \in \mathcal{D}_X$ and $x, y \in X$, we may naturally define

$$x \leqslant_d y \iff d(x, y) \leqslant \Phi(x, y).$$

Moreover, for any $\leqslant \in \mathcal{E}_X$ and $x, y \in X$, we may also naturally define

$$d_{\leqslant}(x, y) = \Phi(x, y) \quad \text{if} \quad x \leqslant y \qquad \text{and} \qquad d_{\leqslant}(x, y) = +\infty \quad \text{if} \quad x \not\leqslant y.$$

Namely, thus we can show that the mappings

$$d \mapsto \leqslant_d \qquad \text{and} \qquad \leqslant \mapsto d_{\leqslant}$$

establish a decreasing Galois connection between the partially ordered sets $\mathcal{D}_X$ and $\mathcal{E}_X$ in the sense that, for any $d \in \mathcal{D}_X$ and $\leqslant \in \mathcal{E}_X$, we have

$$\leqslant \subset \leqslant_d \iff d \leqslant d_{\leqslant}.$$

Therefore, the extensive theory of Galois connections can be applied to investigate the induced relations $\leqslant_d$ and $d_{\leqslant}$.

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In particular, assume now that $X = S \times \mathbb{R}$ for some set S. Then, having in mind the results of DeMarr [15], for any $\rho \in \mathcal{D}_S$ and $(p, \lambda), (q, \mu) \in X$, we may naturally define

$$d_\rho\big((p, \lambda), (q, \mu)\big) = \rho(p, q).$$

Moreover, for any $d \in \mathcal{D}_X$ and $p, q \in S$, we may also naturally define

$$\rho_d(p, q) = \inf\big\{ d\big((p, \lambda), (q, \mu)\big) : \quad \lambda, \mu \in \mathbb{R} \big\}.$$

Namely, thus we can show that the mappings

$$\rho \mapsto d_\rho \qquad \text{and} \qquad d \mapsto \rho_d$$

establish an increasing Galois connection between the partially ordered sets $\mathcal{D}_S$ and $\mathcal{D}_X$ in the sense that, for any $\rho \in \mathcal{D}_S$ and $d \in \mathcal{D}_X$, we have

$$d_\rho \leqslant d \quad \Longleftrightarrow \quad \rho \leqslant \rho_d.$$

The importance of the latter Galois connection lies mainly in the fact that if in particular

$$\Phi\big((p, \lambda), (q, \mu)\big) = \mu - \lambda.$$

for all $(p, \lambda), (q, \mu) \in X$, then for any $\rho \in \mathcal{D}_S$ and $(p, \lambda), (q, \mu) \in X$ we have

$$(p, \lambda) \leqslant_{d_\rho} (q, \mu) \quad \Longleftrightarrow \quad \rho(p, q) \leqslant \mu - \lambda.$$

Moreover, for any $\leqslant \in \mathcal{E}_X$ and $p, q \in S$, we have

$$\rho_{d_\leqslant}(p, q) = \inf\big\{ \mu - \lambda : \quad (p, \lambda) \leqslant (q, \mu) \big\}.$$

Therefore, the relations $\leqslant_{d_\rho}$ and $\rho_{d_\leqslant}$ are straightforward generalizations of those considered by DeMarr [15] and the second author [24].

Thus, it is not surprising that the mappings

$$\rho \mapsto \leqslant_{d_\rho} \qquad \text{and} \qquad \leqslant \mapsto \rho_{d_\leqslant}$$

establish a decreasing Galois connection between the partially ordered sets $\mathcal{D}_S$ and $\mathcal{E}_X$. For this, it is enough to note now only that, for any $\rho \in \mathcal{D}_S$ and $\leqslant \in \mathcal{E}_X$, we have

$$\leqslant \subset \leqslant_{d_\rho} \quad \Longleftrightarrow \quad d_\rho \leqslant d_\leqslant \quad \Longleftrightarrow \quad \rho \leqslant \rho_{d_\leqslant}.$$

Therefore, the extensive theory of Galois connections can also be applied to investigate the composite relations $\leqslant_{d_\rho}$ and $\rho_{d_\leqslant}$.

The most important basic facts on Galois connections, which may be unfamiliar to the reader, will be briefly laid out in next preparatory sections with the help of the notion of Pataki connections. This new notion lies strictly between those of closure operations and Galois connections.

1. The definitions of Galois and Pataki connections

If X is a set and $\leqslant$ is a reflexive, transitive and antisymmetric relation on X, then the ordered pair $X(\leqslant) = (X, \leqslant)$ is called a poset [3].

If $X(\leqslant)$ is a poset, then by taking $X' = X$ and $\leqslant' = \geqslant$, we can at once get another poset $X'(\leqslant')$. This poset is called the dual of the former one.

Now, a function f of one poset X to another Y may be naturally called increasing if $x_1 \leqslant x_2$ implies $f(x_1) \leqslant f(x_2)$ for all $x_1, x_2 \in X$.

Moreover, a function f of one poset X to another Y may be naturally called decreasing if it is an increasing function of X to the dual Y' of Y.

The following definition is a dual to Schmidt's ingenious reformulation [20, p. 205] of Ore's definition of Galois connections [17]. (See also [19], [9, p. 155] and [35].)

Definition 1.1. If X and Y are posets and $*$ and $\star$ are functions of X and Y to Y and X, respectively, such that

$$x^* \leqslant y \iff x \leqslant y^\star$$

for all $x \in X$ and $y \in Y$, then we say that the functions $*$ and $\star$ establish an increasing Galois connection between X and Y.

Remark 1.1. Now, we may naturally say that the functions $*$ and $\star$ establish a decreasing Galois connection between X and Y if they establish an increasing Galois connection between X and Y'.

Theoretically, increasing Galois connections are more natural than the decreasing ones. However, in the practical applications one usually encounters with the decreasing ones.

Example 1.1. Let R be a relation on one set X to another Y. For any $A \subset X$ and $B \subset Y$, define

$$\mathrm{ub}(A) = \{ y \in Y : \quad \forall\ x \in A : \quad x R y \}$$

and

$$\mathrm{lb}(B) = \{ x \in X : \quad \forall\ y \in B : \quad x R y \}.$$

Then, it can be easily seen that the mappings

$$A \mapsto \mathrm{ub}(A) \qquad \text{and} \qquad B \mapsto \mathrm{lb}(B),$$

where $A \subset X$ and $B \subset Y$, establish a decreasing Galois connection between the posets $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

Remark 1.2. The above construction was first considered by Birkhoff [3, p. 122] in 1940 under the name polarities. (For some further studies, see also Ore [17] and Everett [12].)

Ordered triples (X, Y, R), consisting of two sets X and Y and a relation R on X to Y, have recently been also studied by several authors under the name formal contexts [13]. These are important particular cases of relator spaces investigated by the second author in [25].

Concerning the present definition of Galois connections, it is also worth mentioning the following

THEOREM 1.1. *If* $*$ *and* $\star$ *establish an increasing Galois connection between* X *and* Y, *then* $\star$ *and* $*$ *establish an increasing Galois connection between* Y' *and* X'.

PROOF. Namely, for any $y \in Y$ and $x \in X$, we have

$$g(y) \leqslant' x \iff x \leqslant g(y) \iff f(x) \leqslant y \iff y \leqslant' f(x).$$

□

The following analogue of Definition 1.1 has mainly been suggested to us by the various structures derived from relators and the operations induced by these structures. (See [21], [22], [18] and [28].)

