SCIENTIA
Series A: *Mathematical Sciences*, Vol. 17 (2009), 67–82
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

The connected edge geodetic number of a graph

A.P. Santhakumaran[a] and J. John[b]

Abstract. For a non-trivial connected graph G, a set $S \subseteq V(G)$ is called an edge geodetic set of G if every edge of G is contained in a geodesic joining some pair of vertices in S. The edge geodetic number $g_1(G)$ of G is the minimum order of its edge geodetic sets and any edge geodetic set of order $g_1(G)$ is an edge geodetic basis. A connected edge geodetic set of G is an edge geodetic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected edge geodetic set of G is the connected edge geodetic number of G and is denoted by $g_{1c}(G)$. A connected edge geodetic set of cardinality $g_{1c}(G)$ is called a g_{1c} - set of G or a connected edge geodetic basis of G. Some general properties satisfied by this concept are studied. The connected edge geodetic number of certain classes of graphs are determined. Connected graphs of order p with connected edge geodetic number 2 are characterized. Various necessary conditions for the connected edge geodetic number of a graph to be $p-1$ or p are given. It is shown that every pair k, p of integers with $3 \leqslant k \leqslant p$ is realizable as the connected edge geodetic number and order of some connected graph. For positive integers r, d and $n \geqslant d+1$ with $r \leqslant d \leqslant 2r$, there exists a connected graph of radius r, diameter d and connected edge geodetic number n. If p, d and n are integers such that $2 \leqslant d \leqslant p-1$ and $d+1 \leqslant n \leqslant p$, then there exists a connected graph G of order p, diameter d and $g_{1c}(G) = n$. Also if p, a and b are positive integers such that $2 \leqslant a < b \leqslant p$, then there exists a connected graph G of order p, $g_1(G) = a$ and $g_{1c}(G) = b$.

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [4]. The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. An $u-v$ path of length $d(u, v)$ is called an $u-v$ *geodesic*. It is known

2000 *Mathematics Subject Classification.* Primary 05C12.
Key words and phrases. distance, geodesic, geodetic number, connected geodetic number, edge geodetic number, connected edge geodetic number.

that this distance is a metric on the vertex set $V(G)$. A vertex x is said to *lie on* an $u-v$ geodesic P if x is a vertex of P including the vertices u and v. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, *rad* G and the maximum eccentricity is its *diameter*, *diam* G of G. For a cut-vertex v in a connected graph G and a component H of $G-v$, the subgraph H and the vertex v together with all edges joining v to $V(H)$ is called a *branch* of G at v. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbours is complete. For any real number x, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

A *geodetic set* of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The *geodetic number* $g(G)$ of G is the minimum order of its geodetic sets and any set of order $g(G)$ is a *geodetic basis*. The geodetic number of a graph was introduced in [1, 5] and further studied in [3]. It was shown in [5] that determining the geodetic number of a graph is NP-hard problem.

The connected geodetic number was studied by Santhakumaran, Titus and John in [9]. A *connected geodetic set* of G is a geodetic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected geodetic set of G is the *connected geodetic number* of G and is denoted by $g_c(G)$. A connected geodetic set of cardinality $g_c(G)$ is called a *g_c-set* of G or *connected geodetic basis* of G. The edge geodetic number was studied by Santhakumaran and John in [8]. A set $S \subseteq V(G)$ is called an *edge geodetic set* of G if every edge of G is contained in a geodesic joining some pair of vertices in S. The *edge geodetic number* $g_1(G)$ of G is the minimum order of its edge geodetic sets and any edge geodetic set of order $g_1(G)$ is an *edge geodetic basis*. Throughout the following G denotes a connected graph with at least two vertices.

The following theorems are used in the sequel.

THEOREM 1.1. [8] Each extreme vertex of a connected graph G belongs to every edge geodetic set of G. In particular, each end vertex of G belongs every edge geodetic set of G.

THEOREM 1.2. [8] For a connected graph G, $g_1(G) = 2$ if and only if there exist peripheral vertices u and v such that every edge of G is on a diametral path joining u and v.

THEOREM 1.3. [8] For any non-trivial tree T with k end vertices, $g_1(T) = k$.

THEOREM 1.4. [9] Every extreme vertex of a connected graph G belongs to every connected geodetic set of G. In particular, each end vertex of G belongs to every connected geodetic set of G.

THEOREM 1.5. [9] Every cut vertex of a connected graph G belongs to every connected geodetic set of G.

THEOREM 1.6. [9] For any non-trivial tree T of order p, $g_c(T) = p$.

THEOREM 1.7. [6] For a connected graph G, $g_c(G) \geqslant 1 + diam(G)$.

2. The connected edge geodetic number of a graph

Definition 2.1. Let G be a connected graph with at least two vertices. A *connected edge geodetic set* of G is an edge geodetic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected edge geodetic set of G is the *connected edge geodetic number* of G and is denoted by $g_{1c}(G)$. A connected edge geodetic set of cardinality $g_{1c}(G)$ is called a *g_{1c}-set* of G or a *connected edge geodetic basis* of G.

EXAMPLE 2.2. For the graph G given in Figure 1 $S_1 = \{v_1, v_2, v_3, v_4\}$ is a g_{1c}-set so that $g_{1c}(G) = 4$. Also $S_2 = \{v_1, v_2, v_3, v_5\}$ is another g_{1c}-set of G.

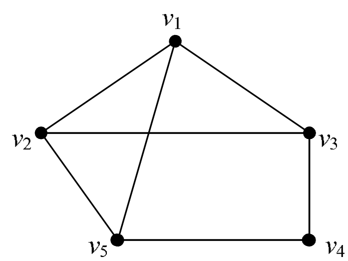

FIGURE 1. G

REMARK 2.3. For the graph G given in Figure 1, $S = \{v_1, v_2, v_4\}$ is an edge geodetic basis of G so that $g_1(G) = 3$. Thus the edge geodetic number and the connected edge geodetic number are different.

