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On the blow-up semidiscretizations in time of some non-local parabolic problems with Neumann boundary conditions

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Abstract. In this paper, we address the following initial value problem

$$u_t = \int_\Omega J(x-y)(u(y,t)-u(x,t))dy + f(u) \quad \text{in} \quad \overline{\Omega} \times (0,T),$$
$$u(x,0) = \varphi(x) \geqslant 0 \quad \text{in} \quad \overline{\Omega},$$

where $f : [0,\infty) \to [0,\infty)$ is a C^1 nondecreasing function, $\int^\infty \frac{d\sigma}{f(\sigma)} < \infty$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $J : \mathbb{R}^N \to \mathbb{R}$ is a kernel which is nonnegative and bounded in $\mathbb{R}^N$. Under some conditions, we show that the solution of a semidiscrete form of the above problem blows up in a finite time and estimate its semidiscrete blow-up time. We also prove that the semidiscrete blow-up time converges to the real one when the mesh size goes to zero. Finally, we give some numerical results to illustrate our analysis.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the following initial value problem

$$u_t(x,t) = \int_\Omega J(x-y)(u(y,t)-u(x,t))dy + f(u) \quad \text{in} \quad \overline{\Omega} \times (0,T), \tag{1.1}$$

$$u(x,0) = \varphi(x) \geqslant 0 \quad \text{in} \quad \overline{\Omega}, \tag{1.2}$$

where $f : [0,\infty) \to [0,\infty)$ is a C^1 nondecreasing function, $\int^\infty \frac{d\sigma}{f(\sigma)} < \infty$, $J : \mathbb{R}^N \to \mathbb{R}$ is a kernel which is nonnegative and bounded in $\mathbb{R}^N$. In addition, J is symmetric ($J(z) = J(-z)$) and $\int_{\mathbb{R}^N} J(z)dz = 1$. The initial datum $\varphi(x)$ is nonnegative and continuous in $\overline{\Omega}$.

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Here, $(0,T)$ is the maximal time interval on which the solution u exists. The time T may be finite or infinite. When T is infinite, then we say that the solution u exists globally. When T is finite, then the solution u develops a singularity in a finite time, namely,

$$\lim_{t\to T}\|u(\cdot,t)\|_\infty=\infty,$$

where $\|u(\cdot,t)\|_\infty=\sup_{x\in\overline{\Omega}}|u(x,t)|$. In this last case, we say that the solution u blows up in a finite time, and the time T is called the blow-up time of the solution u. Recently, nonlocal diffusion has been the subject of investigation of many authors (see, [1]-[7], [10]-[12], [14]-[18], [20], and the references cited therein). Nonlocal evolution equations of the form

$$u_t=\int_{\mathbb{R}^N}J(x-y)(u(y,t)-u(x,t))dy,$$

and variations of it have been used by several authors to model diffusion processes (see, [3], [4], [17], [18]). The solution $u(x,t)$ can be interpreted as the density of a single population at the point x, at the time t, and $J(x-y)$ as the probability distribution of jumping from location y to location x. Then, the convolution $(J*u)(x,t)=\int_{\mathbb{R}^N}J(x-y)u(y,t)dy$ is the rate at which individuals are arriving to position x from all other places, and $-u(x,t)=-\int_{\mathbb{R}^N}J(x-y)u(y,t)dy$ is the rate at which they are leaving location x to travel to any other site (see, [17]). Let us notice that the reaction term $f(u)$ in equation (1.1) can be rewritten as follows

$$f(u(x,t))=\int_{\mathbb{R}^N}J(x-y)f(u(x,t))dy.$$

Therefore, in view of the above equality, the reaction term $f(u)$ can be interpreted as a force that increases the rate at which individuals are arriving to location x from all other places. Due to the presence of the term $f(u(x,t))$, we shall see later the blow-up of the density $u(x,t)$. On the other hand, the integral in (1.1) is taken over Ω. Thus there is no individuals that enter or leave the domain Ω. It is the reason why in the title of the paper, we have added Neumann boundary condition. For the problem described in (1.1)-(1.2), the local in time existence and uniqueness of solutions have been proved by Perez-LLanos and Rossi in [27], where one may also find some results about blow-up solutions. In the current paper, we are interested in the numerical study of the phenomenon of blow-up using a semidiscrete form of (1.1)-(1.2). We start by the construction of an explicit adaptive scheme as follows. Approximate the solution u of (1.1)-(1.2) by the solution U_n of the following semidiscrete equations