DEFINITION 1.2. If $*$ is a function of one poset X to another Y and $-$ is a function of X to itself such that

$$x_1 \leqslant x_2^- \iff x_1^* \leqslant x_2^*$$

for all $x_1, x_2 \in X$, then we say that the functions $*$ and $-$ establish an increasing Pataki connection between X and Y.

REMARK 1.3. Now, we may naturally say that the functions $*$ and $-$ establish a decreasing Pataki connection between X and Y if they establish an increasing Pataki connection between X and Y'.

A close relationship between Galois and Pataki connections can be revealed by the following

THEOREM 1.2. *If* $*$ *and* $\star$ *establish an increasing Galois connection between* X *and* Y *and* $- = *\star$, *then* $*$ *and* $-$ *establish an increasing Pataki connection between* X *and* Y.

PROOF. Namely, for any $x_1, x_2 \in X$, we have

$$x_1 \leqslant x_2^- \iff x_1 \leqslant x_2^{*\star} \iff x_1 \leqslant (x_2^*)^\star \iff x_1^* \leqslant x_2^*.$$

□

REMARK 1.4. From the above theorem, we can see that several properties of Galois connections can be immediately derived from those of Pataki connections.

Moreover, by using Theorem 1.2, we can easily establish the following

EXAMPLE 1.2. Under the notation of Example 1.3, the mappings

$$A \mapsto \operatorname{ub}(A) \qquad \text{and} \qquad A \mapsto \operatorname{lb}\big(\operatorname{ub}(A)\big),$$

where $A \subset X$, establish a decreasing Pataki connection between the posets $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

According to Erné [11, p. 50], the origins of the following brief reformulation of the usual definition of closure operations [3, p. 111] goes back to R. Dedekind. (A closely related characterization was also observed by Everett [12].)

DEFINITION 1.3. A function $-$ of a poset X to itself is called a closure operation on X if $-$ and $-$ establish an increasing Pataki connection between X and X.

REMARK 1.5. Now, a function $\circ$ of a poset X to itself may be naturally called an interior operation on X if it is a closure operation on the dual X' of X.

Thus, Pataki connections are more general objects than closure and interior operations. Moreover, we shall see that they are less general than Galois connections.

2. Some basic properties of Pataki connections

Simple applications of Definition 1.2 immediately yield the following

THEOREM 2.1. *If $*$ and $-$ establish an increasing Pataki connection between X and Y, then*

(1) $-$ is extensive; *(2) $*$ is increasing;* *(3) $* = - *$.*

PROOF. For any $x \in X$, we have $x^* \leqslant x^*$ and $x^- \leqslant x^-$. Hence, by using Definition 1.2, we can infer that $x \leqslant x^-$ and $x^{-*} \leqslant x^*$. Therefore, the assertion (1) and the inequality $- * \leqslant *$ are true.

Moreover, if $x_1, x_2 \in X$ such that $x_1 \leqslant x_2$, then because of the inequality $x_2 \leqslant x_2^-$ we also have $x_1 \leqslant x_2^-$. Hence, by using Definition 1.2, we can infer that $x_1^* \leqslant x_2^*$. Therefore, the assertion (2) is also true. Now, if $x \in X$, then from the inequality $x \leqslant x^-$ we can see that $x^* \leqslant x^{-*}$. Therefore, the inequality $* \leqslant - *$, and thus the assertion (3) is also true. ☐

Now, by using the above theorem and Definitions 1.2 and 1.3, we can also easily prove the following

THEOREM 2.2. *If $*$ is a function of one poset X to another Y and $-$ is a function of X to itself, then the following assertions are equivalent:*

(1) $*$ and $-$ establish an increasing Pataki connection between X and Y;

(2) $-$ is a closure operation on X, and $x_1^- \leqslant x_2^-$ is equivalent to $x_1^* \leqslant x_2^*$ for all $x_1, x_2 \in X$.

PROOF. If (2) holds, then by Definitions 1.3 and 1.2, for any $x_1, x_2 \in X$, we have

$$x_1 \leqslant x_2^- \iff x_1^- \leqslant x_2^- \iff x_1^* \leqslant x_2^*.$$

Therefore, by Definition 1.2, (1) also holds.

While, if (1) holds, then by Definition 1.2 and Theorem 2.1, for any $x_1, x_2 \in X$, we have

$$x_1 \leqslant x_2^- \iff x_1^{-*} \leqslant x_2^* \iff x_1^* \leqslant x_2^*.$$

Therefore, by Definition 1.3, the first part of (1) is true. Moreover, from the above equivalences, we can also see that the second part of (2) is also true. ☐

From this theorem, by Definition 1.3, it is clear that in particular we also have

Corollary 2.1. *If* $-$ *is a function of a poset* X *to itself, then the following assertions are equivalent:*

(1) $-$ *is a closure operation on* X*;*

(3) $*$ *and* $-$ *establish an increasing Pataki connection between* X *and* Y *for some function* $*$ *of* X *to another poset* Y*.*

Moreover, by using Theorem 2.1, we can also easily prove the following theorem which shows the equivalence of the two definitions of closures.

Theorem 2.3. *If* $-$ *is a function of a poset* X *to itself, then the following assertions are equivalent:*

(1) $-$ *is a closure operation on* X*;*

(2) $-$ *is increasing, extensive, and idempotent.*

Remark 2.1. Hence, we can see that if $-$ is an increasing and extensive function of a poset X to itself such that $x^{--} \leqslant x^-$ for all $x \in X$, then $-$ is already a closure operation on X.

Definition 2.1. If $-$ is a closure operation on a poset X, then an element x of X is called closed if $x^- \leqslant x$.

By Theorem 2.3, it is clear that, under the notation $X^- = \{ x^- : \ x \in X \}$, we have the following

Theorem 2.4. *If* $-$ *is a closure operation on a poset* X*, then for any* $x \in X$ *the following assertions are equivalent:*

(1) x *is closed;* *(2)* $x = x^-$*;* *(3)* $x \in X^-$*.*

Remark 2.2. Now, by Theorems 2.3 and 2.4, we can also state that $x^- = \min\{ y \in X^- : \ x \leqslant y \}$ for all $x \in X$. Therefore, the closed elements of X uniquely determine the closure operation on X.

In [35], the second author has also proved a straightforward extension of the following

Theorem 2.5. *If* $*$ *is a function of one poset* X *to another* Y *and* $-$ *is a function of* X *to itself, then the following assertions are equivalent:*

(1) $*$ *and* $-$ *establish an increasing Pataki connection;*

(2) $*$ *is increasing and* $x^- = \max\{ u \in X : \ u^* \leqslant x^* \}$ *for all* $x \in X$*.*

Hence, it is clear that in particular we also have

Corollary 2.2. *If* $*$ *is a function of one poset* X *to another* Y*, then there exists at most one function* $-$ *of* X *to itself such that* $*$ *and* $-$ *establish an increasing Pataki connection.*

Remark 2.3. Moreover, from Theorem 2.5 we can see that a function $-$ of a poset X to itself is a closure operation on X if and only if it is increasing and $x^{-} = \max\{ u \in X : \ u^{-} \leqslant x^{-} \}$ for all $x \in X$.

Finally, we note that the following theorem is also true.

Theorem 2.6. *If* $*$ *and* $-$ *establish an increasing Pataki connection between* X *and* Y, *then the following assertions are equivalent:*

(1) $*$ *is injective;* *(2)* $-$ *is the identity.*

Remark 2.4. Hence, it is clear that a closure operation on X is injective if and only if it is the identity.

