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The integrals in Gradshteyn and Ryzhik. Part 10: The digamma function

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Abstract. The table of Gradshteyn and Rhyzik contains some integrals that can be expressed in terms of the digamma function $\psi(x) = \frac{d}{dx} \log \Gamma(x)$. In this note we present some of these evaluations.

1. Introduction

The table of integrals [**2**] contains a large variety of definite integrals that involve the *digamma* function

$$\psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{1.1}$$

Here $\Gamma(x)$ is the gamma function defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt. \tag{1.2}$$

Many of the analytic properties can be derived from those of $\Gamma(x)$. The next theorem represents a collection of the important properties of $\Gamma(x)$ that are used in the current paper. The reader will find in [**1**] detailed proofs.

Theorem 1.1. The gamma function satisfies:

a) the functional equation

$$\Gamma(x+1) = x\Gamma(x). \tag{1.3}$$

b) For $n \in \mathbb{N}$, the interpolation formula $\Gamma(n) = (n-1)!$.

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c) The *Euler constant* γ, defined by

$$\gamma = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k} - \ln n, \tag{1.4}$$

is also given by $\gamma = -\Gamma'(1)$. This appears as the special case $a = 1$ of formula **4.331.1**:

$$\int_0^\infty e^{-ax} \ln x \, dx = -\frac{\gamma + \ln a}{a}. \tag{1.5}$$

This was established in [**3**]. The change of variables $t = ax$ shows that the case $a = 1$ is equivalent to the general case. This is an instance of a *fake parameter*.

d) The infinite product representation

$$\Gamma(x) = \frac{e^{-\gamma x}}{x} \prod_{k=1}^{\infty} \left[\left(1 + \frac{x}{k} \right)^{-1} e^{x/k} \right] \tag{1.6}$$

is valid for $x \in \mathbb{C}$ away from the poles at $x = 0, \, -1, \, -2, \ldots$

e) For $n \in \mathbb{N}$ we have

$$\Gamma\left(n + \tfrac{1}{2}\right) = \frac{(2n)!}{2^{2n} \, n!} \sqrt{\pi} \tag{1.7}$$

and

$$\Gamma\left(\tfrac{1}{2} - n\right) = (-1)^n \frac{2^{2n} \, n!}{(2n)!} \sqrt{\pi}. \tag{1.8}$$

f) For $x \in \mathbb{C}$, $x \notin \mathbb{Z}$ we have the reflection rule

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{1.9}$$

Several properties of the digamma function $\psi(x)$ follow directly from the gamma function.

Theorem 1.2. The digamma function $\psi(x)$ satisfies
a) the functional equation

$$\psi(x+1) = \psi(x) + \frac{1}{x}. \tag{1.10}$$

b) For $n \in \mathbb{N}$, we have

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k}. \tag{1.11}$$

In particular, $\psi(1) = -\gamma$.

c) For $x \in \mathbb{C}$ away from $x = 0, -1, -2, \cdots$ we have

$$\psi(x) = -\gamma - \frac{1}{x} + x\sum_{k=1}^{\infty} \frac{1}{k(x+k)} \tag{1.12}$$
$$= -\gamma - \sum_{k=0}^{\infty} \left(\frac{1}{x+k} - \frac{1}{k+1} \right)$$

d) The derivative of ψ is given by

$$\psi'(x) = \sum_{k=0}^{\infty} \frac{1}{(x+k)^2}. \tag{1.13}$$

In particular, $\psi'(1) = \pi^2/6$.

e) For $n \in \mathbb{N}$ we have

$$\psi(\tfrac{1}{2} \pm n) = -\gamma - 2\ln 2 + 2\sum_{k=1}^{n} \frac{1}{2k-1}. \tag{1.14}$$

In particular,

$$\psi(\tfrac{1}{2}) = -\gamma - 2\ln 2. \tag{1.15}$$

f) For $x \in \mathbb{C}$, $x \notin \mathbb{Z}$ we have the reflection rule

$$\psi(1-x) = \psi(x) + \pi \cot \pi x. \tag{1.16}$$

2. A first integral representation

In this section we establish the integral evaluation **3.429**. Several direct consequences of this formulas are also described.

Proposition 2.1. Assume $a > 0$. Then

$$\int_0^\infty \left[e^{-x} - (1+x)^{-a} \right] \frac{dx}{x} = \psi(a). \tag{2.1}$$

PROOF. We begin with the double integral

$$\int_0^\infty \int_1^s e^{-tz} dt\, dz = \int_0^\infty \frac{e^{-z} - e^{-sz}}{z} dz. \tag{2.2}$$

On the other hand,

$$\int_1^s \int_0^\infty e^{-tz} dz\, dt = \int_1^s \frac{dt}{t} = \ln s. \tag{2.3}$$

We conclude that

$$\int_0^\infty \frac{e^{-z} - e^{-sz}}{z}\, dz = \ln s. \tag{2.4}$$

This evaluation is equivalent to:

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx = \ln \frac{b}{a}, \tag{2.5}$$

that appears as formula **3.434.2** in [**2**]. The reader will find a proof in [**3**].

We now establish the result: start with

$$\begin{aligned}\Gamma'(a) &= \int_0^\infty e^{-s} s^{a-1} \ln s \, ds \\ &= \int_0^\infty e^{-s} s^{a-1} \int_0^\infty \frac{e^{-z} - e^{-zs}}{z} \, dz \, ds \\ &= \int_0^\infty \left(e^{-z} \int_0^\infty s^{a-1} e^{-s} \, ds - \int_0^\infty s^{a-1} e^{-s(1+z)} \, ds \right) \frac{dz}{z}.\end{aligned}$$

This formula can be rewritten as

$$\Gamma'(a) = \Gamma(a) \int_0^\infty \left(e^{-z} - (1+z)^{-a} \right) \frac{dz}{z}.$$

This establishes (2.1). □

Example 2.2. The special case $a = 1$ yields

$$\int_0^\infty \left(e^{-x} - \frac{1}{1+x} \right) \frac{dx}{x} = -\gamma. \tag{2.6}$$

This appears as **3.435.3**.

Example 2.3. The change of variables $w = -\ln x$ gives the value of **4.275.2**:

$$\int_0^1 \left[x - \left(\frac{1}{1 - \ln x} \right)^q \right] \frac{dx}{x \ln x} = -\int_0^\infty \left[e^{-w} - (1+w)^{-q} \right] \frac{dw}{w} = -\psi(q). \tag{2.7}$$

Example 2.4. The change of variables $t = 1/(x+1)$ in (2.1) yields **3.471.14**:

$$\int_0^1 \frac{e^{(1-1/t)} - t^a}{t(1-t)} \, dt = \psi(a) \tag{2.8}$$

Example 2.5. The result of Example 2.2 can be used to prove **3.435.4**:

$$\int_0^\infty \left(e^{-bx} - \frac{1}{1+ax} \right) \frac{dx}{x} = \ln \frac{a}{b} - \gamma. \tag{2.9}$$

Indeed, the change of variables $t = bx$ yields from (2.2) the identity

$$\begin{aligned}\int_0^\infty \left(e^{-bx} - \frac{1}{1+ax} \right) \frac{dx}{x} &= \int_0^\infty \left(e^{-t} - \frac{1}{1+at/b} \right) \frac{dt}{t} \\ &= \int_0^\infty \frac{e^{-t} - e^{-at/b}}{t} \, dt + \int_0^\infty \left(e^{-at/b} - \frac{1}{1+at/b} \right) \frac{dt}{t}.\end{aligned}$$

Formula (2.5) shows the first integral is $\ln \frac{a}{b}$ and the value of the second one comes from (2.2).