REMARK 2.4. There can be more than one g_{1c}-set for a graph G. For the graph G given in Figure 1, S_1 and S_2 are two different g_{1c}-sets for G.

THEOREM 2.5. Every extreme vertex of a connected graph G belongs to every connected edge geodetic set of G. In particular, every end vertex of G belongs to every connected edge geodetic set of G.

PROOF. Since every connected edge geodetic set is also an edge geodetic set, the result follows from Theorem 1.1. □

THEOREM 2.6. For any connected graph G of order p, $2 \leqslant g_1(G) \leqslant g_{1c}(G) \leqslant p$.

PROOF. An edge geodetic set needs at least two vertices and therefore $g_1(G) \geqslant 2$. Since every connected edge geodetic set is also an edge geodetic set, it follows that $g_1(G) \leqslant g_{1c}(G)$. Also, since $V(G)$ induces a connected edge geodetic set of G, it is clear that $g_{1c}(G) \leqslant p$. □

REMARK 2.7. The bounds for $g_{1c}(G)$ in Theorem 2.6 are sharp. For the complete graph K_2, $g_{1c}(G) = g_1(G) = 2$ and for the complete graph $K_p(p \geqslant 2)$, $g_{1c}(G) = p$. Also, all the inequalities in the theorem are strict. For the graph G given in Figure 1, $g_1(G) = 3$, $g_{1c}(G) = 4$ and $p = 5$.

COROLLARY 2.8. Let G be any connected graph. If $g_{1c}(G) = 2$, then $g_1(G) = 2$.

COROLLARY 2.9. Let G be any connected graph. If $g_1(G) = p$, then $g_{1c}(G) = p$.

COROLLARY 2.10. For any connected graph G with k extreme vertices, $g_{1c}(G) \geqslant max\{2, k\}$.

PROOF. This follows from Theorem 2.5 and Theorem 2.6. □

COROLLARY 2.11. For the complete graph $K_p(p \geqslant 2)$, $g_{1c}(K_p) = p$.

THEOREM 2.12. Let G be a connected graph with cut vertices and let S be a connected edge geodetic set of G. If v is a cut vertex of G, then every component of $G - v$ contains an element of S.

PROOF. Let v be a cut vertex of G and S be a connected edge geodetic set of G. Suppose that there exists a component say G_1 of $G - v$ such that G_1 contains no vertex of S. By Theorem 2.5, S contains all the extreme vertices of G and hence it follows that G_1 does not contain any extreme vertex of G. Thus G_1 contains at least one edge say xy. Since S is a connected edge geodetic set of G, there exist $u, w \in S$ such that xy lies in some $u - w$ geodesic $P : u = u_0, u_1....x, y...u_l = w$ in G. Since the $u - x$ subpath of P and the $x - w$ subpath of P both contain v, it follows that P is not a path, contrary to assumption. □

COROLLARY 2.13. Let G be a connected graph with cut vertices and let S be a connected edge geodetic set of G. Then every branch of G contains an element of S.

THEOREM 2.14. Every cut-vertex of a connected graph G belongs to every connected edge geodetic set of G.

PROOF. Let G be a connected graph and S be a connected edge geodetic set of G. Let v be any cut vertex of G and let $G_1, G_2, \ldots, G_r(r \geqslant 2)$ be the components of $G - v$. By Theorem 2.12, S contains at least one vertex from each $G_i(1 \leqslant i \leqslant r)$. Since $G[S]$ is connected, it follows that $v \in S$. □

COROLLARY 2.15. For any connected graph G with k extreme vertices and l cut vertices, $g_{1c}(G) \geqslant max\{2, k + l\}$.

PROOF. This follows from Theorems 2.5, 2.6 and 2.14. □

COROLLARY 2.16. For any non-trivial tree T of order p, $g_{1c}(T) = p$.

PROOF. This follows from Corollary 2.15. □

THEOREM 2.17. For every pair k, p of integers with $3 \leqslant k \leqslant p$, there exists a connected graph G of order p such that $g_{1c}(G) = k$.

PROOF. Let $P_k : u_1, u_2, u_3, \ldots, u_k$ be a path on k vertices. Add new vertices $v_1, v_2, \ldots, v_{p-k}$ and join each $v_i (1 \leqslant i \leqslant p-k)$ with u_1 and u_3, thereby obtaining the graph G in Figure 2 Then G has order p and $S = \{u_3, u_4, \ldots, u_k\}$ is the set of all cut vertices and extreme vertices of G. By Theorems 2.5 and 2.14, $g_{1c}(G) \geqslant k-2$. Clearly S is not a connected edge geodetic set of G and so $g_{1c}(G) > k-2$. Now, neither $S \cup \{v_i\} (1 \leqslant i \leqslant p-k)$ nor $S \cup \{u_2\}$ is an edge geodetic set of G. But $T = S \cup \{u_1\}$ is an edge geodetic set of G such that $G[T]$ is dis-connected. It is clear that $T \cup \{u_2\}$ is a connected edge geodetic set of G and hence it follows that $g_{1c}(G) = k$. □

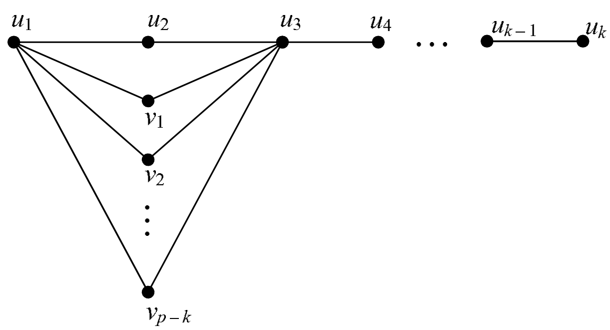

FIGURE 2. G

THEOREM 2.18. For the complete bipartite graph $G = K_{m,n}$,
i) $g_{1c}(G) = 2$ if $m = n = 1$.
ii) $g_{1c}(G) = n + 1$ if $m = 1, n \geqslant 2$.
iii) $g_{1c}(G) = min\{m, n\} + 1$ if $m, n \geqslant 2$.