Note that the corresponding statements for interior operations and decreasing Pataki connections can be easily derived from the former results by appropriate dualizations.

3. Some basic properties of Galois connections

By Theorems 1.2 and 2.2, we evidently have the first parts of the following assertions. The second parts can be obtained by using Theorem 1.1.

Theorem 3.1. *If* $*$ *and* $\star$ *establish an increasing Galois connection between* X *and* Y, *then*

(1) $*$ *and* $\star$ *are increasing;*

(2) $*\star$ *is a closure and* $\star *$ *is an interior operation;*

(3) $* = * \star *$ *and* $\star = \star * \star$.

Hence, by using Theorem 2.4, we can easily derive the following

Corollary 3.1. *If* $*$ *and* $\star$ *establish an increasing Galois connection between* X *and* Y, *then* $Y^{\star}$ *is the family of all closed elements of* X *with respect to the closure operation* $*\star$.

Proof. Note that $X^{*} \subset Y$, and thus $X^{*\star} \subset Y^{\star}$. Moreover, $Y^{\star} \subset X$, and thus $Y^{\star} = Y^{\star * \star} = \left(Y^{\star}\right)^{*\star} \subset X^{*\star}$. Therefore, $Y^{\star} = X^{*\star}$, and thus by Theorem 2.4 the required assertion is also true. □

Moreover, by using Theorem 3.1, we can also easily establish the following

Example 3.1. Under the notation of Example 1.3, the mappings

$$A \mapsto \operatorname{lb}\left(\operatorname{ub}(A)\right) \qquad \text{and} \qquad B \mapsto \operatorname{ub}\left(\operatorname{lb}(B)\right),$$

where $A \subset X$ and $B \subset Y$, are closure operations on the posets $\mathcal{P}(X)$ and $\mathcal{P}(Y)$, respectively.

Remark 3.1. If $-$ is a closure operation on a complete poset P, then it can be shown that the closed elements of P form a complete poset. (See [3, p. 112] or [9, p. 146].)

Thus, in particular the cuts $\operatorname{lb}\big(\operatorname{ub}(A)\big)$, where $A \subset X$, form a complete poset. In the $X = Y$ particular case, it is called the Dedekind–McNeille completion of X. (See [3, p. 126] or [9, p. 166].)

By using Theorem 3.1, we can also easily prove the following theorem which shows the equivalence of the two definitions of Galois connections.

Theorem 3.2. *If X and Y are posets and $*$ and $\star$ are functions of X and Y to Y and X, respectively, then the following assertions are equivalent:*

(1) $*$ and $\star$ establish an increasing Galois connection;

(2) $*$ and $\star$ are increasing and $x \leqslant x^{*\star}$ and $y^{\star *} \leqslant y$ for all $x \in X$ and $y \in Y$.

In [35], the second author has also proved a straightforward extension of the following theorem whose origins go back to Pickert [19].

Theorem 3.3. *If X and Y are posets and $*$ and $\star$ are functions of X and Y to Y and X, respectively, then the following assertions are equivalent:*

(1) $*$ and $\star$ establish an increasing Galois connection;

(2) $*$ is increasing and $y^{\star} = \max\{x \in X : \ x^{*} \leqslant y\}$ for all $y \in Y$.

Hence, it is clear that in particular we also have

Corollary 3.2. *If $*$ is a function of one poset X to another Y, then there exists at most one function $\star$ of Y to X such that $*$ and $\star$ establish an increasing Galois connection.*

Now, as an certain converse to Theorem 1.2, we can also prove the following

Theorem 3.4. *If $*$ and $-$ establish an increasing Pataki connection between X and Y such that $*$ is onto Y, then there exists a unique function $\star$ of Y to X such that $*$ and $\star$ establish an increasing Galois connection between X and Y. Moreover, we have $- = *\star$.*

Proof. By the axiom of choice, there exists a function φ of Y to X such that $y = \varphi(y)^{*}$ for all $y \in Y$. Define $y^{\star} = \varphi(y)^{-}$ for all $y \in Y$. Then, by the corresponding definitions, it is clear that

$$x \leqslant y^{\star} \iff x \leqslant \varphi(y)^{-} \iff x^{*} \leqslant \varphi(y) \iff x^{*} \leqslant y\,.$$

Therefore, $*$ and $\star$ establish an increasing Galois connection between X and Y.

Now, to complete the proof, it remains to note only that by Corollary 3.2 $\star$ is unique. Moreover, by Theorem 1.2, $*$ and $*\star$ establish an increasing Pataki connection between X and Y. Therefore, by Corollary 2.2, $- = *\star$ also holds. $\square$

REMARK 3.2. The above theorem shows that Galois connections are somewhat more general objects than Pataki connections.

However, in the theory of relator spaces the latter ones have proved to be more natural and important tools than the former ones.

Finally, we note that the following theorem is also true.

THEOREM 3.5. *If* $*$ *and* $\star$ *establish an increasing Galois connection between* X *and* Y, *then the following assertions are equivalent:*

(1) $*$ *is onto* Y; *(2)* $\star$ *is injective; (3)* $\star *$ *is the identity.*

Hence, it is clear that in particular we also have

COROLLARY 3.3. *If* $*$ *and* $\star$ *establish an increasing Galois connection between* X *and* Y *such that* $*$ *is injective and onto* Y, *then* $\star$ *is just the inverse of* $*$.

REMARK 3.3. The corresponding statements for decreasing Galois connections can be easily derived from the preceding results by dualizations.

4. The induced inequalities and écarts

DEFINITION 4.1. Let X be a set and $\bar{\mathbb{R}} = \mathbb{R} \cup \{ -\infty , +\infty \}$. Then a function d of X^2 to $\bar{\mathbb{R}}$ is called an écart on X.

REMARK 4.1. In particular, an écart d on X is called a distance function if $0 \leqslant d(x, y)$ for all $x, y \in X$.

Moreover, a distance function d on X is called a quasi-pseudo-metric if $d(x, x) = 0$ and $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in X$. In most of the existing literature the finite valuedness of d is also required.

The following definition has mainly been suggested by Brøndsted [7] and Altman [1].

DEFINITION 4.2. Let Φ be a fixed écart on X. For any écart d on X and $x, y \in X$, define

$$x \leqslant_d y \quad \Longleftrightarrow \quad d(x, y) \leqslant \Phi(x, y).$$

Concerning the induced inequality relation $\leqslant_d$, in [34], the second author has established the following propositions.

PROPOSITION 4.1. *The following assertions are equivalent:*

(1) $\leqslant_d$ *is reflexive on* X; *(2)* $d(x, x) \leqslant \Phi(x, x)$ *for all* $x \in X$.

PROPOSITION 4.2. *If*

$$d(x, z) \leqslant d(x, y) + d(y, z) \qquad and \qquad \Phi(x, y) + \Phi(y, z) \leqslant \Phi(x, z)$$

for all $x, y, z \in X$, *then the relation* $\leqslant_d$ *is transitive.*

Proposition 4.3. *If for any* $x, y \in X$ *we have*

$$0 \leqslant d(x, y) \qquad \text{and} \qquad \Phi(x, y) \leqslant -\Phi(y, x),$$

and moreover $d(x, y) = 0$, $d(y, x) = 0$, $\Phi(x, y) = 0$ *and* $\Phi(y, x) = 0$ *imply* $x = y$, *then the relation* $\leqslant_d$ *is antisymmetric.*

Proof. If $x \leqslant y$ and $y \leqslant x$, then

$$d(x, y) \leqslant \Phi(x, y) \qquad \text{and} \qquad d(y, x) \leqslant \Phi(y, x),$$

and thus $-\Phi(y, x) \leqslant -d(y, x)$. Hence, we can already see that

$$0 \leqslant d(x, y) \leqslant \Phi(x, y) \leqslant -\Phi(y, x) \leqslant -d(y, x) \leqslant 0,$$

and thus

$$d(x, y) = 0, \quad d(y, x) = 0, \quad \Phi(x, y) = 0, \quad \Phi(y, x) = 0.$$

Therefore, $x = y$ also holds. □

Remark 4.2. Note that if in particular d is a quasi-metric and $-\Phi$ is a quasi-pseudo-metric on X, then $\leqslant_d$ is already a partial order on X.

