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On the uniformly convexity of N-functions

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Abstract. This paper discussed the characterizations of uniformly convexity of N-functions.

Definition 1. *A function $M(u)$: $R \to R^+$ is called an N-function if it has the following properties:*

(1) *M is even, continuous, convex*;
(2) *$M(0) = 0$ and $M(u) > 0$ for all $u = 0$*;
(3) $\lim\limits_{u \to 0} \frac{M(u)}{u} = 0$ *and* $\lim\limits_{u \to +\infty} \frac{M(u)}{u} = +\infty$.

The N-function generates the Orlicz spaces. So it is important to analysis it. It is well-known that $M(u)$ is an N-function iff $M(u) = \int_0^{|u|} p(t)dt$, where the right derivative $p(t)$ of $M(u)$ satisfies:

(1) $p(t)$ is right-continuous and nondecreasing;
(2) $p(t) > 0$ whenever $t > 0$;
(3) $p(0) = 0$ and $\lim\limits_{t \to \infty} p(t) = +\infty$.

Definition 2. *A continuous function $M : R \to R$ is called convex if*

$$M\left(\frac{u+v}{2}\right) \leqslant \frac{M(u) + M(v)}{2}$$

for all $u, v \in R$. If, in addition, the two sides of the above inequality are not equal for all $u \neq v$, then we call M strictly convex.

Definition 3. *For a continuous function $M : R \to R$.*

(1) *If for any $\varepsilon > 0$ and any $u_0 > 0$, there exists some $\delta > 0$ such that*

$$M\left(\frac{u+v}{2}\right) \leqslant (1-\delta)\frac{M(u) + M(v)}{2}$$

for all $u, v \in R$ satisfying $|u - v| \geqslant \varepsilon \max\{|u|, |v|\} \geqslant \varepsilon u_0$, then M is said to be uniformly convex for larger argument .

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(2) *If for any* $\varepsilon > 0$ *and any* $u_0 > 0$, *there exists some* $\delta > 0$ *such that*

$$M\left(\frac{u+v}{2}\right) \leqslant (1-\delta)\frac{M(u)+M(v)}{2}$$

for all $u, v \in R$ *satisfying* $|u| \leqslant u_0, |v| \leqslant u_0$ *and* $|u-v| \geqslant \varepsilon \max\{|u|, |v|\}$, *then* M *is said to be uniformly convex near origin.*

There were many fragmentary discussions about uniformly convexity for the exploration of the properties of Orlicz spaces. This paper will show the equivalence of the various definitions and characterizations of uniformly convexity. In the sequel, we only deal with the uniformly convexity near origin.

Theorem 1. M *is uniformly convex iff for any* $\varepsilon > 0$ *and any* $[a, b]$ *contained in* $(0, 1)$, *there exists some* $\delta > 0$ *such that* $M(\alpha u + (1-\alpha)v) \leqslant (1-\delta)[\alpha M(u) + (1-\alpha)M(v)]$ *for all* $\alpha \in [a, b]$ *and all* $u, v \in R$ *satisfying* $|u| \leqslant u_0, |v| \leqslant u_0$ *and* $|u-v| \geqslant \varepsilon \max\{|u|, |v|\}$.

Proof. Sufficiency. It suffices to let $\alpha = \frac{1}{2}$.

Necessity. For any $\varepsilon > 0$ and any $[a, b]$ contained in $(0, 1)$, pick $c > 0$ such that $0 \leqslant a - c < b + c \leqslant 1$. Let $\varepsilon' = 2c\varepsilon$ and $u_0' = c\varepsilon u_0$, then for any $\alpha \in [a, b]$ and all u, v satisfying $|u| \leqslant u_0, |v| \leqslant u_0, |u-v| \geqslant \varepsilon \max\{|u|, |v|\}$, let

$$u^* = (a-c)u + (1-a+c)v, \ v^* = (a+c)u + (1-a-c)v.$$

Since $a \pm c \in [0, 1], \ u \leqslant u^* \leqslant v, \ u \leqslant v^* \leqslant v$, and

$$|u^* - v^*| = 2c|u-v| \geqslant 2c\varepsilon \max\{|u|, |v|\} \geqslant \varepsilon' \max\{|u^*|, |v^*|\}.$$