$$\delta_tU_n(x)=\int_\Omega J(x-y)(U_n(y)-U_n(x))dy+f(U_n(x))\quad\text{in}\quad\overline{\Omega},\tag{1.3}$$

$$U_0(0) = \varphi_h(x) \geqslant 0 \quad \text{in} \quad \overline{\Omega}, \tag{1.4}$$

where $n \geqslant 0$, φ_h is nonnegative and continuous in $\overline{\Omega}$, $\lim_{h\to 0} \varphi_h = \varphi$

$$\delta_t U_n(x) = \frac{U_{n+1}(x) - U_n(x)}{\Delta t_n}.$$

In order to permit the semidiscrete solution to reproduce the properties of the continuous one when the time t approaches the blow-up time T, we need to adapt the size of the step so that we take

$$\Delta t_n = \min\left\{\Delta t, \frac{\tau \|U_n\|_\infty}{f(\|U_n\|_\infty)}\right\},$$

where $\|U_n\|_\infty = \sup_{x\in\overline{\Omega}} |U_n(x)|$, $\tau \in (0,1)$ and $\Delta t \in (0,1)$ is a parameter. Let us notice that the restriction on the time step ensures the nonnegativity of the discrete solution.
To facilitate our discussion, let us define the notion of semidiscrete blow-up time.

Definition 1.1. We say that the semidiscrete solution U_n of (1.3)-(1.4) blows up in a finite time if $\lim_{n\to\infty} \|U_n\|_\infty = \infty$, and the series $\sum_{n=0}^{\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{\infty} \Delta t_n$ is called the semidiscrete blow-up time of the semidiscrete solution U_n.

In the present paper, under some conditions, we show that the semidiscrete solution blows up in a finite time and estimate its semidiscrete blow-up time. We also show that the semidiscrete blow-up time converges to the real one when the mesh size goes to zero. A similar result has been obtained by Le Roux in [21]-[22], and the same author and Mainge in [23] within the framework of local parabolic problems. One may also consult the papers [24] and [25] for numerical studies of the phenomenon of blow-up where semidiscretizations in space have been utilized. The remainder of the paper is organized as follows. In the next section, we give some results about the semidiscrete maximum principle for nonlocal problems. In the third and fourth sections, we prove our main results, and finally, in the last section, we show some numerical experiments to illustrate our analysis.

2. Properties of the semidiscrete scheme

In this section, we gather some results about the semidiscrete maximum principle of nonlocal problems for our subsequent use.
The following result is a version of the maximum principle for semidiscrete nonlocal problems.

LEMMA 2.1. For $n \geqslant 0$, let $U_n, a_n \in C^0(\overline{\Omega})$ be such that

$$\delta_t U_n(x) \geqslant \int_{\Omega} J(x-y)(U_n(y)-U_n(x))dy + a_n(x)U_n(x) \quad \text{in} \quad \overline{\Omega}, \quad n \geqslant 0,$$

$$U_0(x) \geqslant 0 \quad \text{in} \quad \overline{\Omega}.$$

Then, we have $U_n(x) \geqslant 0$ in $\overline{\Omega}$, $n > 0$ when $\Delta t_n \leqslant \frac{1}{1+\|a_n\|_\infty}$.

Proof. If $U_n(x) \geqslant 0$ in $\overline{\Omega}$, then a straightforward computation reveals that

$$U_{n+1}(x) \geqslant U_n(x)(1-\Delta t_n - \|a_n\|_\infty \Delta t_n) \quad \text{in} \quad \overline{\Omega}, \quad n \geqslant 0. \tag{2.1}$$

To obtain the above inequality, we have used the fact that

$$\int_{\Omega} J(x-y)U_n(y)dy \geqslant 0 \quad \text{in} \quad \overline{\Omega}, \text{ and } \int_{\Omega} J(x-y)dy \leqslant \int_{\mathbb{R}^N} J(x-y)dy = 1.$$

Making use of (2.1) and an argument of recursion, we easily check that $U_{n+1}(x) \geqslant 0$ in $\overline{\Omega}$, $n \geqslant 0$. This finishes the proof. □

An immediate consequence of the above result is the following comparison lemma. Its proof is straightforward.

LEMMA 2.2. For $n \geqslant 0$, let U_n, V_n and $a_n \in C^0(\overline{\Omega})$ be such that

$$\delta_t U_n(x) - \int_{\Omega} J(x-y)(U_n(y)-U_n(x))dy + a_n(x)U_n(x)$$

$$\geqslant \delta_t V_n(x) - \int_{\Omega} J(x-y)(V_n(y)-V_n(x))dy + a_n(x)V_n(x) \quad \text{in} \quad \overline{\Omega}, \quad n \geqslant 0,$$

$$U_0(x) \geqslant V_0(x) \quad \text{in} \quad \overline{\Omega}.$$

Then, we have $U_n(x) \geqslant V_n(x)$ in $\overline{\Omega}$, $n > 0$ when $\Delta t_n \leqslant \frac{1}{1+\|a_n\|_\infty}$.