Namely, in this case Φ satisfies the converse of the triangle inequality. Thus, in particular, for any $x, y \in X$, we have $\Phi(x, y) + \Phi(y, x) \leqslant \Phi(x, x) = 0$, and hence $\Phi(x, y) \leqslant -\Phi(y, x)$.

In addition to Definition 4.2, we may also naturally introduce the following

Definition 4.3. For any relation $\leqslant$ on X and $x, y \in X$, define

$$d_{\leqslant}(x, y) = \Phi(x, y) \quad \text{if} \quad x \leqslant y \qquad \text{and} \qquad d_{\leqslant}(x, y) = +\infty \quad \text{if} \quad x \not\leqslant y.$$

Concerning the induced écart $d_{\leqslant}$, we can easily establish the following propositions.

Proposition 4.4. *The following assertions are equivalent:*

(1) $0 \leqslant d_{\leqslant}(x, y)$ *for all* $x, y \in X$*;*

(2) $0 \leqslant \Phi(x, y)$ *for all* $x, y \in X$ *with* $x \leqslant y$.

Proposition 4.5. *The following assertions are equivalent:*

(1) $d_{\leqslant}(x, x) = 0$ *for all* $x \in X$*;*

(2) $\leqslant$ *is reflexive on* X *and* $\Phi(x, x) = 0$ *for all* $x \in X$.

Proposition 4.6. *If* Φ *is finite-valued, then following assertions are equivalent:*

(1) $d_{\leqslant}(x, z) \leqslant d_{\leqslant}(x, y) + d_{\leqslant}(y, z)$ *for all* $x, y, z \in X$*;*

(2) $\leqslant$ *is transitive and* $\Phi(x, z) \leqslant \Phi(x, y) + \Phi(y, z)$ *for all* $x, y, z \in X$ *with* $x \leqslant y$ *and* $y \leqslant z$.

PROOF. If $x, y, z \in X$, such that $x \leqslant y$ and $y \leqslant z$, then by (1), Definition 4.3, and the assumption $+\infty \notin \Phi(X^2)$, we have

$$d_{\leqslant}(x, z) \leqslant d_{\leqslant}(x, y) + d_{\leqslant}(y, z) = \Phi(x, y) + \Phi(y, z) < +\infty .$$

Hence, by using Definition 4.3, we can infer that $x \leqslant z$ and $d_{\leqslant}(x, z) = \Phi(x, z)$. Therefore, (1) implies (2).

To prove the converse implication, note that if $x, y, z \in X$ such that either $x \not\leqslant y$ or $y \not\leqslant z$, then by Definition 4.3 and the assumption $-\infty \notin \Phi(X^2)$, we have $d_{\leqslant}(x, y) + d_{\leqslant}(y, z) = +\infty$. Therefore, the required inequality automatically holds. While if $x \leqslant y$ and $y \leqslant z$, then by (2) and Definition 4.3 we have $x \leqslant z$ and

$$d_{\leqslant}(x, z) = \Phi(x, z) \leqslant \Phi(x, y) + \Phi(y, z) = d_{\leqslant}(x, y) + d_{\leqslant}(y, z) .$$

□

REMARK 4.3. In this respect it is also worth noticing that if $+\infty \notin \Phi(X^2)$ and $\leqslant$ is an antisymmetric relation on X such that $x < y$ for some $x, y \in X$, then the écart $d_{\leqslant}$ is not symmetric.

Namely, in this case $x \leqslant y$ and $x \neq y$. Hence, by the antisymmetry of $\leqslant$, we have $y \not\leqslant x$. Therefore, $d_{\leqslant}(y, x) = +\infty$, but $d_{\leqslant}(x, y) = \Phi(x, y) \neq +\infty$. Thus, $d_{\leqslant}(x, y) \neq d_{\leqslant}(y, x)$.

5. Galois connections between écarts and inequalities

DEFINITION 5.1. Let X be a set, and denote by $\mathcal{D}$ and $\mathcal{E}$ the families of all écarts d and relations $\leqslant$ on X, respectively.

Moreover, consider the families $\mathcal{D}$ and $\mathcal{E}$ to be partially ordered by the pointwise inequality and the ordinary set inclusion, respectively.

REMARK 5.1. Thus, we have $\mathcal{D} = \bar{\mathbb{R}}^{X^2}$ and $\mathcal{E} = \mathcal{P}(X^2)$. Hence, it is clear that $\mathcal{D}$ and $\mathcal{E}$ are complete posets.

Moreover, to clear up the appropriateness of our former Definition 4.3, we can easily prove the following

THEOREM 5.1. *The mappings*

$$d \mapsto \leqslant_d \qquad \text{and} \qquad \leqslant \mapsto d_{\leqslant}$$

establish a decreasing Galois connection between $\mathcal{D}$ and $\mathcal{E}$.

PROOF. If $d \in \mathcal{D}$ and $\leqslant \in \mathcal{E}$, then by Definitions 4.2 and 4.3 it is clear that $\leqslant_d \in \mathcal{E}$ and $d_{\leqslant} \in \mathcal{D}$. Therefore, by Definition 1.1 and Remark 1.1, we need only show that

$$\leqslant \subset \leqslant_d \iff d \leqslant d_{\leqslant} .$$

For this, note that if $x, y \in X$ such that $x \leqslant y$, then by Definition 4.3 we have $d_{\leqslant}(x, y) = \Phi(x, y)$. Hence, if $d \leqslant d_{\leqslant}$, we can infer that $d(x, y) \leqslant \Phi(x, y)$. Therefore, by Definition 4.2, we have $x \leqslant_d y$. This show that $d \leqslant d_{\leqslant}$ implies $\leqslant \subset \leqslant_d$.

To prove the converse implication, note that if $d \not\leqslant d_{\leqslant}$, then there exist $x, y \in X$ such that $d(x, y) \not\leqslant d_{\leqslant}(x, y)$, and hence $d_{\leqslant}(x, y) < d(x, y)$. Thus, in particular $d_{\leqslant}(x, y) < +\infty$. Hence, by using Definition 4.3, we can infer that $x \leqslant y$ and $d_{\leqslant}(x, y) = \Phi(x, y)$. Thus, in particular $\Phi(x, y) < d(x, y)$, and hence $d(x, y) \not\leqslant \Phi(x, y)$. Therefore, by Definition 4.2, we have $x \not\leqslant_d y$. Consequently, $\leqslant \not\subset \leqslant_d$. This shows that $\leqslant \subset \leqslant_d$ also implies $d \leqslant d_{\leqslant}$. □

Remark 5.2. From this theorem, by Corollary 3.2, we can see that the definition of $d_{\leqslant}$ cannot be altered without disturbing the validity of Theorem 5.1.

Moreover, from Theorem 5.1, by using the results of Section 3, we can easily derive several useful facts about the induced inequalities and écarts.