PROOF. i) and ii) follow from Corollary 2.16. (iii) Let $m, n \geqslant 2$. First assume that $m < n$. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be a bipartition of G. Let $S = U \cup \{w_1\}$. We prove that S is a connected edge geodetic basis of G. Any edge $u_i w_j (1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n)$ lies on the geodesic $u_i w_j u_k$ for any $k \neq i$ so that S is an edge geodetic set of G. Since $G[S]$ is connected, S is a connected edge geodetic set of G. Let T be any set of vertices such that $|T| < |S|$. If $T \subsetneq U$, $G[T]$ is not connected and so T is not a connected edge geodetic set of G. If $T \subsetneq W$, again T is not a connected edge geodetic set of G by a similar argument. If $T \supseteq U$, then since $|T| < |S|$, we have $T = U$, which is not a connected geodetic set of G. Similarly, since $|T| < |S|$, T cannot contain W. For if $T \supseteq W$, then $|T| \geqslant n > m \geqslant m+1 = |S|$, which is a contradiction. Thus $T \subsetneq U \cup W$ such that T contains at least one vertex from each of S and W. Then since $|T| < |S|$, there exists vertices $u_i \in U$ and $w_j \in W$

such that $u_i \notin T$ and $w_j \notin T$. Then clearly the edge u_iw_j does not lie on a geodesic connecting two vertices of T so that T is not a connected edge geodetic set. Thus in any case T is not a connected edge geodetic set of G. Hence S is a connected edge geodetic basis so that $g_1(K_{m,n}) = |S| = m+1$. Now, if $m = n$, we can prove similarly that $S = U \cup \{y\}$, where $y \in W$ is a connected edge geodetic basis of G. Hence the theorem follows. □

THEOREM 2.19. For the cycle $C_p (p \geqslant 3)$, $g_{1c}(C_p) = \begin{cases} \frac{p}{2} + 1 & \text{if p is even} \\ \lfloor \frac{p}{2} \rfloor + 2 & \text{if p is odd.} \end{cases}$

PROOF. If p is even, let $p = 2n$. Let $C_{2n} : v_1, v_2, v_3, \ldots, v_{2n}, v_1$ be the cycle of order $2n$. Then v_{n+1} is the antipodal vertex of v_1. Let $S = \{v_1, v_{n+1}\}$. Clearly S is an edge geodetic set of G. It is clear that $G[S]$ is not connected. But $S_1 = \{v_1, v_2, \ldots, v_{n+1}\}$ is a connected edge geodetic set of G so that $g_{1c}(G) \leqslant n+1$. If S' is any connected set of vertices of G with $|S'| < |S_1|$, then S' contains atmost n elements. Hence no two vertices of S' are pairwise antipodal. Thus S' is not an edge geodetic set of G. It follows that $g_{1c}(G) = n+1$.

Let p be odd. If $p = 3$, then by Corollary 2.11, $g_{1c}(G) = 3 = \lfloor \frac{p}{2} \rfloor + 2$. Let $p \geqslant 5$ and let $p = 2n+1$. Let $C_{2n+1} : v_1, v_2, \ldots, v_{2n+1}, v_1$ be the cycle of order $2n+1$. Then v_{n+1} and v_{n+2} are antipodal vertices of v_1. Let $S = \{v_1, v_{n+1}, v_{n+2}\}$. It is clear that S is an edge geodetic set of G and $G[S]$ is not connected. But $S_1 = \{v_1, v_2, \ldots, v_n, v_{n+1}, v_{n+2}\}$ is a connected edge geodetic set of G so that $g_{1c}(G) \leqslant n+2$. If S' is any connected set of vertices of G with $|S'| < |S_1|$, then S' contains at most $n+1$ elements. Hence S' contains at most two vertices say u and v which are antipodal to each other. Let $w \neq v$ be the antipodal vertex of u. Then the edge vw does not lie on any geodesic joining a pair of vertices of S'. Thus S' is not an edge geodetic set of G. It follows that $g_{1c}(G) = n+2 = \lfloor \frac{p}{2} \rfloor + 2$. □

The following theorem characterizes graphs for which the connected edge geodetic number is 2.

THEOREM 2.20. For any connected graph G, $g_{1c}(G) = 2$ if and only if $G = K_2$.

PROOF. If $G = K_2$, then $g_{1c}(G) = 2$. Conversely, let $g_{1c}(G) = 2$. Let $S = \{u, v\}$ be a minimum connected edge geodetic set of G. Then uv is an edge. If $G \neq K_2$, then there exists an edge xy different from uv. Then xy cannot lie on any $u - v$ geodesic so that S is not a g_{1c}-set, which is a contradiction. Thus $G = K_2$. □

We give below some necessary conditions on a graph G for which $g_{1c}(G) = p-1$ and $g_{1c}(G) = p$.

THEOREM 2.21. Let G be a connected graph of order $p \geqslant 3$. If G contains exactly one vertex of degree $p-1$ and which is not a cut vertex of G, then $g_{1c}(G) = p-1$.