Example 2.6. The evaluation **3.476.2**:

$$\int_0^\infty \left(e^{-x^p} - e^{-x^q} \right) \frac{dx}{x} = \frac{p-q}{pq}\gamma \tag{2.10}$$

comes directly from (2.1). Indeed, the change of variables $u = x^p$ yields

$$I := \int_0^\infty \left(e^{-x^p} - e^{-x^q} \right) \frac{dx}{x} = \frac{1}{p} \int_0^\infty \left(e^{-u} - e^{-u^{q/p}} \right) \frac{du}{u}.$$

Now write

$$I = \frac{1}{p} \int_0^\infty \left(e^{-u} - \frac{1}{1+u} \right) \frac{du}{u} + \frac{1}{p} \int_0^\infty \left(\frac{1}{1+u} - e^{-u^{q/p}} \right) \frac{du}{u}.$$

The first integral is $-\gamma$ by (2.6) and the change of variables $v = u^{q/p}$ gives

$$\begin{aligned} I &= -\frac{\gamma}{p} + \frac{1}{q} \int_0^\infty \left(\frac{1}{1+v^{p/q}} - e^{-v} \right) \frac{dv}{v} \\ &= -\frac{\gamma}{p} + \frac{1}{q} \int_0^\infty \left(\frac{1}{1+v} - e^{-v} \right) \frac{dv}{v} + \frac{1}{q} \int_0^\infty \frac{v - v^{p/q}}{v(1+v)(1+v^{p/q})} \, dv. \end{aligned}$$

Split the last integral from $[0,1]$ to $[1,\infty)$ and use the change of variables $x \mapsto 1/x$ in the second part to check that the whole integral vanishes. Formula (2.10) has been established.

Example 2.7. Formula **3.463**:

$$\int_0^\infty \left(e^{-x^2} - e^{-x} \right) \frac{dx}{x} = \frac{\gamma}{2} \tag{2.11}$$

corresponds to the choice $p = 2$ and $q = 1$ in (2.10).

Example 2.8. Formula **3.469.2**:

$$\int_0^\infty \left(e^{-x^4} - e^{-x} \right) \frac{dx}{x} = \frac{3\gamma}{4} \tag{2.12}$$

corresponds to the choice $p = 4$ and $q = 1$ in (2.10).

Example 2.9. Formula **3.469.3**:

$$\int_0^\infty \left(e^{-x^4} - e^{-x^2} \right) \frac{dx}{x} = \frac{\gamma}{4} \tag{2.13}$$

corresponds to the choice $p = 4$ and $q = 2$ in (2.10).

Example 2.10. Formula **3.475.3**:

$$\int_0^\infty \left(e^{-x^{2^n}} - e^{-x} \right) \frac{dx}{x} = (1 - 2^{-n})\gamma \tag{2.14}$$

corresponds to the choice $p = 2^n$ and $q = 1$ in (2.10).

The case $p = q$ in (2.10) is now modified to include a parameter.

Proposition 2.11. Let a, b, $p \in \mathbb{R}^+$. Then **3.476.1** in [**2**] states that

$$\int_0^\infty \left[e^{-ax^p} - e^{-bx^p} \right] \frac{dx}{x} = \frac{\ln b - \ln a}{p}. \tag{2.15}$$

Proof. The change of variables $t = ax^p$ gives

$$\int_0^\infty \left[e^{-ax^p} - e^{-bx^p} \right] \frac{dx}{x} = \frac{1}{p} \int_0^\infty \left(e^{-t} - e^{-bt/a} \right) \frac{dt}{t}.$$

Introduce the term $1/(1+t)$ to obtain

$$\begin{aligned} I &= \frac{1}{p} \int_0^\infty \left(e^{-t} - \frac{1}{1+t} \right) \frac{dt}{t} - \frac{1}{p} \int_0^\infty \left(e^{-bt/a} - \frac{1}{1+t} \right) \frac{dt}{t} \\ &= -\frac{\gamma}{p} - \frac{1}{p} \int_0^\infty \left(e^{-s} - \frac{b}{b+as} \right) \frac{ds}{s}. \end{aligned}$$

Adding and subtracting the term $1/(1+s)$ produces

$$I = \frac{1}{p} \int_0^\infty \left(\frac{b}{b+as} - \frac{1}{1+s} \right) \frac{ds}{s}. \tag{2.16}$$

The final result now comes from evaluating the last integral. □

We now present another integral representation of the digamma function.

Proposition 2.12. The digamma function is given by

$$\psi(a) = \int_0^\infty \left(\frac{e^{-x}}{x} - \frac{e^{-ax}}{1-e^{-x}} \right) dx. \tag{2.17}$$

This expression appears as **3.427.1** in [**2**].

Proof. The representation (2.1) is written as

$$\psi(a) = \lim_{\delta \to 0} \int_\delta^\infty \frac{e^{-z}}{z}\, dz - \int_\delta^\infty \frac{dz}{z(1+z)^a}, \tag{2.18}$$

to avoid the singularity at $z = 0$. The change of variables $z = e^t - 1$ in the second integral gives

$$\psi(a) = \lim_{\delta \to 0} \int_\delta^\infty \frac{e^{-z}}{z}\, dz - \int_{\ln(1+\delta)}^\infty \frac{e^{-at}\, dt}{1-e^{-t}}. \tag{2.19}$$

Now observe that

$$\left| \int_\delta^{\ln(1+\delta)} \frac{e^{-t}}{t}\, dt \right| \leqslant \int_{\ln(1+\delta)}^\delta \frac{dt}{t} \to 0, \tag{2.20}$$

as $\delta \to 0$. This completes the proof. □

Example 2.13. The special case $a = 1$ in (2.17) gives **3.427.2**:

$$\int_0^\infty \left(\frac{1}{1-e^{-x}} - \frac{1}{x} \right) e^{-x}\, dx = \gamma. \tag{2.21}$$

Example 2.14. The change of variables $t = e^{-x}$ in (2.17) produces **4.281.4**:

$$\int_0^1 \left(\frac{1}{\ln t} + \frac{t^{a-1}}{1-t} \right) dt = -\psi(a). \tag{2.22}$$

Example 2.15. The special case $a = 1$ in (2.22) yields **4.281.1**:

$$\int_0^1 \left(\frac{1}{\ln t} + \frac{1}{1-t} \right) dt = \gamma. \tag{2.23}$$

Proposition 2.16. Let $p,\, q \in \mathbb{R}$. Then

$$\int_0^1 \left(\frac{x^{p-1}}{\ln x} + \frac{x^{q-1}}{1-x} \right) dx = \ln p - \psi(q). \tag{2.24}$$

This appears as **4.281.5** in [**2**].

Proof. Write

$$\int_0^1 \left(\frac{x^{p-1}}{\ln x} + \frac{x^{q-1}}{1-x} \right) dx = \int_0^1 \left(\frac{1}{\ln x} + \frac{x^{q-1}}{1-x} \right) dx + \int_0^1 \frac{x^{p-1}-1}{\ln x}\, dx. \tag{2.25}$$

The first integral is $-\psi(q)$ from (2.22) and to evaluate the second one, differentiate with respect to p, to produce

$$\frac{d}{dp} \int_0^1 \frac{x^{p-1}-1}{\ln x}\, dx = \int_0^1 x^{p-1}\, dx = \frac{1}{p}. \tag{2.26}$$

The value at $p = 1$ shows that the constant of integration vanishes. The formula (2.24) has been established. □

3. The difference of values of the digamma function

In this section we establish an integral representation for the difference of values of the digamma function. The expression appears as **3.231.5** in [**2**].