For instance, by Theorems 5.1 and 3.1 and Remarks 1.1 and 1.5, we have the following

Theorem 5.2. *The following assertions are true:*

(1) the mappings $d \mapsto \leqslant_d$ *and* $\leqslant \mapsto d_{\leqslant}$ *are decreasing;*

(2) the mappings $d \mapsto d_{\leqslant_d}$ *and* $\leqslant \mapsto \leqslant_{d_{\leqslant}}$ *are closure operations;*

(3) $\leqslant_d = \leqslant_{d_{\leqslant_d}}$ *for all* $d \in \mathcal{D}$ *and* $d_{\leqslant} = d_{\leqslant_{d_{\leqslant}}}$ *for all* $\leqslant \in \mathcal{E}$.

Hence, by Corollary 3.1, it is clear that we also have the following

Corollary 5.1. *The following assertions are true:*

(1) d *is a closed member of* $\mathcal{D}$ *if and only if* $d = d_{\leqslant}$ *for some* $\leqslant \in \mathcal{E}$;

(2) $\leqslant$ *is a closed member of* $\mathcal{E}$ *if and only if* $\leqslant = \leqslant_d$ *for some* $d \in \mathcal{D}$.

In addition to Theorem 5.2, we can also easily establish the following

Theorem 5.3. *For any* $d \in \mathcal{D}$ *and* $x, y \in X$, *we have*

(1) $d_{\leqslant_d}(x, y) = \Phi(x, y)$ *whenever* $d(x, y) \leqslant \Phi(x, y)$;

(2) $d_{\leqslant_d}(x, y) = +\infty$ *whenever* $\Phi(x, y) < d(x, y)$.

Proof. If $d(x, y) \leqslant \Phi(x, y)$, then by Definition 4.2 $x \leqslant_d y$. Hence, by Definition 4.3, $d_{\leqslant_d}(x, y) = \Phi(x, y)$.

While, if $\Phi(x, y) < d(x, y)$, then $d(x, y) \not\leqslant \Phi(x, y)$. Thus, by Definition 4.2, $x \not\leqslant_d y$. Hence, by Definition 4.3, $d_{\leqslant_d}(x, y) = +\infty$. □

From this theorem, by using Definition 1.3, we can immediately derive

Corollary 5.2. *A member* d *of* $\mathcal{D}$ *is closed if and only if*

(1) $d(x, y) \leqslant \Phi(x, y)$ *implies* $d(x, y) = \Phi(x, y)$;

(2) $\Phi(x, y) < d(x, y)$ *implies* $d(x, y) = +\infty$.

Proof. By Definition 1.1 and Theorem 5.2, d is closed if and only $d_{\leqslant_d} \leqslant d$. That is, for any $x, y \in X$, we have $d_{\leqslant_d}(x, y) \leqslant d(x, y)$. However, the latter inequality, by Theorem 5.3, means only that $\Phi(x, y) \leqslant d(x, y)$ whenever $d(x, y) \leqslant \Phi(x, y)$, and $+\infty \leqslant d(x, y)$ whenever $\Phi(x, y) < d(x, y)$. Therefore, the required assertion is also true. □

From this corollary, it is clear that in particular we also have

Corollary 5.3. *If $d \in \mathcal{D}$ such that $+\infty \notin d(X^2)$, then d is a closed member of $\mathcal{D}$ if and only if $d = \Phi$.*

Proof. Namely, if d is a closed member of $\mathcal{D}$, then by Corollary 5.2 $\Phi(x, y) < d(x, y)$ implies $d(x, y) = +\infty$. Therefore, by the assumption $+\infty \notin d(X^2)$, for any $x, y \in X$ we have $\Phi(x, y) \not< d(x, y)$, and thus $d(x, y) \leqslant \Phi(x, y)$. Hence, by Corollary 5.2, it follows that $d(x, y) = \Phi(x, y)$. Therefore, $d = \Phi$. Moreover, by Corollary 5.2, we can see Φ is a closed member of $\mathcal{D}$. □

In addition to Theorem 5.2, we can also easily establish the following

Theorem 5.4. *For any $\leqslant \in \mathcal{E}$ and $x, y \in X$, the following assertions are equivalent:*

(1) $x \leqslant_{d_\leqslant} y$; *(2)* $\Phi(x, y) = +\infty$ *if* $x \not\leqslant y$.

Proof. If (1) holds, then by Definition 4.2 $d_\leqslant(x, y) \leqslant \Phi(x, y)$. Thus, by Definition 4.3, $+\infty \leqslant \Phi(x, y)$ if $x \not\leqslant y$. Therefore, (2) also holds.

Conversely, if (2) holds, then by using Definition 4.3 we can at once see that $d_\leqslant(x, y) \leqslant \Phi(x, y)$. Therefore, by Definition 4.2, (1) also holds. □

From this theorem, by using Definition 1.3, we can immediately derive

Corollary 5.4. *A member $\leqslant$ of $\mathcal{E}$ is closed if and only if $x \leqslant y$ whenever $x \not\leqslant y$ implies $\Phi(x, y) = +\infty$.*

Proof. By Definition 1.11 and Theorem 5.2, $\leqslant$ is closed if and only $\leqslant_{d_\leqslant} \subset \leqslant$. That is, $x \leqslant_{d_\leqslant} y$ implies $x \leqslant y$. However, by Theorem 5.4, the former inequality means only that $\Phi(x, y) = +\infty$ if $x \not\leqslant y$. Therefore, the required assertion is also true. □

Moreover, by using Theorem 5.4, we can also easily prove the following

Theorem 5.5. *If $+\infty \notin \Phi(X^2)$, then $\leqslant = \leqslant_{d_\leqslant}$ for all $\leqslant \in \mathcal{E}$.*

Proof. If $\leqslant \in \mathcal{E}$, then by Theorem 5.2 we always have $\leqslant \subset \leqslant_{d_\leqslant}$. Moreover, if $x, y \in X$ such that $x \leqslant_{d_\leqslant} y$, then by Theorem 5.4 we have $\Phi(x, y) = +\infty$ whenever $x \not\leqslant y$. Hence, by the assumption $\Phi(x, y) \neq +\infty$, it is clear that $x \leqslant y$. Therefore, the inclusion $\leqslant_{d_\leqslant} \subset \leqslant$ also holds. □

Remark 5.3. The above theorem shows that if $+\infty \notin \Phi(X^2)$, then each member of $\mathcal{E}$ is closed.

Moreover, as an immediate consequence of Theorem 5.5, we can also state

Corollary 5.5. *If* $+\infty \notin \Phi(X^2)$, *then the mapping* $d \mapsto \leqslant_d$ *is onto* $\mathcal{E}$ *and the mapping* $\leqslant \mapsto d_{\leqslant}$ *is injective.*

6. A Galois connection between écarts on S and $S\times\mathbb{R}$

Definition 6.1. Let S be a set and $X = S\times\mathbb{R}$. For any écart ρ on S and $(p, \lambda), (q, \mu) \in X$, define

$$d_\rho\big((p, \lambda), (q, \mu)\big) = \rho(p, q).$$

Concerning the induced écart d_ρ, we can easily establish the following

Proposition 6.1. d_ρ *is a pseudo-metric on* X *if and only if* ρ *is a pseudo-metric on* S.

In this respect, it is also worth proving the following

Proposition 6.2. *If* d *is a pseudo-metric on* X *such that the mapping*

$$\lambda \mapsto d\big((p, \lambda)(q, \mu)\big),$$

is increasing for any $p, q \in S$ *and* $\mu \in \mathbb{R}$, *then* $d = d_\rho$ *for some pseudo-metric* ρ *on* S.