PROOF. Let v be the unique vertex of degree $p-1$. Let $S = V - \{v\}$. Let vu be any edge incident with v. Since v is the only vertex of degree $p-1$, there exists at least one vertex say u' such that u and u' are not adjacent. Then vu lies on the geodesic uvu' joining u and u' in S. Any edge $e = xy$ which is not incident with v lies on the geodesic xy itself joining two vertices of S. Thus S is an edge geodetic set of G. Since v is not a cut vertex of G, $G[S]$ is connected so that $g_{1c}(G) \leqslant p-1$. We claim that $g_{1c}(G) = p-1$. Let T be any set of vertices with $|T| \leqslant p-2$. Then there exist at least two vertices say u and w which are not in T. If $v \notin T$, then $v \neq u$ or $v \neq w$ so that the edge vu or vw cannot lie on any geodesic joining two vertices of T. If $v \in T$, again the edge vu or vw cannot lie on any geodesic joining two vertices of T. In any case, T is not an edge geodetic set of G. Hence $g_{1c}(G) = p-1$. □

COROLLARY 2.22. If a connected graph G has exactly one vertex v of degree $p-1$ and which is not a cut vertex, then $g_{1c}(G) = p-1$ and G has a unique connected edge geodetic basis consisting of all the vertices of G other than v.

PROOF. The proof is contained in the proof of Theorem 2.21. □

COROLLARY 2.23. For the wheel $W_{1,p-1}$, $g_{1c}(W_{1,p-1}) = p-1$.

REMARK 2.24. The converse of Theorem 2.21 is false. For the graph G given in Figure 3, $S = \{v_1, v_2, v_3, v_4, v_5\}$ is a g_{1c}-set of G. Therefore $g_{1c}(G) = 5 = p-1$ and no vertex has degree $p-1$.

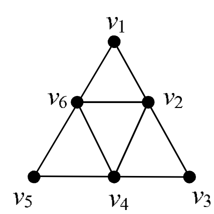

FIGURE 3. G

THEOREM 2.25. Let G be a connected graph. If every vertex of G is either an extreme vertex or a cut-vertex of G, then $g_{1c}(G) = p$.

PROOF. Let G be a connected graph with every vertex of G is either an extreme vertex or a cut-vertex of G. Then the result follows from Corollary 2.15. □

COROLLARY 2.26. Let G be a connected graph of order $p \geqslant 3$ such that $G = K_1 + \cup m_j K_j$, where $\sum m_j \geqslant 2$, then $g_{1c}(G) = p$.

REMARK 2.27. The converse of the Theorem 2.25 is false. For the graph G given in Figure 4, $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a connected g_{1c}-set of G so that $g_{1c}(G) = 6 = p$. But G has vertices which are neither cut vertices nor extreme vertices.

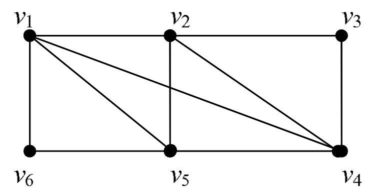

FIGURE 4. G

THEOREM 2.28. If a connected graph G has more than one vertex of degree $p-1$, then every connected edge geodetic set contains all those vertices of degree $p-1$.

PROOF. Let G be a graph with more than one vertex of degree $p-1$. If u and v are two vertices of degree $p-1$, then uv is an edge and it does not lie on any geodesic joining two vertices of G other than u and v. Hence it follows that both u and v belong to every connected edge geodetic set of G. □

THEOREM 2.29. For any connected graph G with at least two vertices of degree $p-1$, $g_{1c}(G) = p$.

PROOF. If all the vertices are of degree $p-1$, then $G = K_p$ and so $g_{1c}(G) = p$. Otherwise, let $v_1, v_2, \ldots, v_k (2 \leqslant k \leqslant p-2)$ be the vertices of degree $p-1$. Suppose $g_{1c}(G) < p$. Let S be a connected edge geodetic basis of G such that $|S| < p$. By Theorem 2.28, S contains all the vertices $v_1, v_2, \ldots, v_k$. Let v be a vertex such that $v \notin S$. Then $deg(v) < p-1$. Since any two of $v_1, v_2, \ldots, v_k$ are adjacent, the edge $vv_i (1 \leqslant i \leqslant k)$ cannot lie on a geodesic joining a pair of vertices v_j and $v_l (j \neq l)$. Similarly, since any v_j is adjacent to any vertex of S, which is different from $v_1, v_2, \ldots, v_k$, the edge $vv_i (1 \leqslant i \leqslant k)$ cannot lie on a geodesic joining a vertex v_j and a vertex of S, which is different from $v_1, v_2, \ldots, v_k$. Now, let u and w be vertices of S different from $v_1, v_2, \ldots, v_k$. Since v_i is adjacent to both u and w and $d(u, v) \leqslant 2$, the edge vv_i cannot lie on a geodesic joining u and w. Thus we see that the edges $vv_i (1 \leqslant i \leqslant k)$ do not lie on any geodesic joining a pair of vertices of S, which is a contradiction to the fact that S is a connected edge geodetic basis of G. Hence $g_{1c}(G) = p$. □

REMARK 2.30. The converse of Theorem 2.29 is false. For the graph G given in Figure 4, $S = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a connected edge geodetic basis of G so that $g_{1c}(G) = 6 = p$ and G has no vertex of degree $p-1$.

THEOREM 2.31. If G is a connected graph of order $p \geqslant 3$ such that G contains a cut vertex v of degree $p-1$, then $g_{1c}(G) = p$.

PROOF. Let S be any connected edge geodetic set of G. Then, by Theorem 2.14, $v \in S$. Claim $S = V(G)$ is the connected edge geodetic basis of G. Otherwise, there exists a proper subset T of V such that T is a connected edge geodetic basis of G. By

Theorem 2.14, $v \in T$. Since $T \subsetneq V$, there exists a vertex $u \in V$ such that $u \notin T$. Since T is a connected edge geodetic set of G, the edge vu lies on a geodesic joining a pair of vertices x and y of T. Let the geodesic be $P : x, \ldots, v, u, \ldots, y$. We have $u \neq x, y$. If $x = v$, then, since v is adjacent to every vertex of G, vy is the only geodesic joining v and y. Similarly if $x \neq v$, then xvy is the only geodesic joining x and y. Thus in any case P is not a $x - y$ geodesic, which is a contradiction. □

Remark 2.32. The converse of Theorem 2.31 is false. For the graph G given in Figure 5, $S = \{v_1, v_2, v_3, v_4, v_5\}$ is a connected edge geodetic basis. Therefore $g_{1c}(G) = 5 = p$. But G has no cut-vertex of degree $p - 1$.