Then from definition 3(2), there exists $\delta' > 0$ such that

$$\begin{aligned}
& M(\alpha u + (1-\alpha)v) \\
= \;& M\left(\frac{u^*+v^*}{2}\right) \\
\leqslant \;& (1-\delta')\frac{M(u^*)+M(v^*)}{2} \\
= \;& \frac{1-\delta'}{2}[M((a-c)u + (1-a+c)v) + M((a+c)u + (1-a-c)v] \\
\leqslant \;& \frac{1-\delta'}{2}[(a-c)M(u) + (1-a+c)M(v) + (a+c)M(u) + (1-a-c)M(v)] \\
= \;& (1-\delta')[aM(u) + (1-a)M(v)].
\end{aligned}$$

We complete the proof by setting $\delta = \delta'$.

Since an N-function is even, we only discuss the positive variables.

Theorem 2. Let M *is an* N*-function, then the following are equivalent*:

(1) *For any* $a > 0$ *and* $a \neq 1$, *there exists* $\delta(a) > 0$ *such that* $M(\frac{u+au}{2}) \leqslant (1-\delta(a))\frac{M(u)+M(au)}{2}$ *for all* $au, u \in [0, u_0]$;

(2) *For any* $a \in (0, 1)$, *there exists* $\delta(a) > 0$ *such that* $M(\frac{u+au}{2}) \leqslant (1-\delta(a))\frac{M(u)+M(au)}{2}$ *for all* $u \in [0, u_0]$;

(3) *M is uniformly convex on $[0,u_0]$, namely, for any $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that* $M(\frac{u+v}{2})\leqslant(1-\delta(\varepsilon))\frac{M(u)+M(v)}{2}$ $(|u-v|\geqslant\varepsilon\max\{u,v\})$;

(4) *For any $\beta\in[0,u_0)$, $b\geqslant u_0$ and $\varepsilon>0$, there exists $\delta>0$ such that* $M(\frac{u+v}{2})\leqslant(1-\delta(\varepsilon))\frac{M(u)+M(v)}{2}$ $(\max\{u,v\}\leqslant b,\ 0\leqslant\min\{u,v\}\leqslant\beta,\ |u-v|\geqslant\varepsilon\max\{u,v\})$;

(5) *For any $m\in N^*$, $m\geqslant 2$, $\beta\in[0,u_0)$, and $b\geqslant u_0$, $\varepsilon>0$, there exists $\delta>0$ such that* $M(\frac{1}{m}\sum\limits_{j=1}^{m}u_j)\leqslant(1-\delta(\varepsilon))\frac{1}{m}\sum\limits_{j=1}^{m}M(u_j)$ $(\max\{u_j:1\leqslant j\leqslant m\}\leqslant b,\ 0\leqslant\min\{u_j:1\leqslant j\leqslant m\}\leqslant\beta,\ \max\{|u_i-u_j|:1\leqslant i,\ j\leqslant m\}\geqslant\varepsilon\max\{u_j:1\leqslant j\leqslant m\})$.

Proof. (1) $\Rightarrow$ (2). Trivial.

(2) $\Rightarrow$ (1). For any $a>0$, it is known if $a\in(0,1)$.
if $a=1$, $M(\frac{u+u}{2})=M(u)<\frac{M(u)+M(u)}{2}$, the equal sign don't hold.
if $a>1$, then $\frac{1}{a}<1$.

$$\begin{aligned}M\left(\frac{u+au}{2}\right) &= M\left(\frac{a}{2}\left(a+\frac{u}{a}\right)\right)\\ &= M\left(\frac{au+\frac{1}{a}(au)}{2}\right)\\ &\leqslant (1-\delta)\frac{M(au)+M\left(\frac{1}{a}\cdot au\right)}{2}\\ &= (1-\delta)\frac{M(au)+M(u)}{2}.\end{aligned}$$

(3) $\Rightarrow$ (2) For any $a\in(0,1)$, let $v=au$, then $|u-v|=(1-a)u$. Pick $\varepsilon=1-a$, then there exists $\delta>0$ such that $M(\frac{u+v}{2})\leqslant(1-\delta)\frac{M(u)+M(v)}{2}$ or equivalently, $M(\frac{u+au}{2})\leqslant(1-\delta)\frac{M(u)+M(au)}{2}$.