Proof. If $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 \leqslant \lambda_2$, then by the assumed increasingness, for any $p, q \in S$ and $\mu \in \mathbb{R}$, we have

$$d\big((p, \lambda_1), (q, \mu)\big) \leqslant d\big((p, \lambda_2), (q, \mu)\big).$$

Moreover, since d is a pseudo-metric, we also have

$$d\big((p, \lambda_2), (q, \mu)\big) \leqslant d\big((p, \lambda_2), (p, \lambda_1)\big) + d\big((p, \lambda_1), (q, \mu)\big)$$

and

$$d\big((p, \lambda_2), (p, \lambda_1)\big) = d\big((p, \lambda_1), (p, \lambda_2)\big) \leqslant d\big((p, \lambda_2), (p, \lambda_2)\big) = 0.$$

Therefore,

$$d\big((p, \lambda_2), (q, \mu)\big) \leqslant d\big((p, \lambda_1), (q, \mu)\big)$$

also holds. Thus, we necessarily have

$$d\big((p, \lambda_1), (q, \mu)\big) = d\big((p, \lambda_2), (q, \mu)\big),$$

even if $\lambda_2 \leqslant \lambda_1$. Now, we can also easily see that

$$d\big((p, \lambda_1), (q, \mu_1)\big) = d\big((p, \lambda_2), (q, \mu_1)\big) =$$
$$= d\big((q, \mu_1), (p, \lambda_2)\big) = d\big((q, \mu_2), (p, \lambda_2)\big) = d\big((p, \lambda_2), (q, \mu_2)\big)$$

for all $p, q \in S$ and $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$. Hence, it clear that, by defining

$$\rho(p, q) = d\big((p, 0), (q, 0)\big)$$

for all $p, q \in S$, we can get an écart ρ on S such that

$$d_\rho\big((p, \lambda), (q, \mu)\big) = \rho(p, q) = d\big((p, 0), (q, 0)\big) = d\big((p, \lambda), (q, \mu)\big)$$

for all $p, q \in S$ and $\lambda, \mu \in \mathbb{R}$, and thus $d_\rho = d$. Hence, by Proposition 6.1, we can see that ρ is also a pseudo-metric. □

In addition to Definition 6.1, we may also naturally introduce the following

DEFINITION 6.2. For any écart d on X and $p, q \in S$, define

$$\rho_d(p, q) = \inf\left\{ d\big((p, \lambda), (q, \mu)\big) : \quad \lambda, \mu \in \mathbb{R} \right\}.$$

Unfortunately, now an analogue of Proposition 6.1 does not hold. However, by denoting the family of all écarts on S by $\mathcal{D}_S$, we can easily prove the following

THEOREM 6.1. *The mappings*

$$\rho \ \mapsto \ d_\rho \qquad \textit{and} \qquad d \ \mapsto \ \rho_d$$

establish an increasing Galois connection between $\mathcal{D}_S$ *and* $\mathcal{D}_X$.

PROOF. If $\rho \in \mathcal{D}_S$ and $d \in \mathcal{D}_X$, then by Definitions 6.1 and 6.2 it is clear that $d_\rho \in \mathcal{D}_X$ and $\rho_d \in \mathcal{D}_S$. Therefore, by Definition 1.1, we need only show that

$$d_\rho \leqslant d \quad \Longleftrightarrow \quad \rho \leqslant \rho_d.$$

For this, note that if $d_\rho \leqslant d$ and $p, q \in S$, then for any $\lambda, \mu \in \mathbb{R}$ we have

$$\rho(p, q) = d_\rho\big((p, \lambda), (q, \mu)\big) \leqslant d\big((p, \lambda), (q, \mu)\big).$$

Therefore,

$$\rho(p, q) \leqslant \inf\left\{ d\big((p, \lambda), (q, \mu)\big) : \quad \lambda, \mu \in \mathbb{R} \right\} = \rho_d(p, q).$$

Thus, $\rho \leqslant \rho_d$ also holds.

Conversely, if $\rho \leqslant \rho_d$ holds, then for any $(p, \lambda), (q, \mu) \in X$, we have

$$d_\rho\big((p, \lambda), (q, \mu)\big) = \rho(p, q) \leqslant \rho_d(p, q) =$$
$$= \inf\left\{ d\big((p, \alpha), (q, \beta,)\big) : \quad \alpha, \beta \in \mathbb{R} \right\} \leqslant d\big((p, \lambda), (q, \mu)\big).$$

Therefore, $d_\rho \leqslant d$ also holds. □

REMARK 6.1. From this theorem, by Corollary 3.2, we can see that the definition of ρ_d cannot be altered without disturbing the validity of Theorem 6.5.

Moreover, from Theorem 6.1, by Theorem 3.1, we can immediately get the following assertion which is also quite obvious from Definitions 6.1 and 6.2.

COROLLARY 6.1. *The mappings considered in Theorem 6.1 are increasing.*

By using the above mentioned definitions, we can also easily establish the following

THEOREM 6.2. *For any* $\rho \in \mathcal{D}_S$, *we have* $\rho = \rho_{d_\rho}$.

PROOF. Namely,

$$\rho_{d_\rho}(p, q) = \inf\left\{ d_\rho\big((p, \lambda), (q, \mu)\big) : \quad \lambda, \mu \in \mathbb{R} \right\} =$$
$$= \inf\left\{ \rho(p, q) : \quad \lambda, \mu \in \mathbb{R} \right\} = \rho(p, q)$$

for all $p, q \in S$. Therefore, the required equality is also true. □

REMARK 6.2. The above theorem shows that each member of $\mathcal{D}_S$ is closed.

Moreover, as an immediate consequence of Theorems 6.2, we can state

Corollary 6.2. *The mapping* $\rho \mapsto d_\rho$ *is injective and the mapping* $d \mapsto \rho_d$ *is onto* $\mathcal{D}_S$.

Remark 6.3. Finally, we note that from Theorem 6.1, by Theorem 3.1, we can also state that the mapping $d \mapsto d_{\rho_d}$ is an interior operation on $\mathcal{D}_X$.

7. A Galois connection between écarts on S and inequalities on $S \times \mathbb{R}$

Definition 7.1. By using the notation $X = S \times \mathbb{R}$, we define

$$\Phi\big((p, \lambda), (q, \mu)\big) = \mu - \lambda .$$

for any $(p, \lambda), (q, \mu) \in X$.

Moreover, by using Definitions 6.1, 4.2, 4.3 and 6.2, we define

$$\leqslant_\rho = \leqslant_{d_\rho} \qquad \text{and} \qquad \rho_{\leqslant} = \rho_{d_{\leqslant}}$$

for any $\rho \in \mathcal{D}_S$ and $\leqslant \in \mathcal{E}_X$.

Remark 7.1. Unfortunately, the latter notations may cause some confusions. However, we trust the reader's good sense to avoid them.

Now, concerning the induced inequality relation $\leqslant_\rho$, we can easily establish the following

Proposition 7.1. *For any any* $\rho \in \mathcal{D}_S$ *and* $(p, \lambda), (q, \mu) \in X$, *we have*

$$(p, \lambda) \leqslant_\rho (q, \mu) \iff \rho(p, q) \leqslant \mu - \lambda .$$

Proof. Note that

$$(p, \lambda) \leqslant_\rho (q, \mu) \iff (p, \lambda) \leqslant_{d_\rho} (q, \mu) \iff$$
$$\iff d_\rho\big((p, \lambda), (q, \mu)\big) \leqslant \Phi\big((p, \lambda), (q, \mu)\big) \iff \rho(p, q) \leqslant \mu - \lambda .$$

□

Remark 7.2. Thus, $\leqslant_\rho$ is a straightforward generalization of the induced inequality relations considered by DeMarr [15] and the second author [24]. (Baranga [2] has used a slightly different definition.)