3. The semidiscrete blow-up time

In this section, under some assumptions, we show that the semidiscrete solution blows up in a finite time and estimate its semidiscrete blow-up time.

We need the following lemma.

LEMMA 3.1. Let a and b be two positive numbers such that, $b > 1$, $f(0) = 0$, and $f(s)$ is convex for positive values of s. Then, the following estimate holds

$$\sum_{n=0}^{\infty} \frac{ab^n}{f(ab^n)} \leqslant \frac{a}{f(a)} + \frac{1}{\ln(b)} \int_a^\infty \frac{d\sigma}{f(\sigma)}.$$

Proof. We observe that

$$\int_0^\infty \frac{ab^\sigma d\sigma}{f(ab^\sigma)} = \sum_{n=0}^{\infty}\int_n^{n+1} \frac{ab^\sigma d\sigma}{f(ab^\sigma)} \geqslant \sum_{n=0}^{\infty} \frac{ab^{n+1}}{f(ab^{n+1})},$$

because $\frac{f(s)}{s}$ is nondecreasing for nonnegative values of s. We infer that

$$\int_0^\infty \frac{ab^\sigma d\sigma}{f(ab^\sigma)} \geqslant -\frac{a}{f(a)} + \sum_{n=0}^{\infty} \frac{ab^n}{f(ab^n)}. \tag{3.1}$$

On the other hand, by a change of variables, we see that

$$\int_0^\infty \frac{ab^\sigma d\sigma}{f(ab^\sigma)} = \frac{1}{\ln(b)}\int_0^a \frac{d\sigma}{f(\sigma)}. \tag{3.2}$$

Use (3.1) and (3.2) to complete the rest of the proof. □

Now, let us give our result about the semidiscrete blow-up time which is stated in the following theorem.

THEOREM 3.1. Suppose that $f(0) = 0$, $f(s)$ is convex for positive values of s and $\varphi_{\min} = \min_{x\in\overline{\Omega}} \varphi(x) > 0$. Let $A = \frac{\varphi_{\min}}{f(\varphi_{\min})}.$ If $A < 1$, then the solution U_n of (1.3)-(1.4) blows up in a finite time, and its semidiscrete blow-up time $T_h^{\Delta t}$ is estimated as follows

$$T_h^{\Delta t} \leqslant \frac{\tau\|\varphi\|_\infty}{f(\|\varphi\|_\infty)} + \frac{\tau}{\ln(1+\tau')}\int_{\|\varphi\|_\infty}^\infty \frac{d\sigma}{f(\sigma)},$$

where $\tau' = (1-A)\min\{\frac{\Delta t f(\|\varphi\|_\infty)}{\|\varphi\|_\infty}, \tau\}$.

Proof. Due to the fact that $U_n(x)$ is nonnegative in $\overline{\Omega}$, and $\int_\Omega J(x-y)dy \leqslant \int_{\mathbb{R}^N} J(x-y)dy = 1$, a straightforward computation reveals that

$$\delta_t U_n(x) \geqslant -U_n(x) + f(U_n(x)) \quad \text{in} \quad \overline{\Omega}, \quad n \geqslant 0,$$

which implies that

$$\delta_t U_n(x) \geqslant f(U_n(x))\left(1 - \frac{U_n(x)}{f(U_n(x))}\right) \quad \text{in} \quad \overline{\Omega}, \quad n \geqslant 0.$$

Since $f(0) = 0$, then $\frac{s}{f(s)}$ is nonincreasing for positive values of s. We have $\delta_t U_0(x) > 0$ in $\overline{\Omega}$, and we claim that $\delta_t U_n(x) > 0$ in $\overline{\Omega}$, $n \geqslant 0$. To prove the claim, we argue by contradiction. Let N be the first integer such that $\delta_t U_N(x_0) \leqslant 0$ for a certain $x_0 \in \overline{\Omega}$. Since $\delta_t U_0(x) > 0$ in $\overline{\Omega}$, we know that $N \geqslant 1$. Due to the fact that $\delta_t U_{N-1}(x) > 0$ in $\overline{\Omega}$, we get $U_N(x) \geqslant U_{N-1}(x) \geqslant U_0(x) \geqslant \varphi_{\min}$ in $\overline{\Omega}$. Consequently,