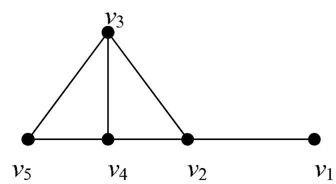

Figure 5. G

Theorem 2.33. If G is a connected non complete graph such that it has a minimum cutset of G consisting of i independent vertices, then $g_{1c}(G) \leqslant p - i + 1$.

Proof. Since G is non complete, it is clear that $1 \leqslant i \leqslant p - 2$. Let $U = \{v_1, v_2, \ldots, v_i\}$ be a minimum independent cutset of vertices of G. Let $G_1, G_2, \ldots,$ $G_r (r \geqslant 2)$ be the components of $G - U$ and let $S = V(G) - U$. Then every vertex $v_j (1 \leqslant j \leqslant i)$ is adjacent to at least one vertex of G_t for every $t (1 \leqslant t \leqslant r)$. Let uv be any edge of G. If uv lies in one of the G_t for any $(1 \leqslant t \leqslant r)$, then clearly uv lies on the geodesic (uv itself) joining two vertices u and v of S. Otherwise, uv is of the form $vju (1 \leqslant j \leqslant i)$, where $u \in G_t$ for some t such that $1 \leqslant t \leqslant r$. As $r \geqslant 2$, v_j is adjacent to some w in G_s for some $s \neq t$ such that $1 \leqslant s \leqslant r$. Thus $v_j u$ lies on the geodesic $u v_j w$ of length 2. Thus S is an edge geodetic set such that $G[S]$ is not connected. Now, it is clear that $S \cup \{x\}$, where $x \in U$ is a connected edge geodetic set of G so that $g_{1c}(G) \leqslant |S \cup \{x\}| = p - i + 1$. □

Corollary 2.34. If G is a connected non complete graph such that it has a minimum cutset of G consisting of i independent vertices, then $g_{1c}(G) \leqslant p - \kappa + 1$, where κ is the vertex connectivity of G.

Proof. By Theorem 2.33, $g_{1c}(G) \leqslant p - i + 1$. Since $\kappa \leqslant i$, it follows that $g_{1c}(G) \leqslant p - \kappa + 1$. □

Corollary 2.35. If G is a connected non complete graph such that every minimum cutset of vertices of G is independent, then $g_{1c}(G) \leqslant p - \kappa + 1$.

Proof. This follows from Theorem 2.33. □

3. Connected geodetic number and connected edge geodetic number of a graph

THEOREM 3.1. Every connected edge geodetic set of a connected graph G is a connected geodetic set of G.

PROOF. Let G be a connected graph and S be a connected edge geodetic set of G. Let $v \in V(G)$. Let uv be an edge of G. Then uv lies on a geodesic joining a pair of vertices of S. Thus v lies on a geodesic joining a pair of vertices of S so that S is a geodetic set of G. Since S is connected edge geodetic set of G, $G[S]$ is connected and so S is a connected geodetic set of G. □

THEOREM 3.2. For any connected graph G, $2 \leqslant g_c(G) \leqslant g_{1c}(G) \leqslant p$.

PROOF. Any connected edge geodetic set needs atleast two vertices and so $g_{1c}(G) \geqslant 2$. Let S be any connected edge geodetic set of G with minimum cardinality. Then $g_{1c} = |S|$. By Theorem 3.1, S is a connected geodetic set of G so that $g_c(G) \leqslant |S| = g_{1c}(G)$. Also, since $V(G)$ induces a connected edge geodetic set of G, it is clear that $g_{1c}(G) \leqslant p$. Thus $2 \leqslant g_c(G) \leqslant g_{1c}(G) \leqslant p$. □

REMARK 3.3. For the graph K_2, $g_{1c}(K_2) = 2$. For the graph G given in Figure 1, $S = \{v_1, v_3, v_5\}$ is a g_c-set so that $g_c(G) = 3$ and $S_1 = \{v_1, v_2, v_3, v_5\}$ is a g_{1c}-set so that $g_{1c}(G) = 4$ and so $g_c(G) < g_{1c}(G)$. Also for any non-trivial tree T, $g_c(T) = g_{1c}(T)$, by Theorem 1.6 and Corollary 2.16.

PROBLEM 3.4. Characterize graphs G for which $g_c(G) = g_{1c}(G)$.

THEOREM 3.5. For a connected graph G, $g_{1c}(G) \geqslant 1 + diam(G)$.

PROOF. This follows from Theorems 1.7 and 3.2. □

THEOREM 3.6. If G is a connected graph such that $g_1(G) = 2$, then $g_{1c}(G) = 1 + diam(G)$.

PROOF. Let $g_1(G) = 2$. Then by Theorem 1.2, there exist peripheral vertices u and v such that every edge of G lies on a diametral path joining u and v. Let $P : u = u_0, u_1, u_2, \ldots, u_n = v$ be a diametral path of G. Let $S = \{u_0, u_1, u_2, \ldots, u_n\}$. Then it is clear that S is a connected edge geodetic set of G so that $g_{1c}(G) \leqslant |S| = 1 + diam(G)$. Now the theorem follows from Theorem 3.5. □

THEOREM 3.7. For any positive integers $3 \leqslant a \leqslant b$, there exists a connected graph G such that $g_c(G) = a$ and $g_{1c}(G) = b$.