Proposition 3.1. Let $p,\, q \in \mathbb{R}$. Then

$$\int_0^1 \frac{x^{p-1} - x^{q-1}}{1-x}\, dx = \psi(q) - \psi(p). \tag{3.1}$$

Proof. Consider first

$$I(\epsilon) = \int_0^1 x^{p-1}(1-x)^{\epsilon-1}\, dx - \int_0^1 x^{q-1}(1-x)^{\epsilon-1}\, dx, \tag{3.2}$$

that avoids the apparent singularity at $x = 1$. The integral $I(\epsilon)$ can be expressed in terms of the beta function

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx \tag{3.3}$$

as $I(\epsilon) = B(p,\epsilon) - B(q,\epsilon)$, and using the relation

$$B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} \tag{3.4}$$

we obtain

$$I(\epsilon) = \Gamma(\epsilon)\left(\frac{\Gamma(p)}{\Gamma(p+\epsilon)} - \frac{\Gamma(q)}{\Gamma(q+\epsilon)}\right). \tag{3.5}$$

Now use $\Gamma(1+\epsilon) = \epsilon\Gamma(\epsilon)$ to write

$$I(\epsilon) = \Gamma(1+\epsilon)\left(\frac{\Gamma(p)-\Gamma(p+\epsilon)}{\epsilon}\,\frac{1}{\Gamma(p+\epsilon)} - \frac{\Gamma(q)-\Gamma(q+\epsilon)}{\epsilon}\,\frac{1}{\Gamma(q+\epsilon)}\right), \tag{3.6}$$

and obtain (3.1) by letting $\epsilon \to 0$. □

Example 3.2. The special value $\psi(1) = -\gamma$ produces

$$\int_0^1 \frac{1-x^{q-1}}{1-x}\,dx = \gamma + \psi(q). \tag{3.7}$$

This appears as **3.265** in [**2**].

Example 3.3. A second special value appears in **3.268.2**:

$$\int_0^1 \frac{1-x^a}{1-x}\,x^{b-1}\,dx = \psi(a+b) - \psi(b). \tag{3.8}$$

It is obtained from (3.1) by choosing $p = b$ and $q = a+b$.

Example 3.4. Now let $q = 1-p$ in (3.1) to produce

$$\int_0^1 \frac{x^{p-1}-x^{-p}}{1-x}\,dx = \psi(1-p) - \psi(p) = \pi\cot\pi p. \tag{3.9}$$

This appears as **3.231.1** in [**2**].

Example 3.5. The special case $p = a+1$ and $q = 1-a$ produces

$$\int_0^1 \frac{x^a - x^{-a}}{1-x}\,dx = \psi(1-a) - \psi(1+a) = \pi\cot\pi a - \frac{1}{a}, \tag{3.10}$$

where we have used (1.10) and (1.16) to simplify the result. This is **3.231.3** in [**2**].

Example 3.6. The change of variables $x = t^a$ in (3.1) produces

$$\int_0^1 \frac{t^{ap-1}-t^{aq-1}}{1-t^a}\,dt = \frac{\psi(q)-\psi(p)}{a}. \tag{3.11}$$

Now let $p = 1, a = \nu$ and $q = \frac{\mu}{\nu}$ and the replace μ by p and ν by q to obtain **3.244.3** in [**2**]:

$$\int_0^1 \frac{t^{q-1}-t^{p-1}}{1-t^q}\,dt = \frac{1}{q}\left(\gamma + \psi\left(\frac{p}{q}\right)\right). \tag{3.12}$$

Example 3.7. The special case $p = b/a$ and $q = 1 - b/a$ in (3.11) produces

$$\int_0^1 \frac{x^{b-1} - x^{a-b-1}}{1 - x^a} \, dx = \frac{1}{a} \left(\psi(1 - b/a) - \psi(b/a) \right). \tag{3.13}$$

The result is now simplified using (1.16) to produce

$$\int_0^1 \frac{x^{b-1} - x^{a-b-1}}{1 - x^a} \, dx = \frac{\pi}{a} \cot \frac{\pi b}{a}. \tag{3.14}$$

This is **3.244.2** in [**2**].

Example 3.8. The special case $a = 2$ in (3.11) yields

$$\int_0^1 \frac{t^{2\mu-1} - t^{2\nu-1}}{1 - t^2} \, dt = \frac{1}{2} \left(\psi(\nu) - \psi(\mu) \right). \tag{3.15}$$

The choice $\mu = 1 + p/2$ and $\nu = 1 - p/2$:

$$\int_0^1 \frac{x^p - x^{-p}}{1 - x^2} \, x \, dx = \frac{1}{2} \left(\psi(1 + p/2) - \psi(1 - p/2) \right). \tag{3.16}$$

The identities $\psi(x+1) = \psi(x) + 1/x$ and $\psi(1-x) - \psi(x) = \pi \cot \pi x$ produce

$$\int_0^1 \frac{x^p - x^{-p}}{1 - x^2} \, x \, dx = \frac{\pi}{2} \cot \left(\frac{p\pi}{2} \right) - \frac{1}{p}. \tag{3.17}$$

This appears as **3.269.1** in [**2**].

Example 3.9. The choice $\mu = \frac{a+1}{2}$ and $\nu = \frac{b+1}{2}$ in (3.15) gives **3.269.3**:

$$\int_0^1 \frac{x^a - x^b}{1 - x^2} \, dx = \frac{1}{2} \left(\psi\left(\frac{b+1}{2} \right) - \psi\left(\frac{a+1}{2} \right) \right). \tag{3.18}$$

4. Integrals over a half-line

In this section we consider integrals over the half-line $[0, \infty)$ that can be evaluated in terms of the digamma function.

Proposition 4.1. Let p, $q \in \mathbb{R}$. Then

$$\int_0^\infty \left(\frac{t^p}{(1+t)^p} - \frac{t^q}{(1+t)^q} \right) \frac{dt}{t} = \psi(q) - \psi(p). \tag{4.1}$$

This is **3.219** in [**2**]. Also

$$\int_0^\infty \left(\frac{1}{(1+t)^p} - \frac{1}{(1+t)^q} \right) \frac{dt}{t} = \psi(q) - \psi(p). \tag{4.2}$$

Proof. Let $t = x/(1-x)$ in (3.1). The second form comes from the first by the change of variables $x \mapsto 1/x$. □

Example 4.2. The special case $p = 1$ yields

$$\int_0^\infty \left(\frac{1}{1+t} - \frac{1}{(1+t)^q} \right) \frac{dt}{t} = \psi(q) + \gamma. \tag{4.3}$$

This appears as **3.233** in [**2**].

Example 4.3. The evaluation of **3.235**:

$$\int_0^\infty \frac{(1+x)^a - 1}{(1+x)^b}\frac{dx}{x} = \psi(b) - \psi(b-a) \tag{4.4}$$

can be established directly from (4.3). Simply write

$$\int_0^\infty \frac{(1+x)^a - 1}{(1+x)^b}\frac{dx}{x} = \int_0^\infty \left(\frac{1}{1+x} - \frac{1}{(1+x)^b} \right) \frac{dx}{x} - \int_0^\infty \left(\frac{1}{1+x} - \frac{1}{(1+x)^{b-a}} \right) \frac{dx}{x}$$

to obtain the result.