$$0 \geqslant \delta_t U_N(x) \geqslant f(\varphi_{\min})\left(1 - \frac{\varphi_{\min}}{f(\varphi_{\min})}\right) > 0 \quad \text{in} \quad \overline{\Omega},$$

which is a contradiction, and the claim is proved. According to the fact that $\delta_t U_n(x) > 0$ in $\overline{\Omega}$ for $n > 0$, we note that $U_n(x) \geqslant U_0(x) \geqslant \varphi_{\min}$ in $\overline{\Omega}$, $n \geqslant 0$, which implies that

$$\delta_t U_n(x) \geqslant f(U_n(x))\left(1 - \frac{\varphi_{\min}}{f(\varphi_{\min})}\right) \quad \text{in} \quad \overline{\Omega}, \quad n \geqslant 0. \tag{3.3}$$

This entails that

$$\delta_t U_n(x) \geqslant (1-A) f(U_n(x)) \quad \text{in} \quad \overline{\Omega}, \quad n \geqslant 0. \tag{3.4}$$

Consequently we know that the estimate (3.4) may be rewritten as follows

$$U_{n+1}(x) \geqslant U_n(x) + (1-A)\Delta t_n f(U_n(x)) \quad \text{in} \quad \overline{\Omega}, \quad n \geqslant 0. \tag{3.5}$$

Let $x_0 \in \overline{\Omega}$ be such that $U_n(x_0) = \|U_n\|_\infty$. Replacing x by x_0 in (3.5), we find that

$$U_{n+1}(x_0) \geqslant \|U_n\|_\infty + (1-A)\Delta t_n f(\|U_n\|_\infty) \quad \text{in} \quad \overline{\Omega}, \quad n \geqslant 0.$$

Use the fact that $\|U_{n+1}\|_\infty \geqslant U_{n+1}(x_0)$ to arrive at

$$\|U_{n+1}\|_\infty \geqslant \|U_n\|_\infty + (1-A)\Delta t_n f(\|U_n\|_\infty), \quad n \geqslant 0. \tag{3.6}$$

It is not hard to see that

$$(1-A)\Delta t_n f(\|U_n\|_\infty) = (1-A)\min\{\Delta t \frac{f(\|U_n\|_\infty)}{\|U_n\|_\infty}, \tau\}, \quad n \geqslant 0. \tag{3.7}$$

In view of (3.6), we note that $\|U_{n+1}\|_\infty \geqslant \|U_n\|_\infty$, $n \geqslant 0$, and by induction, we discover that $\|U_n\|_\infty \geqslant \|U_0\|_\infty = \|\varphi\|_\infty$, $n > 0$. It follows from (3.7) that

$$(1-A)\Delta t_n \frac{f(\|U_n\|_\infty)}{\|U_n\|_\infty} \geqslant (1-A)\min\{\Delta t \frac{f(\|\varphi\|_\infty)}{\|\varphi\|_\infty}, \tau\} = \tau', \quad n \geqslant 0,$$

and making use of (3.6), we find that

$$\|U_{n+1}\|_\infty \geqslant \|U_n\|_\infty (1+\tau'), \quad n \geqslant 0. \tag{3.8}$$

Using an argument of recursion, we get

$$\|U_n\|_\infty \geqslant \|U_0\|_\infty (1+\tau')^n = \|\varphi\|_\infty (1+\tau')^n, \quad n \geqslant 0. \tag{3.9}$$

Consequently, according (3.9), we discover that $\|U_n\|_\infty$ goes to infinity as n approaches infinity. Now, let us estimate the semidiscrete blow-up time. The restriction on the time step and (3.9) render

$$\sum_{n=0}^{\infty} \Delta t_n \leqslant \sum_{n=0}^{\infty} \frac{\tau \|\varphi\|_\infty (1+\tau')^n}{f(\|\varphi\|_\infty (1+\tau')^n)}.$$

It follows from Lemma 3.1 that

$$\sum_{n=0}^{\infty} \Delta t_n \leqslant \frac{\tau \|\varphi\|_\infty}{f(\|\varphi\|_\infty)} + \frac{\tau}{\ln(1+\tau')} \int_{\|\varphi\|_\infty}^{\infty} \frac{d\sigma}{f(\sigma)}.$$

Use the fact that the quantity on the right hand side of the above inequality is finite to complete the rest of the proof. □