PROOF. If $a = b$, let $G = K_{1,a-1}$. Then by Theorem 1.6 and Corollary 2.16, $g_c(G) = g_{1c}(G) = a$. If $a = 3$ and $b \geqslant 4$, then the graph G in Figure 6 is obtained from the path on three vertices $P : u_1, u_2, u_3$ by adding $b-2$ new vertices $v_1, v_2, \ldots, v_{b-2}$ and joining each $v_i (1 \leqslant i \leqslant b-2)$ with u_1, u_2 and u_3. It is clear that $S = \{u_1, u_2, u_3\}$ is a minimum connected geodetic set of G so that $g_c(G) = 3$. Now u_2 is a full degree vertex such that it is not a cut vertex of G and so by Theorem 2.21, $g_{1c}(G) = b-2+3-1 = b$.

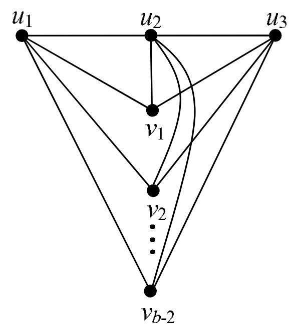

Figure 6. G

If $3 < a < b$, let G be the graph obtained from the path on three vertices $P : u_1, u_2, u_3$ by adding $b - 3$ new vertices $v_1, v_2 \ldots, v_{b-a}, w_1, w_2, \ldots, w_{a-3}$ and joining each $v_i (1 \leqslant i \leqslant b-a)$ with u_1, u_2, u_3 and joining each $w_i (1 \leqslant i \leqslant a-3)$ with u_2 and the graph G is of order b and shown in Figure 7. By Theorems 1.4 and 1.5, every connected geodetic set of G contains all the extreme vertices and all the cut vertices of G. Now, let $S = \{w_1, w_2, \ldots, w_{a-3}, u_2\}$. It is clear that S is not a connected geodetic set of G. It is also easily seen that $S \cup \{v\}$, where $v \in V(G) - S$ is not a connected geodetic set of G. But, it is clear that $S_1 = S \cup \{u_1, u_3\}$ is a connected geodetic set of G so that $g_c(G) = a - 2 + 2 = a$. Now, since G contains the cut vertex u_2, which is of full degree, it follows from Theorem 2.31 that $g_{1c}(G) = b$. □

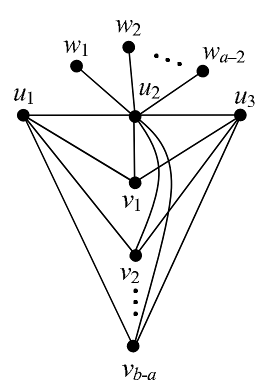

Figure 7. G

For every connected graph G, rad $G \leqslant$ diam $G \leqslant 2$ rad G, Ostrand[7] showed that every two positive integers a and b with $a \leqslant b \leqslant 2a$ are realizable as the radius and diameter, respectively, of some connected graph. Ostrand's theorem can be extended so that the connected edge geodetic number can be prescribed when $g_{1c}(G) \geqslant diam G + 1$.

THEOREM 3.8. For positive integers r, d and $n \geqslant d+1$ with $r \leqslant d \leqslant 2r$, there exists a connected graph G with rad $G = r$, diam $G = d$ and $g_{1c}(G) = n$.

PROOF. If $r = 1$, then $d = 1$ or 2. If $d = 1$, let $G = K_n$. Then by Corollary 2.11, $g_{1c}(G) = n$. If $d = 2$, let $G = K_{1,n-1}$. Then by Corollary 2.16, $g_{1c}(G) = n$. Now, let $r \geqslant 2$. We construct a graph G with the desired properties as follows:
Case 1. Suppose $r = d$. For $n = d+1$, let $G = C_{2r}$. Then it is clear that $r = d$. By Theorem 2.19, $g_{1c}(G) = d + 1 = n$. Now, let $n \geqslant d + 2$. Let $C_{2r} : u_1, u_2, \ldots, u_{2r}, u_1$ be the cycle of order $2r$. Let G be the graph obtained by adding the new vertices $x_1, x_2, \ldots, x_{n-r-1}$ and joining each $x_i (1 \leqslant i \leqslant n-r-1)$ with u_1 and u_2 of C_{2r}. The graph G is shown in Figure 8. It is easily verified that the eccentricity of each vertex of

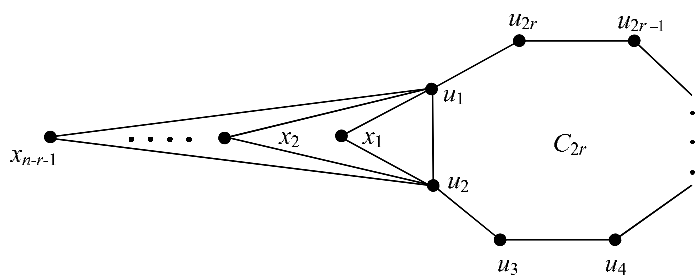

FIGURE 8. G

G is r so that *rad* $G =$ *diam* $G = r$. Let $S = \{x_1, x_2, \ldots, x_{n-r-1}\}$. Then S is the set of all extreme vertices of G with $|S| = n - r - 1$. It is clear that S is not a connected edge geodetic set of G. Let $T = S \cup \{u_1, u_2, u_3, \ldots, u_{r+1}\}$. It is clear that T is a connected edge geodetic set of G and so $g_{1c}(G) \leqslant |T| = n$. Now, if $g_{1c}(G) < n$, then there exists a connected edge geodetic set M of G such that $|M| < n$. By Theorem 2.5, M contains S and since$|M| < n$, M contains at most r vertices of C_{2r}. Since M is a connected edge geodetic set of G, u_1 or u_2 must belong to M. We consider two cases.
Case a. Suppose $u_1 \in M$ and $u_2 \notin M$. Since M is a connected edge geodetic set of G and $|M| < n$, M contains at most the vertices $u_1, u_{2r}, u_{2r-1}, \ldots, u_{r+2}$ of C_{2r}. Then the edge $u_{r+1}u_{r+2}$ does not lie on any geodesic joining a pair of vertices of M and so M is not a connected edge geodetic set of G, which is a contradiction.
Case b. Suppose $u_1, u_2 \in M$. Now we may assume without loss of generality that M contains at most the vertices $u_1, u_2, u_3, \ldots, u_r$ of C_{2r}. Then the edge $u_r u_{r+1}$ does not lie on any geodesic joining a pair of vertices of M and so M is not a connected geodetic set of G, which is a contradiction. Thus $g_{1c}(G) = n$.