Some examples of integrals over $[0,\infty)$ can be reduced to a pair of integrals over $[0,1]$.

Proposition 4.4. The formula **3.231.6** of [**2**] states that

$$\int_0^\infty \frac{x^{p-1} - x^{q-1}}{1-x}\, dx = \pi \left(\cot \pi p - \cot \pi q \right). \tag{4.5}$$

Proof. To evaluate this, make the change of variables $t = 1/x$ in the part over $[1,\infty)$ to produce

$$\int_0^\infty \frac{x^{p-1} - x^{q-1}}{1-x}\, dx = \int_0^1 \frac{x^{p-1} - x^{q-1}}{1-x}\, dx - \int_0^1 \frac{x^{-p} - x^{-q}}{1-x}\, dx. \tag{4.6}$$

Now use the result (3.1) to write

$$\int_0^\infty \frac{x^{p-1} - x^{q-1}}{1-x}\, dx = \psi(q) - \psi(p) - \left[\psi(1-q) - \psi(1-p) \right]. \tag{4.7}$$

The relation $\psi(x) - \psi(1-x) = -\pi\,\cot(\pi q)$ yields the result. □

5. An exponential scale

In this section we present the evaluation of certain definite integrals involving the exponential function. These are integrals that can be evaluated in terms of the digamma function of the parameters involved.

Example 5.1. The simplest one is **3.317.2**:

$$\int_{-\infty}^\infty \left(\frac{1}{(1+e^{-x})^p} - \frac{1}{(1+e^{-x})^q} \right) dx = \psi(q) - \psi(p) \tag{5.1}$$

that comes from (4.2) via the change of variables $x \mapsto e^{-x}$.

Example 5.2. The special case $p = 1$ and $\psi(1) = -\gamma$ produces **3.317.1**:

$$\int_{-\infty}^\infty \left(\frac{1}{1+e^{-x}} - \frac{1}{(1+e^{-x})^q} \right) dx = \psi(q) + \gamma \tag{5.2}$$

Example 5.3. The evaluation of **3.316**:

$$\int_{-\infty}^\infty \frac{(1+e^{-x})^p - 1}{(1+e^{-x})^q}\, dx = \psi(q) - \psi(q-p) \tag{5.3}$$

comes directly from (5.1).

Proposition 5.4. Let p, $q \in \mathbb{R}$. Then

$$\int_0^\infty \frac{e^{-pt}-e^{-qt}}{1-e^{-t}}\,dt = \psi(q)-\psi(p), \tag{5.4}$$

This appears as **3.311.7** in [**2**].

Proof. Make the change of variables $x = e^{-t}$ in (3.1). □

Example 5.5. The evaluation (5.4) can also be written as

$$\int_0^\infty \frac{e^{t(1-p)}-e^{t(1-q)}}{e^t-1}\,dt = \psi(q)-\psi(p), \tag{5.5}$$

Example 5.6. The special case $p=1$ and $q = 1-\nu$ is

$$\int_0^\infty \frac{1-e^{\nu t}}{e^t-1}\,dt = \psi(1-\nu)-\psi(1), \tag{5.6}$$

and using $\psi(1) = -\gamma$ and $\psi(1-\nu) = \psi(\nu)+\pi\,\cot\pi\nu$, yields the form

$$\int_0^\infty \frac{1-e^{\nu t}}{e^t-1}\,dt = \psi(\nu)+\gamma+\pi\,\cot\pi\nu, \tag{5.7}$$

as it appears in **3.311.5**.

Example 5.7. Another special case of (5.4) is **3.311.6**, that corresponds to $p=1$:

$$\int_0^\infty \frac{e^{-t}-e^{-qt}}{1-e^{-t}}\,dt = \psi(q)+\gamma. \tag{5.8}$$

Example 5.8. The evaluation **3.311.11**:

$$\int_0^\infty \frac{e^{px}-e^{qx}}{e^{rx}-e^{sx}}\,dx = \frac{1}{r-s}\left(\psi\left(\frac{r-q}{r-s}\right)-\psi\left(\frac{r-p}{r-s}\right)\right), \tag{5.9}$$

follows directly from (5.4) by the change of variables $t=(r-s)x$.

Example 5.9. The evaluation of **3.311.12**:

$$\int_0^\infty \frac{a^x-b^x}{c^x-d^x}\,dx = \frac{1}{\ln c-\ln d}\left(\psi\left(\frac{\ln c-\ln b}{\ln c-\ln d}\right)-\psi\left(\frac{\ln c-\ln a}{\ln c-\ln d}\right)\right), \tag{5.10}$$

is proved by simply writing the exponentials in natural base.

Example 5.10. The formula **3.311.10** had a sign error in the *sixth* edition of [**2**]. It appears as

$$\int_0^\infty \frac{e^{-px}-e^{-qx}}{1+e^{-(p+q)x}}\,dx = \frac{\pi}{p+q}\cot\left(\frac{p\pi}{p+q}\right). \tag{5.11}$$

It should be

$$\int_0^\infty \frac{e^{-px}-e^{-qx}}{1-e^{-(p+q)x}}\,dx = \frac{\pi}{p+q}\cot\left(\frac{p\pi}{p+q}\right). \tag{5.12}$$

The value (5.9) yields

$$\int_0^\infty \frac{e^{-px}-e^{-qx}}{1-e^{-(p+q)x}}\,dx = \frac{1}{p+q}\left(\psi\left(\frac{q}{p+q}\right)-\psi\left(\frac{p}{p+q}\right)\right), \tag{5.13}$$

and the trigonometric answer follows from (1.16). This has been corrected in the current edition of [**2**].

Example 5.11. The evaluation of **3.312.2**:

$$\int_0^\infty \frac{(1-e^{-ax})(1-e^{-bx})e^{-px}}{1-e^{-x}}\,dx = \psi(p+a)+\psi(p+b)-\psi(p+a+b)-\psi(p) \tag{5.14}$$

follows directly from (3.1). Indeed, the change of variables $t=e^{-x}$ gives

$$I = \int_0^1 \frac{t^{p-1}(1-t^a-t^b+t^{a+b})}{1-t}\,dt \tag{5.15}$$

and now split them as

$$I = \int_0^1 \frac{t^{p-1}-t^{p+a-1}}{1-t}\,dt - \int_0^1 \frac{t^{p+b-1}-t^{p+a+b-1}}{1-t}\,dt \tag{5.16}$$

and use (3.1) to conclude.

6. A singular example

The example discussed in this section is

$$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{b-e^{-x}} = b^{\mu-1}\pi\,\cot(\pi\mu), \tag{6.1}$$

that appears as **3.311.8** in [**2**]. In the case $b>0$ this has to be modified in its presentation to avoid the singularity $x=-\ln b$. The case $b<0$ was discussed in [**4**]. In order to reduce the integral to a previous example, we let $t=e^{-x}$ to obtain

$$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{b-e^{-x}} = \int_0^\infty \frac{t^{\mu-1}\,dt}{b-t}. \tag{6.2}$$

The change of variables $t=by$ yields

$$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{b-e^{-x}} = b^{\mu-1}\int_0^\infty \frac{y^{\mu-1}\,dy}{1-y}. \tag{6.3}$$

Now separate the range of integration into $[0,1]$ and $[1,\infty)$. Then make the change of variables $y=1/z$ in the second part. This produces

$$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{b-e^{-x}} = b^{\mu-1}\int_0^1 \frac{z^{\mu-1}-z^{-\mu}}{1-z}\,dz. \tag{6.4}$$

This last integral has been evaluated as $\cot(\pi\mu)$ in (3.9).