REMARK 3.1. Using (3.8) and an argument of recursion, we discover that

$$\|U_n\|_\infty \geqslant \|U_q\|_\infty (1+\tau')^{n-q}, \quad n \geqslant q.$$

The restriction on the time step leads us to

$$\sum_{n=q}^{\infty} \Delta t_n \leqslant \sum_{n=q}^{\infty} \frac{\tau \|U_q\|_\infty (1+\tau')^{n-q}}{f(\|U_q\|_\infty (1+\tau')^{n-q})}.$$

An application of Lemma 3.1 gives

$$\sum_{n=q}^{\infty} \Delta t_n \leqslant \frac{\tau \|U_q\|_\infty}{f(\|U_q\|_\infty)} + \frac{\tau}{\ln(1+\tau')} \int_{\|U_q\|_\infty}^{\infty} \frac{d\sigma}{f(\sigma)},$$

or equivalently

$$T^{\Delta t} - t_q \leqslant \frac{\tau \|U_q\|_\infty}{f(\|U_q\|_\infty)} + \frac{\tau}{\ln(1+\tau')} \int_{\|U_q\|_\infty}^{\infty} \frac{d\sigma}{f(\sigma)}.$$

If we take $\tau = \Delta t$, then we note that

$$\frac{\tau'}{\tau} = (1-A) \min\{\frac{f(\|\varphi\|_\infty)}{\|\varphi\|_\infty}, 1\}.$$

Consequently, applying Taylor's expansion we see that $\frac{\tau}{\ln(1+\tau')} = O(1)$ with the choice $\tau = \Delta t$.

In the sequel, we pick $\tau = \Delta t$.

4. Convergence of the semidiscrete blow-up time

In this section, under some hypotheses, we show that the semidiscrete solution blows up in a finite time, and its semidiscrete blow-up time converges to the real one when the mesh size goes to zero. To do this, we firstly show that the semidiscrete solution approaches the real one in any interval $\overline{\Omega} \times [0, T-\tau]$ with $\tau \in (0, T)$. This result is stated in the following theorem.

THEOREM 4.1. Assume that the problem (1.1)-(1.2) admits a solution $u \in C^{0,2}(\overline{\Omega} \times [0, T-\tau])$ with $\tau \in (0, T)$. Then, the problem (1.3)-(1.4) admits a unique solution $U_n \in C^0(\overline{\Omega})$ for Δt and h small enough, $n \leqslant J$, and the following relation holds

$$\sup_{0 \leqslant n \leqslant J} \|U_n - u(\cdot, t_n)\|_\infty = O(\Delta t + \|\varphi_h - \varphi\|_\infty) \quad \text{as} \quad (\Delta t, h) \to (0,0), \tag{4.1}$$

where J is a positive integer such that $\sum_{j=0}^{J-1} \Delta t_j \leqslant T - \tau$, and $t_n = \sum_{j=0}^{n-1} \Delta t_j$.

Proof. The problem (1.3)-(1.4) admits for each $n \geqslant 0$, a unique solution $U_n \in C^0(\overline{\Omega})$. Let $N \leqslant J$ be the greatest integer such that

$$\|U_n - u(\cdot, t_n)\|_\infty < 1 \quad \text{for} \quad n < N. \tag{4.2}$$

Making use of the fact that (4.2) holds when $n = 0$, we note that $N \geqslant 1$. Since $u \in C^{0,2}$, then there exists a positive constant M such that $\|u(\cdot, t_n)\|_\infty \leqslant M$ for $n < N$. An application of the triangle inequality gives

$$\|U_n\|_\infty \leqslant \|u(\cdot, t_n)\|_\infty + \|U_n - u(\cdot, t_n)\|_\infty \leqslant 1 + M \quad \text{for} \quad n < N. \tag{4.3}$$

Apply Taylor's expansion to obtain

$$\delta_t u(x, t_n) = u_t(x, t_n) + \frac{\Delta t_n}{2} u_{tt}(x, \widetilde{t_n}) \quad \text{in} \quad \overline{\Omega}, \quad n < N,$$

which implies that

$$\delta_t u(x, t_n) = \int_\Omega J(x-y)(u(y, t_n) - u(x, t_n))dy + f(u(x, t_n))$$

$$+\frac{\Delta t_n}{2} u_{tt}(x, \widetilde{t_n}) \quad \text{in} \quad \overline{\Omega}, \quad n < N.$$

Introduce the error e_n defined as follows

$$e_n(x) = U_n(x) - u(x, t_n) \quad \text{in} \quad \overline{\Omega}, \quad n < N.$$