Case 2. Suppose $r < d \leqslant 2r$. Let $C_{2r} : v_1, v_2, \ldots, v_{2r}, v_1$ be a cycle of order $2r$ and let $P_{d-r+1} : u_0, u_1, \ldots, u_{d-r}$ be a path of order $d-r+1$. Let H be a graph obtained from C_{2r} and P_{d-r+1} by identifying v_1 in C_{2r} and u_0 in P_{d-r+1}. Now, we add $n-d-1$ new vertices $w_1, w_2, \ldots, w_{n-d-1}$ to the graph H and join each vertex $w_i (1 \leqslant i \leqslant n-d-1)$ to the vertex u_{d-r-1} and obtain the graph G of Figure 9. Then $rad\ G = r$ and $diam$

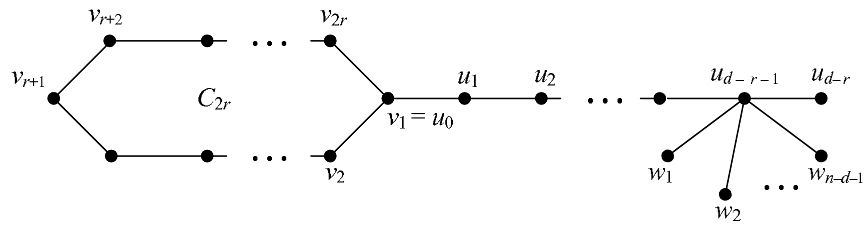

FIGURE 9. G

$G = d$. Let $S = \{v_1, u_1, u_2, \ldots, u_{d-r}, w_1, w_2, \ldots, w_{n-d-1}\}$ be the set of all cut vertices and extreme vertices of G. By Theorems 2.5 and 2.14, every connected edge geodetic set of G contains S. It is clear that S is not a connected edge geodetic set of G. Let $T = S \cup \{v_2, v_3, \ldots, v_{r+1}\}$. It is clear that T is a connected edge geodetic set of G and so $g_{1c}(G) \leqslant |T| = n$. Then by an argument similar to that given in the proof of case 1 of this theorem, it can be proved that $g_{1c}(G) = n$. □

THEOREM 3.9. If p, d and n are integers such that $2 \leqslant d \leqslant p-1$ and $d+1 \leqslant n \leqslant p$, then there exists a connected graph G of order p, diameter d and $g_{1c}(G) = n$.

PROOF. We prove this theorem by considering three cases.
Case 1. Let $d = 2$. Let P_3 be the path on three vertices u_1, u_2 and u_3. Now add $p-3$ new vertices $w_1, w_2, \ldots, w_{p-n}, v_1, v_2, \ldots, v_{n-3}$. Let G be the graph obtained by joining each $v_i (1 \leqslant i \leqslant n-3)$ to u_1 and u_2 and each $w_i (1 \leqslant i \leqslant p-n)$ to both u_1 and u_3. The graph G is shown in Figure 10 and has order p with diameter $d = 2$. Let $S = \{v_1, v_2, \ldots, v_{n-3}\}$ be the set of extreme vertices of G. By Theorem 4.5, every connected edge geodetic set contains S. It is clear that S is not a connected edge geodetic set of G. It is easily seen that $S \cup \{v\}$ or $S \cup \{u, v\}$, where $u, v \notin S$, is not a connected edge geodetic set of G. Now, it is clear that $S \cup \{u_1, u_2, u_3\}$ is a connected edge geodetic set of G so that $g_{1c}(G) = n$.

Case 2. Let $3 \leqslant d \leqslant p-2$. Let $P_{d+1} : u_0, u_1, u_2, \ldots, u_d$ be a path of length d. Add $p-d-1$ new vertices $w_1, w_2, \ldots, w_{p-n}, v_1, v_2, \ldots, v_{n-d-1}$ to P_{d+1} and join $w_1, w_2, \ldots, w_{p-n}$ to both u_0 and u_2 and join $v_1, v_2, \ldots, v_{n-d-1}$ to u_{d-1}, there by producing the graph G of Figure 11. Then G has order p and diameter d. Let $S = \{u_2, u_3, \ldots, u_d, v_1, v_2, \ldots, v_{n-d-1}\}$ be the set of all cut vertices and all extreme vertices of G. By Theorems 2.5 and 2.14, every connected edge geodetic set of G contains S. It is clear that S is not a connected edge geodetic set of G. Clearly

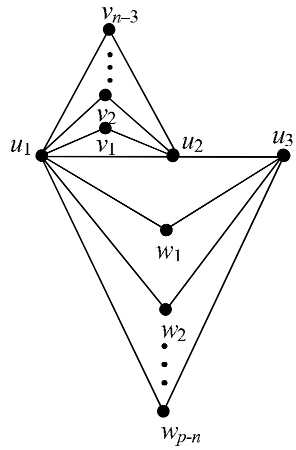

FIGURE 10. G

$S \cup \{x\}$, where $x \in \{u_1, w_1, w_2, \ldots, w_{p-n}\}$ is not a connected edge geodetic set of G. Now $S \cup \{u_0\}$ is an edge geodetic set of G but not a connected edge geodetic set of G. Since $S \cup \{u_0, u_1\}$ is a connected edge geodetic set of G, it follows that $g_{1c}(G) = n$.