7. An integral with a fake parameter

The example considered in this section is **3.234.1**:

$$\int_0^1 \left(\frac{x^{q-1}}{1-ax} - \frac{x^{-q}}{a-x}\right)dx = \frac{\pi}{a^q}\cot\pi q. \tag{7.1}$$

We show that the parameter a is *fake*, in the sense that it can be easily scaled out of the formula. The integral is written as $\lim_{\epsilon \to 0} I(\epsilon)$ where

$$\begin{aligned} I(\epsilon) &= \int_0^1 \left(\frac{x^{q-1}}{(1-ax)^{1-\epsilon}} - \frac{x^{-q}}{(a-x)^{1-\epsilon}} \right) dx \\ &= \int_0^1 \frac{x^{q-1}}{(1-ax)^{1-\epsilon}} \, dx - \int_0^1 \frac{x^{-q}}{(a-x)^{1-\epsilon}} \, dx. \end{aligned}$$

Make the change of variables $t = ax$ in the first integral and $x = at$ in the second one to produce

$$I(\epsilon) = a^{-q} \int_0^a \frac{t^{q-1}\, dt}{(1-t)^{1-\epsilon}} - a^{-q+\epsilon} \int_0^{1/a} \frac{t^{-q}\, dt}{(1-t)^{1-\epsilon}},$$

and then let $\epsilon \to 0$ to produce

$$\int_0^1 \left(\frac{x^{q-1}}{1-ax} - \frac{x^{-q}}{a-x} \right) dx = a^{-q} \left(\int_0^a \frac{t^{q-1}\, dt}{1-t} - \int_0^{1/a} \frac{t^{-q}\, dt}{1-t} \right).$$

Differentiation with respect to the parameter a, shows that the expression in parenthesis is independent of a. It is now evaluated by using $a = 1$ to obtain

$$\int_0^1 \left(\frac{x^{q-1}}{1-ax} - \frac{x^{-q}}{a-x} \right) dx = a^{-q} \left(\int_0^1 \frac{t^{q-1} - t^{-q}}{1-t} \, dt \right).$$

The evaluation (3.1) now yields

$$\begin{aligned} \int_0^1 \left(\frac{x^{q-1}}{1-ax} - \frac{x^{-q}}{a-x} \right) dx &= a^{-q} \left(\psi(1-q) - \psi(q) \right) \\ &= a^{-q} \pi \cot \pi q. \end{aligned}$$

Formula (7.1) has been established.

8. The derivative of ψ

In a future publication we will discuss the evaluation of definite integrals in terms of the *polygamma function*

$$\text{PolyGamma}[n, x] := \left(\frac{d}{dx} \right)^n \psi(x). \tag{8.1}$$

In this section, we simply describe some integrals in [**2**] that comes from direct differentiation of the examples described above.

Example 8.1. Differentiating (3.1) with respect to the parameter p produces **4.251.4**:

$$\int_0^1 \frac{x^{p-1}\, \ln x}{1-x} \, dx = -\psi'(p). \tag{8.2}$$

Example 8.2. The change of variables $x = t^q$ in (8.2), followed by the change of parameter $p \mapsto \frac{p}{q}$ yields **4.254.1**:

$$\int_0^1 \frac{t^{p-1}\,\ln t}{1-t^q}\,dx = -\frac{1}{q^2}\psi'\left(\frac{p}{q}\right). \tag{8.3}$$

Example 8.3. Replace q by $2q$ and p by q in (8.3) to produce

$$\int_0^1 \frac{t^{q-1}\,\ln t}{1-t^{2q}}\,dt = -\frac{1}{4q^2}\psi'\left(\tfrac{1}{2}\right). \tag{8.4}$$

To evaluate this last term, differentiate the logarithm of the identity

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\,\Gamma(x+\tfrac{1}{2}), \tag{8.5}$$

to obtain

$$2\psi(2x) = 2\ln 2 + \psi(x) + \psi(x+\tfrac{1}{2}). \tag{8.6}$$

One more differentitation produces

$$4\psi'(2x) = \psi'(x) + \psi'(x+\tfrac{1}{2}). \tag{8.7}$$

The value $x = \frac{1}{2}$ gives

$$\psi'(\tfrac{1}{2}) = 3\psi'(1) = \frac{\pi^2}{2}. \tag{8.8}$$

Therefore we obtain **4.254.6**:

$$\int_0^1 \frac{x^{q-1}\,\ln x}{1-x^{2q}}\,dx = -\frac{\pi^2}{8q^2}. \tag{8.9}$$

Example 8.4. Differentiating (3.12) n-times with respect to the parameter p produces **4.271.15**:

$$\int_0^1 \ln^n x\,\frac{x^{p-1}\,dx}{1-x^q} = -\frac{1}{q^{n+1}}\psi^{(n)}\left(\frac{p}{q}\right). \tag{8.10}$$

9. A family of logarithmic integrals

Several of the integrals appearing in [**2**] are particular examples of the family evaluated in the next proposition.

Proposition 9.1. Let $a,\, b \in \mathbb{R}^+$. Then

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,\ln x\,dx = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}\left(\psi(a) - \psi(a+b)\right). \tag{9.1}$$

Proof. Differentiate the identity

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} \tag{9.2}$$

with respect to the parameter a and recall that $\Gamma'(x) = \psi(x)\Gamma(x)$. □

The next corollary appears as **4.253.1** in [**2**].

Corollary 9.2. Let a, b, $c \in \mathbb{R}^+$. Then

$$\int_0^1 x^{a-1}(1-x^c)^{b-1} \ln x \, dx = \frac{\Gamma(a/c)\,\Gamma(b)}{c^2\,\Gamma(a/c+b)} \left(\psi\left(\frac{a}{c}\right) - \psi\left(\frac{a}{c}+b\right)\right). \tag{9.3}$$

Proof. Let $t = x^c$ in the integral (9.1). ☐

Example 9.3. The formula in the previous corollary also appears as **4.256** in the form

$$\int_0^1 \ln\left(\frac{1}{x}\right) \frac{x^{\mu-1}\,dx}{\sqrt[n]{(1-x^n)^{n-m}}} = \frac{1}{n^2} B\left(\frac{\mu}{n}, \frac{m}{n}\right)\left[\psi\left(\frac{\mu+m}{n}\right) - \psi\left(\frac{\mu}{n}\right)\right]. \tag{9.4}$$

Example 9.4. The integral

$$\int_0^1 \frac{x^{2n}\,\ln x}{\sqrt{1-x^2}}\,dx = \int_0^1 x^{2n}(1-x^2)^{-1/2}\,\ln x\,dx \tag{9.5}$$

that appears as **4.241.1** in [**2**], corresponds to $a = 2n+1$, $b = \frac{1}{2}$ and $c = 2$ in (9.3). Therefore

$$\int_0^1 \frac{x^{2n}\,\ln x}{\sqrt{1-x^2}}\,dx = \frac{\Gamma(n+\frac{1}{2})\,\Gamma(\frac{1}{2})}{4\Gamma(n+1)} \left[\psi(n+\tfrac{1}{2}) - \psi(n+1)\right]. \tag{9.6}$$

Using (1.7), (1.11) and (1.14) yields

$$\int_0^1 \frac{x^{2n}\,\ln x}{\sqrt{1-x^2}}\,dx = \frac{\binom{2n}{n}\pi}{2^{2n+1}} \left(\sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} - \ln 2 \right). \tag{9.7}$$

This is **4.241.1**.