Invoking the mean value theorem, it is easy to see that

$$\delta_t e_n(x) = \int_\Omega J(x-y)(e_n(y) - e_n(x))dy + f'(\xi_n(x))e_n(x)$$

$$-\frac{\Delta t_n}{2} u_{tt}(x, \widetilde{t_n}) \quad \text{in} \quad \overline{\Omega}, \quad n < N,$$

where $\xi_n(x)$ is an intermediate value between $u(x, t_n)$ and $U_n(x)$. We infer that there exists a positive constant K such that

$$\delta_t e_n(x) \leqslant \int_\Omega J(x-y)(e_n(y) - e_n(x))dy + f'(\xi_n(x))e_n(x)$$

$$+K\Delta t \quad \text{in} \quad \overline{\Omega}, \quad n < N, \tag{4.4}$$

because $u \in C^{0,2}$ and $\Delta t_n = O(\Delta t)$. Introduce the function Z_n defined as follows

$$Z_n(x) = (\|\varphi_h - \varphi\|_\infty + K\Delta t)e^{(L+1)t_n} \quad \text{in} \quad \overline{\Omega}, \quad n < N,$$

where $L = f'(M + 1)$. A straightforward computation reveals that

$$\delta_t Z_n \geqslant \int_\Omega J(x-y)(Z_n(y) - Z_n(x))dy + f'(\xi_n(x))Z_n(x)$$

$$+K\Delta t \quad \text{in} \quad \overline{\Omega}, \quad n < N,$$

$$Z_0(x) \geqslant e_0(x) \quad \text{in} \quad \overline{\Omega}.$$

We deduce from Lemma 2.2 that

$$Z_n(x) \geqslant e_n(x) \quad \text{in} \quad \overline{\Omega}, \quad n < N.$$

In the same way, we also show that

$$Z_n(x) \geqslant -e_n(x) \quad \text{in} \quad \overline{\Omega}, \quad n < N,$$

which implies that

$$|e_n(x)| \leqslant Z_n(x) \quad \text{in} \quad \overline{\Omega}, \quad n < N,$$

or equivalently

$$\|U_n - u(\cdot, t_n)\|_\infty \leqslant (\|\varphi_h - \varphi\|_\infty + K\Delta t)e^{(L+1)t_n}, \quad n < N. \tag{4.5}$$

Now, let us reveal that $N = J$. To prove this result, we argue by contradiction. Assume that $N < J$. Replacing n by N in (4.5), and using (4.2), we discover that

$$1 \leqslant \|U_N - u(\cdot, t_N)\|_\infty \leqslant (\|\varphi_h - \varphi\|_\infty + K\Delta t)e^{(L+1)T}.$$

Since the term on the right hand side of the second inequality goes to zero as Δt and h tend to zero, we deduce that $1 \leqslant 0$, which is impossible. Consequently, $N = J$, and the proof is complete. □

Now, we are in a position to prove the main result of this section

Theorem 4.2. Assume that the problem (1.1)–(1.2) has a solution u which blows up in a finite time T such that $u \in C^{0,2}(\overline{\Omega} \times [0, T))$. Then, the solution U_n of (1.3)-(1.4) blows up in a finite time, and its semidiscrete blow-up time $T_h^{\Delta t}$ obeys the following relation

$$\lim_{(\Delta t, h) \to (0,0)} T_h^{\Delta t} = T.$$

Proof: Let $0 < \varepsilon < T/2$. In view of Remark 3.1, we know that $\frac{\tau}{\ln(1+\tau')}$ is bounded. Thus, there exists a positive constant R such that

$$\frac{\tau R}{f(R)} + \frac{\tau}{\ln(1+\tau')} \int_R^\infty \frac{d\sigma}{f(\sigma)} < \frac{\varepsilon}{2}. \tag{4.6}$$

Since u blows up at the time T, there exists a time $T_0 \in (T - \varepsilon/2, T)$ such that

$$\|u(\cdot, t)\|_\infty \geqslant 2R \quad \text{for} \quad t \in [T_0, T).$$

Let q be a positive integer such that

$$t_q = \sum_{n=0}^{q-1} \Delta t_n \in [T_0, T).$$

Invoking Theorem 4.1, we know that the problem (1.3)-(1.4) admits a unique solution $U_n \in C^0(\overline{\Omega})$ such that $\|U_q - u(\cdot, t_q)\|_\infty \leqslant R$. An application of the triangle inequality gives $\|U_q\|_\infty \geqslant \|u(\cdot,t_q)\|_\infty - \|U_q - u(\cdot,t_q)\|_\infty$, which implies that $\|U_q\|_\infty \geqslant 2R - R = R$. It follows from Remark 3.1 and (4.6) that

$$|T_h^{\Delta t} - T| \leqslant |T^{\Delta t} - t_q| + |t_q - T| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This finishes the proof. □