In this respect, it is also worth noticing that now, by Definitions 7.1 and 4.3, we also have the following

Proposition 7.2. *For any any* $\leqslant \in \mathcal{E}_X$ *and* $(p, \lambda), (q, \mu) \in X$, *we have*

$$d_{\leqslant}\big((p, \lambda), (q, \mu)\big) = \mu - \lambda \qquad \text{if} \qquad (p, \lambda) \leqslant (q, \mu)$$

and

$$d_{\leqslant}\big((p, \lambda), (q, \mu)\big) = +\infty \qquad \text{if} \qquad (p, \lambda) \not\leqslant (q, \mu) .$$

Hence, by using Definitions 7.1 and 6.2, we can immediately derive the following

PROPOSITION 7.3. *For any* $\leqslant \in \mathcal{E}_X$ *and* $p, q \in S$, *we have*

$$\rho_{\leqslant}(p, q) = \inf\left\{ \mu - \lambda : \quad (p, \lambda) \leqslant (q, \mu) \right\}.$$

PROOF. Note that

$$\rho_{\leqslant}(p, q) = \rho_{d_{\leqslant}}(p, q) = \inf\left\{ d_{\leqslant}\big((p, \lambda), (q, \mu)\big) : \quad \lambda, \mu \in \mathbb{R} \right\} = \\ = \inf\left\{ \mu - \lambda : \quad (p, \lambda) \leqslant (q, \mu) \right\}.$$

□

REMARK 7.3. Thus, $\rho_{\leqslant}$ is a straightforward generalization the induced écart considered by the second author in [24].

Therefore, it is not surprising that we have the following

THEOREM 7.1. *The mappings*

$$\rho \ \mapsto \ \leqslant_{\rho} \qquad \text{and} \qquad \leqslant \ \mapsto \ \rho_{\leqslant}$$

establish a decreasing Galois connection between $\mathcal{D}_S$ *and* $\mathcal{E}_X$.

PROOF. If $\rho \in \mathcal{D}_S$ and $\leqslant \in \mathcal{E}_X$, then by the corresponding definitions it is clear that $\leqslant_{\rho} \in \mathcal{E}_X$ and $\rho_{\leqslant} \in \mathcal{D}_S$. Moreover, by Theorems 5.1 and 6.1, we can see that

$$\leqslant \subset \leqslant_{\rho} \iff \leqslant \subset \leqslant_{d_{\rho}} \iff d_{\rho} \leqslant d_{\leqslant} \iff \rho \leqslant \rho_{d_{\leqslant}} \iff \rho \leqslant \rho_{\leqslant}.$$

Therefore, by Definition 1.1 and Remark 1.1, the required assertion is also true. □

REMARK 7.4. From this theorem, by Corollary 3.1, we can see that the definition of $\rho_{\leqslant}$ cannot be altered without disturbing the validity of Theorem 7.1.

Moreover, from Theorem 2.5, by Theorem 3.1 and Remark 1.1, we can immediately get the following assertion which is also quite obvious from the corresponding definitions.

COROLLARY 7.1. *The mappings considered in Theorem 7.1 are decreasing.*

Now, by using Propositions 7.3 and 7.1, we can also easily establish the following

THEOREM 7.2. *For any* $\rho \in \mathcal{D}_S$, *we have* $\rho = \rho_{\leqslant_{\rho}}$.

PROOF. Namely,

$$\rho_{\leqslant_{\rho}}(p, q) = \inf\left\{ \mu - \lambda : \quad (p, \lambda) \leqslant_{\rho} (q, \mu) \right\} = \\ = \inf\left\{ \mu - \lambda : \quad \rho(p, q) \leqslant \mu - \lambda \right\} = \rho(p, q)$$

for all $p, q \in S$. □

REMARK 7.5. The above theorem shows that each member of $\mathcal{D}_S$ is closed.

Moreover, as an immediate consequence of Theorems 7.2, we can also state

COROLLARY 7.2. *The mapping* $\rho \mapsto \leqslant_{\rho}$ *is injective and the mapping* $\leqslant \mapsto \rho_{\leqslant}$ *is onto* $\mathcal{D}_S$.

From Theorem 7.1, by Theorem 3.1 and Remark 1.1, it is clear that we also have the following

THEOREM 7.3. *The mapping* $\leqslant\ \mapsto\ \leqslant_{\rho_{\leqslant}}$ *is a a closure operation on* $\mathcal{E}_X$ *such that* $\rho_{\leqslant} = \rho_{\leqslant_{\rho_{\leqslant}}}$ *for all* $\leqslant\in \mathcal{E}_X$.

Moreover, by using Propositions 7.1 and 7.3, we can also easily establish the following

THEOREM 7.4. *For any* $\leqslant\in \mathcal{E}_X$ *and* $(p, \lambda), (q, \mu) \in X$, *the following assertions are equivalent:*

(1) $(p, \lambda) \leqslant_{\rho_{\leqslant}} (q, \mu)$;

(2) for each $\varepsilon > 0$ *there exist* $\alpha, \beta \in \mathbb{R}$ *such that* $(p, \alpha) \leqslant (q, \beta)$ *and* $\beta - \alpha < \mu - \lambda + \varepsilon$.

PROOF. Namely,

$$(p, \lambda) \leqslant_{\rho_{\leqslant}} (q, \mu) \iff \rho_{\leqslant}(p, q) \leqslant \mu - \lambda \iff$$
$$\iff \inf\big\{ \beta - \alpha : \quad (p, \alpha) \leqslant (q, \beta)\big\} \leqslant \mu - \lambda .$$

Hence, it is clear that assertions (1) and (2) are equivalent. □

From this theorem, by Definition 1.3, it is clear that in particular we have

COROLLARY 7.3. *A member* $\leqslant$ *of* $\mathcal{E}_X$ *is closed if and only if for any* $(p, \lambda), (q, \mu) \in X$, *having the property (2) established in Theorem 7.4, we already have* $(p, \lambda) \leqslant (q, \mu)$.

REMARK 7.6. By Theorem 2.4 and Corollary 3.1, we can also state that a member $\leqslant$ of $\mathcal{E}_X$ is closed if and only if $\leqslant = \leqslant_{\rho_{\leqslant}}$, or equivalently $\leqslant = \leqslant_{\rho}$ for some $\rho \in \mathcal{D}_S$.

However, it is now more important to note that we can also prove the following

THEOREM 7.5. *A member* $\leqslant$ *of* $\mathcal{E}_X$ *is closed if and only if for any* (p, λ), $(q, \mu) \in X$

(1) $(p, \lambda) \leqslant (q, \mu)$ *implies* $(p, \lambda + \nu) \leqslant (q, \mu + \nu)$ *for all* $\nu \in \mathbb{R}$ *;*

(2) $(p, \lambda) \leqslant (q, \mu)$ *if and only if* $(p, \lambda) \leqslant (q, \mu + \varepsilon)$ *for all* $\varepsilon > 0$.

PROOF. If $\leqslant$ is a closed member of $\mathcal{E}$, then by Remark 7.6 we have $\leqslant = \leqslant_{\rho}$ for some $\rho \in \mathcal{D}_S$. Hence, by Proposition 7.1, it is clear that the above properties (1) and (2) hold.