Example 9.5. The integral in **4.241.2** states that

$$\int_0^1 \frac{x^{2n+1}\,\ln x}{\sqrt{1-x^2}}\,dx = \frac{(2n)!!}{(2n+1)!!} \left(\ln 2 + \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} \right). \tag{9.8}$$

Writing the integral as

$$I = \int_0^1 x^{2n+1}(1-x^2)^{-1/2}\,\ln x\,dx \tag{9.9}$$

we see that is corresponds to the case $a = 2n+2$, $b = \frac{1}{2}$, $c = 2$ in (9.3). Therefore

$$I = \frac{\Gamma(n+1)\,\Gamma(\frac{1}{2})}{4\Gamma(n+\frac{3}{2})} \left[\psi(n+1) - \psi(n+\tfrac{3}{2})\right]. \tag{9.10}$$

Using (1.7), (1.11) and (1.14) yields

$$\int_0^1 \frac{x^{2n+1}\,\ln x}{\sqrt{1-x^2}}\,dx = \frac{2^{2n}}{(n+1)\binom{2n+1}{n}} \left(\ln 2 + \sum_{k=1}^{2n+1} \frac{(-1)^k}{k} \right) \tag{9.11}$$

This is equivalent to (9.8).

Example 9.6. The integral **4.241.3** in [**2**] states that

$$\int_0^1 x^{2n}\sqrt{1-x^2}\,\ln x\,dx = \frac{(2n-1)!!}{(2n+2)!!}\cdot\frac{\pi}{2}\left(\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k} - \frac{1}{2n+2} - \ln 2\right). \tag{9.12}$$

To evaluate the integral, we write it as

$$I = \int_0^1 x^{2n}(1-x^2)^{1/2}\,\ln x\,dx \tag{9.13}$$

and we see that is corresponds to the case $a = 2n+1$, $b = \frac{3}{2}$, $c = 2$ in (9.3). Therefore

$$I = \frac{\Gamma(n+\frac{1}{2})\,\Gamma(\frac{3}{2})}{4\Gamma(n+2)}\left[\psi(n+\tfrac{1}{2}) - \psi(n+2)\right]. \tag{9.14}$$

Using (1.7), (1.11) and (1.14) yields

$$\int_0^1 x^{2n}\sqrt{1-x^2}\,\ln x\,dx = -\frac{\binom{2n}{n}\pi}{2^{2n+2}\,(n+1)}\left(\ln 2 + \frac{1}{2n+2} + \sum_{k=1}^{2n}\frac{(-1)^k}{k}\right). \tag{9.15}$$

This is equivalent to (9.12).

Example 9.7. The integral **4.241.4** in [**2**] states that

$$\int_0^1 x^{2n+1}\sqrt{1-x^2}\,\ln x\,dx = \frac{(2n)!!}{(2n+3)!!}\left(\ln 2 + \sum_{k=1}^{2n+1}\frac{(-1)^{k-1}}{k} - \frac{1}{2n+3}\right). \tag{9.16}$$

To evaluate the integral, we write it as

$$I = \int_0^1 x^{2n+1}(1-x^2)^{1/2}\,\ln x\,dx \tag{9.17}$$

and we see that is corresponds to the case $a = 2n+2$, $b = \frac{3}{2}$, $c = 2$ in (9.3). Therefore

$$I = \frac{\Gamma(n+1)\,\Gamma(\frac{3}{2})}{4\Gamma(n+\frac{5}{2})}\left[\psi(n+1) - \psi(n+\tfrac{5}{2})\right]. \tag{9.18}$$

Using (1.7), (1.11) and (1.14) yields

$$\int_0^1 x^{2n+1}\sqrt{1-x^2}\,\ln x\,dx = \frac{2^{2n+1}}{(n+1)(n+2)\binom{2n+3}{n+1}}\left(\ln 2 - \frac{1}{2n+3} + \sum_{k=1}^{2n+1}\frac{(-1)^k}{k}\right). \tag{9.19}$$

This is equivalent to (9.16).

Example 9.8. The integral **4.241.5** in [**2**] states that

$$\int_0^1 \ln x\,\sqrt{(1-x^2)^{2n-1}}\,dx = -\frac{(2n-1)!!\,\pi}{4\,(2n)!!}\left[\psi(n+1) + \gamma + \ln 4\right] \tag{9.20}$$

To evaluate the integral, we write it as

$$I = \int_0^1 (1-x^2)^{n-\frac{1}{2}}\,\ln x\,dx \tag{9.21}$$

and we see that is corresponds to the case $a = 1$, $b = n + \frac{1}{2}$, $c = 2$ in (9.3). Therefore

$$I = \frac{\Gamma(n+\frac{1}{2})\,\Gamma(\frac{1}{2})}{4\Gamma(n+1)}\left[\psi(\tfrac{1}{2}) - \psi(n+\tfrac{1}{2})\right]. \tag{9.22}$$

Using (1.7), (1.11) and (1.14) yields

$$\int_0^1 (1-x^2)^{n-\frac{1}{2}}\,\ln x\,dx = -\frac{\binom{2n}{n}\pi}{2^{2n+2}}\left(2\ln 2 + \sum_{k=1}^{n}\frac{1}{k}\right). \tag{9.23}$$

This is equivalent to (9.20). This integral also appears as **4.246**.

Example 9.9. The case $n = 0$ in (9.7) yields

$$\int_0^1 \frac{\ln x\,dx}{\sqrt{1-x^2}} = -\frac{\pi}{2}\ln 2. \tag{9.24}$$

This appears as **4.241.7** in [**2**].

Example 9.10. Formula **4.241.8** states that

$$\int_1^\infty \frac{\ln x\,dx}{x^2\sqrt{x^2-1}} = 1 - \ln 2. \tag{9.25}$$

To evaluate this, let $t = 1/x$ to obtain

$$I = -\int_0^1 t(1-t^2)^{-1/2}\,\ln t\,dt. \tag{9.26}$$

This corresponds to the case $a = 2$, $b = \frac{1}{2}$, $c = 2$ in (9.3). Therefore

$$I = -\frac{\Gamma(1)\,\Gamma(\frac{1}{2})}{4\,\Gamma(\frac{3}{2})}\left[\psi(1) - \psi(\tfrac{3}{2})\right], \tag{9.27}$$

and the value $1 - \ln 2$ comes from (1.11) and (1.14).

Example 9.11. The case $n = 0$ in (9.15) produces

$$\int_0^1 \sqrt{1-x^2}\,\ln x\,dx = -\frac{\pi}{8}(2\ln 2 + 1). \tag{9.28}$$

This appears as **4.241.9** in [**2**].

Example 9.12. The case $n = 0$ in (9.19) produces

$$\int_0^1 x\sqrt{1-x^2}\,\ln x\,dx = \frac{1}{9}(3\ln 2 - 4). \tag{9.29}$$

This appears as **4.241.10** in [**2**].