5. Numerical results

In this section, we give some computational experiments to illustrate the theory given in the previous section. We consider the problem (1.1)-(1.2) in the case where $\Omega = (-1, 1)$,

$$J(x) = \begin{cases} \frac{3}{2}x^2 & \text{if} \quad |x| < 1, \\ 0 & \text{if} \quad |x| \geqslant 1, \end{cases}$$

$u_0(x) = \frac{2+\varepsilon\cos(\pi x)}{4}$ with $\varepsilon \in [0,1]$. We start by the construction of some adaptive schemes as follows. Let I be a positive integer and let $h = 2/I$. Define the grid $x_i = -1 + ih$, $0 \leqslant i \leqslant I$, and approximate the solution u of (1.1)-(1.2) by the solution $U_h^{(n)} = (U_0^{(n)}, \cdots, U_I^{(n)})^T$ of the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \sum_{j=0}^{I-1} hJ(x_i - x_j)(U_j^{(n)} - U_i^{(n)}) + f(U_i^{(n)}), \quad 0 \leqslant i \leqslant I,$$

$$U_i^{(0)} = \varphi_i, \quad 0 \leqslant i \leqslant I,$$

where $\varphi_i = \frac{2+\varepsilon\cos(\pi x_i)}{4}$. In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches the blow-up time T, we need to adapt the size of the time step so that we take

$$\Delta t_n = \min\left\{h^2, \frac{h^2\|U_h^{(n)}\|_\infty}{f(\|U_h^{(n)}\|_\infty)}\right\}$$

with $\|U_h^{(n)}\|_\infty = \max_{0\leqslant i\leqslant I} |U_i^{(n)}|$. Let us notice that the restriction on the time step ensures the nonnegativity of the discrete solution. We also approximate the solution u of (1.1)-(1.2) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \sum_{j=0}^{I-1} hJ(x_i - x_j)(U_j^{(n+1)} - U_i^{(n+1)}) + f(U_i^{(n)}), \quad 0 \leqslant i \leqslant I,$$

$$U_i^{(0)} = \varphi_i, \quad 0 \leqslant i \leqslant I.$$

As in the case of the explicit scheme, here, we also choose

$$\Delta t_n = \frac{h^2 \|U_h^{(n)}\|_\infty}{f(\|U_h^{(n)}\|_\infty)}.$$

Let us again remark that for the above implicit scheme, existence and nonnegativity of the discrete solution are also guaranteed using standard methods (see, for instance [9]).
We need the following definition.

Definition 5.1. We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme blows up in a finite time if $\lim_{n\to\infty} \|U_h^{(n)}\|_\infty = \infty$, and the series $\sum_{n=0}^{\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{\infty} \Delta t_n$ is called the numerical blow-up time of the discrete solution $U_h^{(n)}$.

In the following tables, in rows, we present the numerical blow-up times, the numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical blow-up time $t_n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when

$$\Delta t_n = |t_{n+1} - t_n| \leqslant 10^{-16}.$$

The order (s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Numerical experiments for $f(s) = s^2$

First case: $\varepsilon = 0$

Table 1.: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	*CPU time*	s
16	2.025369	2101	1	-
32	2.007079	8001	8	-
64	2.001851	30536	58	1.81
128	2.000477	116419	631	1.91

Table 2.: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	$CPU\ time$	s
16	2.025276	2101	2	-
32	2.007076	8001	12	-
64	2.001186	30536	113	1.63
128	2.000477	116419	2760	3.05

Second case: $\varepsilon = 1/10$

Table 3.: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	$CPU\ time$	s
16	1.929188	2143	0.8	-
32	1.911537	8166	8	-
64	1.906539	31196	58	1.82
128	1.905216	119059	734	1.92

Table 4.: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	$CPU\ time$	s
16	1.934435	1822	1.2	-
32	1.908374	7507	12	-
64	1.903592	29886	118	2.45
128	1.903413	116501	2700	4.74

Third case:
$\varepsilon = 1/100$

Table 5.: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	t_n	n	$CPU\ time$	s
16	2.015320	2101	1	-
32	1.997097	8177	8	-
64	1.991902	31241	54	1.81
128	1.995052	119238	869	0.72

Table 6.: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	t_n	n	$CPU\ time$	s
16	2.030313	1905	2	-
32	2.001710	7411	11	-
64	1.992897	29977	120	1.70
128	1.990658	116702	2805	1.87

Remark 5.1. If we consider the problem (1.1)-(1.2) in the case where $u_0(x) = 1/2$, then using standard methods, one may easily check that the blow-up time of the solution u is $T = 2$. We note from Tables 1 to 2 that the numerical blow-up time of the discrete solution is approximately equal 2. We observe in passing the continuity of the numerical blow-up time (see, Tables 3-6).