Next, we show that if (1) and (2) hold, then $\leqslant_{\rho_{\leqslant}} \subset \leqslant$, and thus $\leqslant$ is a closed member of $\mathcal{E}$. For this, note that if $(p, \lambda) \leqslant_{\rho_{\leqslant}} (q, \mu)$, then Theorem 7.3, for each $\varepsilon > 0$, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$(p, \alpha) \leqslant (q, \beta) \qquad \text{and} \qquad \beta - \alpha < \mu - \lambda + \varepsilon .$$

Hence, by (2), we can see that

$$(p, \alpha) \leqslant (q, \mu - \lambda + \varepsilon + \alpha)$$

also holds. However, by (1), this implies that

$$(p, \lambda) \leqslant (q, \mu+\varepsilon).$$

Hence, again by (2), it is clear that $(p, \lambda) \leqslant (q, \mu)$, and thus the required inclusion is also true. □

8. Some further properties of the relations $\leqslant_\rho$ and $\rho_\leqslant$

By using Propositions 7.1 and 7.3 and Remark 7.6, we can also easily prove the following analogues of the results of Section 3.

Actually, some of the following propositions can also be derived from those of Section 4. However, the direct proofs seem more reliable.

Proposition 8.1. *If $\rho \in \mathcal{D}_S$, then the following assertions are equivalent:*

(1) $\leqslant_\rho$ is reflexive on X; *(2) $\rho(p, p) \leqslant 0$ for all $p \in S$.*

Corollary 8.1. *If $\leqslant$ is a closed member of $\mathcal{E}_X$, then the following assertions are equivalent:*

(1) $\leqslant$ is reflexive on X; *(2) $\rho_\leqslant(p, p) \leqslant 0$ for all $p \in S$.*

Proof. Now, by Remark 7.17, we have $\leqslant = \leqslant_{\rho_\leqslant}$. Therefore, the required equivalence can be obtained from Proposition 8.1 by writing $\rho_\leqslant$ in place of ρ. □

Proposition 8.2. *If $\rho \in \mathcal{D}_S$ such that $\rho(p, r) \leqslant \rho(p, q) + \rho(q, r)$ for all $p, q, r \in S$, then the relation $\leqslant_\rho$ is transitive.*

Corollary 8.2. *If $\leqslant$ is a closed member of $\mathcal{E}_X$ such that the écart $\rho_\leqslant$ satisfies the triangle inequality, then the relation $\leqslant$ is transitive.*

Proposition 8.3. *If ρ is a finite-valued écart on S such that the relation $\leqslant_\rho$ is transitive, then ρ satisfies the triangle inequality.*

Proof. In this case, for any $p, q, r \in S$ we have $(p, 0) \leqslant_\rho \big(q, \rho(p, q)\big)$ and $\big(q, \rho(p, q)\big) \leqslant_\rho \big(r, \rho(p, q) + \rho(q, r)\big)$. Therefore, $(p, 0) \leqslant_\rho \big(r, \rho(p, q) + \rho(q, r)\big)$, and thus $\rho(p, r) \leqslant \rho(p, q) + \rho(q, r)$ is also true. □

Corollary 8.3. *If $\leqslant$ is a closed and transitive member of $\mathcal{E}_X$ such that the écart $\rho_\leqslant$ is finite-valued, then $\rho_\leqslant$ satisfies the triangle inequality.*

Proposition 8.4. *If ρ is a distance function on S such that $\rho(p, q) = 0$ and $\rho(q, p) = 0$ imply $p = q$, then the relation $\leqslant_\rho$ is antisymmetric.*

Corollary 8.4. *If $\leqslant$ is a closed member of $\mathcal{E}_X$ such that $\rho_\leqslant(p, q) = 0$ and $\rho_\leqslant(q, p) = 0$ imply $p = q$, then the relation $\leqslant$ is antisymmetric.*

Proposition 8.5. *If $\rho \in \mathcal{D}_S$ such that the relation $\leqslant_\rho$ is antisymmetric, then $\rho(p, q) \leqslant \lambda$ and $\rho(q, p) \leqslant -\lambda$ imply $p = q$ and $\lambda = 0$.*

Proof. If the above inequalities hold, then $(p\,,\,0) \leqslant_\rho (q\,,\,\lambda)$ and $(q\,,\,\lambda) \leqslant_\rho (p\,,\,0)$. Therefore, $(p\,,\,0) = (q\,,\,\lambda)$, and thus the required equalities are also true. □

Corollary 8.5. *If $\leqslant$ is a closed and antisymmetric member of $\mathcal{E}_X$, then $\rho_\leqslant(p\,,\,q) \leqslant \lambda$ and $\rho_\leqslant(q\,,\,p) \leqslant -\lambda$ imply $p = q$ and $\lambda = 0$.*

Proposition 8.6. *If $\rho \in \mathcal{D}_S$ such that $\rho(p\,,\,q) = \rho(q\,,\,p)$ for all $p\,,\,q \in S$, then $(p\,,\,\lambda) \leqslant_\rho (q\,,\,\mu)$ implies $(q\,,\,\lambda) \leqslant_\rho (p\,,\,\mu)$.*

Corollary 8.6. *If $\leqslant$ is a closed member of $\mathcal{E}_X$ such that the écart $\rho_\leqslant$ is symmetric, then $(p\,,\,\lambda) \leqslant (q\,,\,\mu)$ implies $(q\,,\,\lambda) \leqslant (p\,,\,\mu)$.*

Proposition 8.7. *If ρ is a finite-valued écart on S such that the implication established in Proposition 8.6 holds, then ρ is symmetric.*

Proof. In this case, for any $p\,,\,q \in S$ we have $(p\,,\,0) \leqslant_\rho \big(q\,,\,\rho(p\,,\,q)\big)$, and hence $(q\,,\,0) \leqslant_\rho \big(p\,,\,\rho(p\,,\,q)\big)$. Therefore, $\rho(q\,,\,p) \leqslant \rho(p\,,\,q)$ also holds. □

Corollary 8.7. *If $\leqslant$ is a closed member of $\mathcal{E}_X$ such that the implication established in Corollary 8.6 holds, then the écart $\rho_\leqslant$ is symmetric.*

In this respect, it is also worth mentioning that we also have the following

Proposition 8.8. *If $\rho \in \mathcal{D}_S$, then the following assertions are equivalent:*

(1) $\leqslant_\rho$ is symmetric; *(2) $\rho(p\,,\,q) = +\infty$ for all $p\,,\,q \in S$.*

Proof. Note that if (1) holds and there exist $p\,,\,q \in S$ such that $\rho(p\,,\,q) < +\infty$, then by taking $\mu = \rho(p\,,\,q) + 1$ we have $(p\,,\,0) \leqslant_\rho (q\,,\,\mu)$, and hence $(q\,,\,\mu) \leqslant_\rho (p\,,\,0)$. Therefore, $\rho(q\,,\,p) \leqslant -\mu = -\rho(p\,,\,q) - 1$, and thus $0 \leqslant \rho(p\,,\,q) + \rho(q\,,\,p) \leqslant -1$ also holds, which is a contradiction. □

Corollary 8.8. *If $\leqslant$ is a closed member of $\mathcal{E}_X$, then the following assertions are equivalent:*

(1) $\leqslant$ is symmetric; *(2) $\rho_\leqslant(p\,,\,q) = +\infty$ for all $p\,,\,q \in S$.*

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