Example 9.13. Entry **4.241.11** states that

$$\int_0^1 \frac{\ln x\,dx}{\sqrt{x(1-x^2)}} = -\frac{\sqrt{2\pi}}{8}\Gamma^2\left(\tfrac{1}{4}\right). \tag{9.30}$$

To evaluate the integral, write it as

$$I = \int_0^1 x^{-1/2}(1-x^2)^{-1/2}\,\ln x\,dx \tag{9.31}$$

and this corresponds to the case $a = \frac{1}{2}$, $b = \frac{1}{2}$, $c = 2$ in (9.3). Therefore

$$I = \frac{\Gamma(\frac{1}{4})\,\Gamma(\frac{1}{2})}{4\Gamma(\frac{3}{4})}\left[\psi\left(\tfrac{1}{4}\right) - \psi\left(\tfrac{3}{4}\right)\right]. \tag{9.32}$$

The stated form comes from using (1.9) and (1.16).

Example 9.14. The identity

$$\int_0^1 \frac{x\,\ln x}{\sqrt{1-x^4}}\,dx = -\frac{\pi}{8}\ln 2 \tag{9.33}$$

appears as **4.243** in [**2**]. To evaluate it, we write it as

$$I = \int_0^1 x(1-x^4)^{-1/2}\,\ln x\,dx \tag{9.34}$$

that corresponds to $a = 2$, $b = \frac{1}{2}$, $c = 4$ in (9.3). Therefore,

$$I = \frac{1}{16}\Gamma^2(\tfrac{1}{2})\left[\psi(\tfrac{1}{2}) - \psi(1)\right]. \tag{9.35}$$

The values $\psi(1) = -\gamma$ and $\psi(\frac{1}{2}) = -\gamma - 2\ln 2$ gives the result.

Example 9.15. The verification of **4.244.1**:

$$\int_0^1 \frac{\ln x\,dx}{\sqrt[3]{x(1-x^2)^2}} = -\frac{1}{8}\Gamma^3\left(\tfrac{1}{3}\right) \tag{9.36}$$

is achieved by using (9.3) with $a = \frac{2}{3}$, $b = \frac{1}{3}$ and $c = 2$ to obtain

$$I = \frac{\Gamma^2\left(\frac{1}{3}\right)}{4\Gamma(\frac{2}{3})}\left[\psi\left(\tfrac{1}{3}\right) - \psi\left(\tfrac{2}{3}\right)\right]. \tag{9.37}$$

Using (1.9) and (1.16) produces the stated result.

Example 9.16. The usual application of (9.3) shows that **4.244.2** is

$$\int_0^1 \frac{\ln x\,dx}{\sqrt[3]{1-x^3}} = \frac{2\pi}{9\sqrt{3}}\left(\psi\left(\tfrac{1}{3}\right) + \gamma\right), \tag{9.38}$$

where we have used $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = 2\pi/\sqrt{3}$. It remains to evaluate $\psi(\frac{1}{3})$. The identity (1.16) gives

$$\psi\left(\tfrac{1}{3}\right) - \psi\left(\tfrac{2}{3}\right) = -\frac{\pi}{\sqrt{3}}. \tag{9.39}$$

To obtain a second relation among these quantities, we start from the identity

$$\Gamma(3x) = \frac{3^{3x-1/2}}{2\pi}\Gamma(x)\,\Gamma(x+\tfrac{1}{3})\,\Gamma(x+\tfrac{2}{3}) \tag{9.40}$$

that follows directly from (1.6), and differentiate logarithmically to obtain

$$\psi(3x) = \ln 3 + \frac{1}{3}\left(\psi(x) + \psi(x+\tfrac{1}{3}) + \psi(x+\tfrac{2}{3})\right). \tag{9.41}$$

The special case $x = \frac{1}{3}$ yields

$$\psi\left(\tfrac{1}{3}\right) + \psi\left(\tfrac{2}{3}\right) = -2\gamma - 3\ln 3. \tag{9.42}$$

We conclude that

$$\psi(\tfrac{1}{3}) = -\gamma - \frac{3}{2}\ln 3 - \frac{\pi}{2\sqrt{3}} \tag{9.43}$$

and

$$\psi(\tfrac{2}{3}) = -\gamma - \frac{3}{2}\ln 3 + \frac{\pi}{2\sqrt{3}}. \tag{9.44}$$

This gives

$$\int_0^1 \frac{\ln x\,dx}{\sqrt[3]{1-x^3}} = -\frac{\pi}{3\sqrt{3}}\left(\ln 3 + \frac{\pi}{3\sqrt{3}}\right), \tag{9.45}$$

as stated in **4.244.2**.

Example 9.17. The evaluation of **4.244.3**:

$$\int_0^1 \frac{\ln x\,dx}{\sqrt[3]{1-x^3}} = -\frac{\pi}{3\sqrt{3}}\left(\ln 3 - \frac{\pi}{3\sqrt{3}}\right), \tag{9.46}$$

proceeds as in the previous example. The integral is identified as

$$I = \tfrac{1}{9}\Gamma(\tfrac{2}{3})\Gamma(\tfrac{1}{3})\left[\psi\left(\tfrac{2}{3}\right) + \gamma\right]. \tag{9.47}$$

The value (9.44) gives the rest.

Example 9.18. The change of variables $t = x^4$ yields

$$\int_0^1 \frac{x^p\,\ln x\,dx}{\sqrt{1-x^4}} = \frac{1}{16}\int_0^1 t^{(p-3)/4}(1-t)^{-1/2}\,\ln t\,dt. \tag{9.48}$$

The last integral is evaluated using (9.3) with $a = \frac{p+1}{4}$, $b = \frac{1}{2}$ and $c = 1$ to obtain

$$\int_0^1 \frac{x^p\,\ln x\,dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}}{16}\frac{\Gamma(\frac{p+1}{4})}{\Gamma(\frac{p+3}{4})}\left[\psi\left(\frac{p+1}{4}\right) - \psi\left(\frac{p+3}{4}\right)\right]. \tag{9.49}$$

The special case $p = 4n+1$ yields

$$\int_0^1 \frac{x^{4n+1}\,\ln x\,dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}}{16n!}\Gamma(n+\tfrac{1}{2})\left[\psi(n+\tfrac{1}{2}) - \psi(n+1)\right]. \tag{9.50}$$

The special case $p = 4n+1$ yields

$$\int_0^1 \frac{x^{4n+1}\,\ln x\,dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}{16n!}\left[\psi(n+\tfrac{1}{2}) - \psi(n+1)\right].$$

Using (1.7), (1.11) and (1.14) yields **4.245.1** in the form

$$\int_0^1 \frac{x^{4n+1}\,\ln x\,dx}{\sqrt{1-x^4}} = \frac{\pi\binom{2n}{n}}{2^{2n+3}}\left(-\ln 2 + \sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}\right). \tag{9.51}$$

The special case $p = 4n+3$ yields

$$\int_0^1 \frac{x^{4n+3}\,\ln x\,dx}{\sqrt{1-x^4}} = \frac{\sqrt{\pi}\,n!}{16\Gamma(n+\frac{3}{2})}\left[\psi(n+1) - \psi(n+\tfrac{3}{2})\right].$$

Using (1.7), (1.11) and (1.14) yields **4.245.2** in the form

$$\int_0^1 \frac{x^{4n+3}\,\ln x\,dx}{\sqrt{1-x^4}} = \frac{2^{2n-2}}{(2n+1)\binom{2n}{n}}\left(\ln 2+\sum_{k=1}^{2n+1}\frac{(-1)^k}{k}\right). \tag{9.52}$$