In what follows, we also gives some plots to illustrate our analysis. In Figures 1-6, we can appreciate that the discrete solution blows up in a finite time. Let us notice that when the initial datum is constant, then the discrete solution blows up globally (see, Figures 1-2).

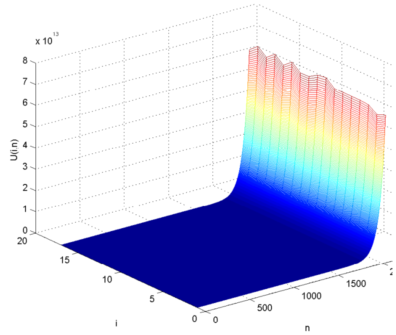

Fig. 1 Explicit scheme
Evolution of the discrete solution
$l = 16, \epsilon = 0, f(s) = s^2$

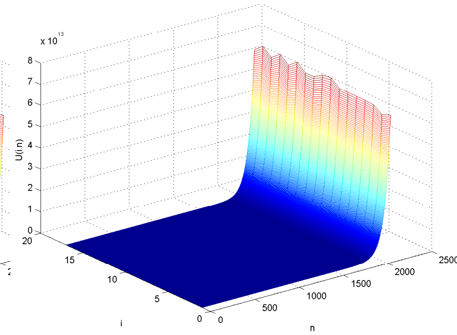

Fig. 2 Implicit scheme
Evolution of the discrete solution
$l = 16, \epsilon = 0, f(s) = s^2$

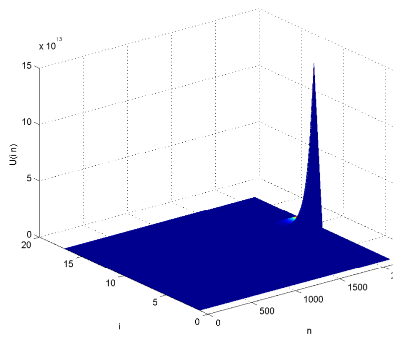

Fig. 3 Explicit scheme
Evolution of the discrete solution
$l = 16, \epsilon = 1/10, f(s) = s^2$

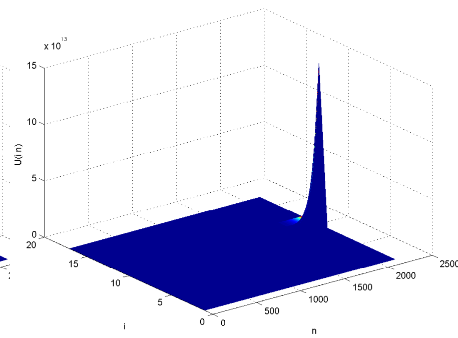

Fig. 4 Implicit scheme
Evolution of the discrete solution
$l = 16, \epsilon = 1/10, f(s) = s^2$

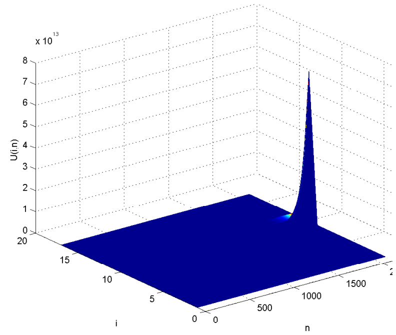

Fig. 5 Explicit scheme
Evolution of the discrete solution
$l = 16, \epsilon = 1/100, f(s) = s^2$

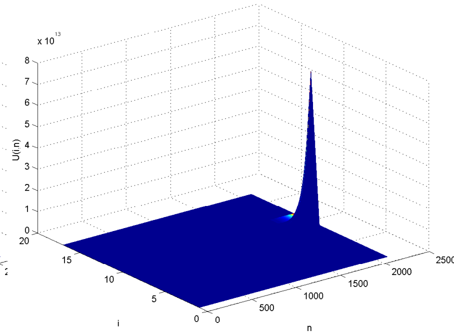

Fig. 6 Implicit scheme
Evolution of the discrete solution
$l = 16, \epsilon = 1/100, f(s) = s^2$

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