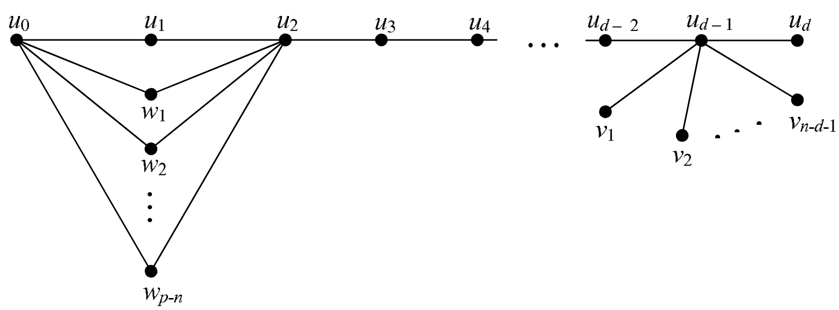

FIGURE 11. G

Case 3. Let $d = p - 1$. Then $n = p$. Let G be the path of order n. Then, by Corollary 2.16, $g_{1c}(G) = n$. □

We proved that $2 \leqslant g_1(G) \leqslant g_{1c}(G) \leqslant p$. The following theorem gives a realization for these parameters when $2 \leqslant a < b \leqslant p$.

THEOREM 3.10. If p, a and b are positive integers such that $2 \leqslant a < b \leqslant p$, then there exists a connected graph G of order p, $g_1(G) = a$ and $g_{1c}(G) = b$.

PROOF. We prove this theorem by considering two cases.
Case 1. $2 \leqslant a < b = p$. Let G be any tree with a pendant vertices. Then by Theorem 1.3, $g_1(G) = a$ and by Corollary 2.16, $g_{1c}(G) = p$.

Case 2. $2 \leqslant a < b < p$. Let $P_{b-a+2} : u_1, u_2, \ldots, u_{b-a+2}$ be a path of length $b - a + 1$. Add $p - b + a - 2$ new vertices $w_1, w_2, \ldots, w_{p-b}, v_1, v_2, \ldots, v_{a-2}$ to P_{b-a+2} and join $w_1, w_2, \ldots, w_{p-b}$ to both u_1 and u_3 and join $v_1, v_2, \ldots, v_{a-2}$ to u_{b-a+1}, there by producing the graph G of Figure 12. Then G has order p and $S = \{u_{b-a+2}, v_1, v_2, \ldots, v_{a-2}\}$ is the set of all extreme vertices of G. It is clear that S is not an edge geodetic set of G. On the other hand, $S \cup \{u_1\}$ is an edge geodetic set of G and it follows from Theorem 1.1 that $g_1(G) = a$. By an argument exactly similar to the one given in Case 2 of Theorem 3.9, it can be proved that $g_{1c}(G) = b$. □

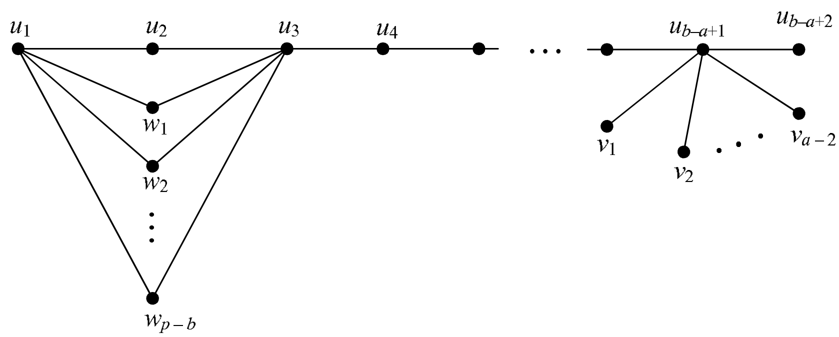

FIGURE 12. G

References

[1] F. Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley, Redwood City, CA, (1990).

[2] F. Buckley, F. Harary, L.V. Quintas, Extremal results on the geodetic number of a graph, *Scientia A2* (1988) 17-26.

[3] G. Chartrand, F. Harary, P. Zhang, On the geodetic number of a graph, *Networks*, (2002) 1-6.

[4] F. Harary, *Graph Theory*, Addison-Wesley, 1969.

[5] F. Harary, E. Loukakis, C. Tsouros, The geodetic number of a graph, *Math. Comput. Modeling* 17(11)(1993) 89-95.

[6] D. A. Mojdeh and N. J. Rad, Connected Geodomination in Graphs, *Journal of Discrete Mathematical Sciences & Cryptography* Vol.9 (2006), No.1, 177-186.

[7] P. A. Ostrand, Graphs with specified radius and diameter, *Discrete Mathematics* 4(1973) 71 - 75.

[8] A.P. Santhakumaran and J. John, Edge Geodetic Number of a Graph, *Journal of Discrete Mathematical Sciences & Cryptograph*, Vol. 10(3) (2007), pp. 415 - 432.

[9] A.P. Santhakumaran, P. Titus and J. John, On the Connected Geodetic Number of a Graph, *J. Combin. Math. Combin. Comput.*, to appear.

Received 12 08 2008, revised 28 01 2009

[a] *Current address: Research Department of Mathematics,*
St. Xavier's College (Autonomous),
Palayamkottai - 627 002,
India

E-mail address: santham kumar <apskumar1953@yahoo.co.in>

[b] *Current address: Alagappa Chettiar Govt,*
College of Engineering Technology,
Karaikudi - 630 004,
India
E-mail address: johnramesh1971@yahoo.co.in