Example 9.19. The change of variables $t = x^{2n}$ produces

$$\int_0^1 \frac{\ln x\,dx}{\sqrt[n]{1-x^{2n}}} = \frac{1}{4n^2}\int_0^1 t^{\frac{1}{2n}-1}(1-t)^{-\frac{1}{n}}\,\ln t\,dt. \tag{9.53}$$

Then (9.3) with $a = \frac{1}{2n}$, $b = 1-\frac{1}{n}$ and $c = 1$ give the value

$$\int_0^1 \frac{\ln x\,dx}{\sqrt[n]{1-x^{2n}}} = \frac{\Gamma(\frac{1}{2n})\,\Gamma(1-\frac{1}{n})}{4n^2\,\Gamma(1-\frac{1}{2n})}\left[\psi\left(\tfrac{1}{2n}\right)-\psi\left(1-\tfrac{1}{2n}\right)\right]. \tag{9.54}$$

Using (1.7) and (1.14) to obtain

$$\int_0^1 \frac{\ln x\,dx}{\sqrt[n]{1-x^{2n}}} = -\frac{\pi}{8}\frac{B\left(\frac{1}{2n},\frac{1}{2n}\right)}{n^2\,\sin\left(\frac{\pi}{2n}\right)}. \tag{9.55}$$

This is **4.247.1** in [**2**].

Example 9.20. The change of variables $t = x^2$ gives

$$\int_0^1 \frac{\ln x\,dx}{\sqrt[n]{x^{n-1}(1-x^2)}} = \frac{1}{4}\int_0^1 t^{\frac{1}{2n}-1}(1-t)^{-\frac{1}{n}}\,\ln t\,dt.$$

Using (9.3) we obtain

$$\int_0^1 \frac{\ln x\,dx}{\sqrt[n]{x^{n-1}(1-x^2)}} = \frac{\Gamma(\frac{1}{2n})\,\Gamma(1-\frac{1}{2n})}{\Gamma(1-\frac{1}{2n})}\left[\psi\left(\tfrac{1}{2n}\right)-\psi\left(1-\tfrac{1}{2n}\right)\right]. \tag{9.56}$$

Proceeding as in the previous example, we obtain

$$\int_0^1 \frac{\ln x\,dx}{\sqrt[n]{x^{n-1}(1-x^2)}} = -\frac{\pi}{8}\frac{B\left(\frac{1}{2n},\frac{1}{2n}\right)}{\sin\left(\frac{\pi}{2n}\right)}. \tag{9.57}$$

This is **4.247.2** in [**2**].

Some integrals in [**2**] have the form of the Corollary 9.2 after an elementary change of variables.

Example 9.21. Formula **4.293.8** in [**2**] states that

$$\int_0^1 x^{a-1}\ln(1-x)\,dx = -\frac{1}{a}\left(\psi(a+1)+\gamma\right). \tag{9.58}$$

This follows directly from (9.3) by the change of variables $x \mapsto 1-x$. The same is true for **4.293.13**:

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,\ln(1-x)\,dx = B(a,b)\left[\psi(b)-\psi(a+b)\right]. \tag{9.59}$$

Example 9.22. The change of variables $t = e^{-x}$ gives

$$\int_0^\infty xe^{-x}(1-e^{2x})^{n-\frac{1}{2}}\,dx = -\int_0^1 (1-t^2)^{n-\frac{1}{2}}\,\ln t\,dt. \tag{9.60}$$

This latter integral is evaluated using (9.3) as

$$I = -\frac{\sqrt{\pi}\Gamma(n+\frac{1}{2})}{4n!}\left(\psi\left(\tfrac{1}{2}\right) - \psi(n+1)\right). \tag{9.61}$$

Using (1.7) and (1.11) we obtain

$$\int_0^\infty xe^{-x}(1-e^{-2x})^{n-\frac{1}{2}}\,dx = \frac{\binom{2n}{n}\pi}{2^{2n+2}}\left(2\ln 2 + \sum_{k=1}^n \frac{1}{k}\right). \tag{9.62}$$

This appears as **3.457.1** in [**2**].

10. An announcement

There are many integrals in [**2**] that contain the term $1+x$ in the denominator, instead of the term $1-x$ seen, for instance, in Section 3. The evaluation of these integrals can be obtained using the *incomplete beta function*, defined by

$$\beta(a) := \int_0^1 \frac{x^{a-1}\,dx}{1+x} \tag{10.1}$$

as it appears in **8.371.2**. This function is related to the digamma function by the identity

$$\beta(a) = \frac{1}{2}\left[\psi\left(\frac{a+1}{2}\right) - \psi\left(\frac{a}{2}\right)\right]. \tag{10.2}$$

These evaluations will be reported in [**5**].

11. One more family

We conclude this collection with a two-parameter family of integrals.

Proposition 11.1. Let a, $b \in \mathbb{R}^+$. Then

$$\int_0^\infty \left(e^{-x^a} - \frac{1}{1+x^b}\right)\frac{dx}{x} = -\frac{\gamma}{a}, \tag{11.1}$$

independently of b.

Proof. Write the integral as

$$\int_0^\infty \left(e^{-x^a} - e^{-x^b}\right)\frac{dx}{x} + \int_0^\infty \left(e^{-x^b} - \frac{1}{1+x^b}\right)\frac{dx}{x}. \tag{11.2}$$

The first integral is $(a-b)\gamma/ab$ according to (2.10). The change of variables $t = x^b$ converts the second one into

$$\frac{1}{b}\int_0^\infty \left(e^{-t} - \frac{1}{1+t}\right)\frac{dt}{t} = -\frac{\gamma}{b}, \tag{11.3}$$

according to (2.6). The formula has been established. □

Example 11.2. The case $a = 2^n$ and $b = 2^{n+1}$ gives **3.475.1**:

$$\int_0^\infty \left(\exp(-x^{2^n}) - \frac{1}{1+x^{2^{n+1}}} \right) \frac{dx}{x} = -\frac{\gamma}{2^n}. \tag{11.4}$$

Example 11.3. The case $a = 2^n$ and $b = 2$ gives **3.475.2**:

$$\int_0^\infty \left(\exp(-x^{2^n}) - \frac{1}{1+x^2} \right) \frac{dx}{x} = -\frac{\gamma}{2^n}. \tag{11.5}$$

Example 11.4. The case $a = 2$ and $b = 2$ gives **3.467**:

$$\int_0^\infty \left(e^{-x^2} - \frac{1}{1+x^2} \right) \frac{dx}{x} = -\frac{\gamma}{2}. \tag{11.6}$$

Example 11.5. Finally, the change of variables $t = ax$ yields

$$\int_0^\infty \left(e^{-px} - \frac{1}{1+a^2x^2} \right) \frac{dx}{x} = \int_0^\infty \frac{e^{-pt/a} - e^{-t}}{t} \, dt + \int_0^\infty \left(e^{-t} - \frac{1}{1+t^2} \right) \frac{dt}{t}. \tag{11.7}$$

The first integral is $\ln \frac{a}{p}$ according to (2.5), the second one is $-\gamma$. This gives the evaluation of **3.442.3**:

$$\int_0^\infty \left(e^{-px} - \frac{1}{1+a^2x^2} \right) \frac{dx}{x} = \gamma + \ln \frac{a}{p}. \tag{11.8}$$

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