(2) $\Rightarrow$ (3) For any $\varepsilon>0$, without loss of generality, we may assume $u>v$ and $u,\ v\in[0,u_0]$. For any u, let $v_u=(1-\varepsilon)u$, then $(1-\varepsilon)\in(0,1)$. From (2), there exists $\delta(\varepsilon)>0$ such that $M(\frac{u+v_u}{2})\leqslant(1-\delta(\varepsilon))\frac{M(u)+M(v_u)}{2}$, then $u-v\geqslant u-v_u=\varepsilon u=\varepsilon\max\{u,v\}$ for any v satisfying $0\leqslant v\leqslant v_u$. Considering $f(v)=M(\frac{u+v}{2})/\frac{M(u)+M(v)}{2}$, it is an increasing function on $[0,(1-\varepsilon)u]$, then $\sup\limits_{0\leqslant v\leqslant(1-\varepsilon)u}f(v)=f((1-\varepsilon)u)=1-\delta(\varepsilon)$, so $M(\frac{u+v}{2})\leqslant(1-\delta(\varepsilon))\frac{M(u)+M(v)}{2}$.

(3) $\Rightarrow$ (4) For any $\varepsilon>0$, we discuss the following two cases considering $u,\ v$.
① If $0\leqslant u,\ v\leqslant u_0$, then from (3) there exists $\delta_1>0$ such that $M(\frac{u+v}{2})\leqslant(1-\delta(\varepsilon))\frac{M(u)+M(v)}{2}$.
② If $0\leqslant\min\{u,v\}\leqslant\beta<u_0$, $u_0\leqslant\max\{u,v\}\leqslant b$ and $|u-v|\geqslant\varepsilon\max\{u,v\}$. Since M is uniformly convex on $[\beta,u_0]$, it is strictly convex, so M is not affine, then $M(\frac{u+v}{2})\leqslant\frac{M(u)+M(v)}{2}$ (otherwise, M is affine on $[u,v]$ or $[v,u]$). So the bivariate continuous function $f(u,v)=M(\frac{u+v}{2})/\frac{M(u)+M(v)}{2}<1$ on bounded closed field(compact set) $A=\{(u,v):|u-v|\geqslant\varepsilon\max\{u,v\};\ 0\leqslant\min\{u,v\}\leqslant\beta,\ u_0\leqslant\max\{u,v\}\leqslant b\}$. Then the maximum of $f(u,v)$ is less than 1 and there exists $\delta_2>0$ such that $f(u,v)\leqslant 1-\delta_2$. Let $\delta=\min\{\delta_1,\delta_2\}$, then for any $u,\ v$ satisfying our condition, we have $M(\frac{u+v}{2})\leqslant(1-\delta(\varepsilon))\frac{M(u)+M(v)}{2}$.

(4) $\Rightarrow$ (3) It suffices to let $b = u_0$.

(4) $\Rightarrow$ (5) Let $\max\{u_j : 1 \leqslant j \leqslant m\} \leqslant b$, $0 \leqslant \min\{u_j : 1 \leqslant j \leqslant m\} \leqslant \beta$, $\max\{|u_j - v_j| : 1 \leqslant i,\ j \leqslant m\} \geqslant \varepsilon \max\{u_j : 1 \leqslant j \leqslant m\}$. We use 'forward-backward' complete induction.

Firstly, if $m = 2^k (k \in N^*)$,

① It is obvious when $k = 1$.

② Assume the inequality is correct when $m = 2^k$, namely,

$$M\left(\frac{1}{2^k}\sum_{j=1}^{2^k} u_j\right) \leqslant (1-\delta(\varepsilon))\frac{\sum\limits_{j=1}^{2^k} M(u_j)}{2^k}$$

then we have

$$\begin{aligned} M\left(\frac{1}{2^{k+1}}\sum_{j=1}^{2^{k+1}} u_j\right) &= M\left(\frac{1}{2^{k+1}}\left(\sum_{j=1}^{2^k} u_j + \sum_{j=2^k+1}^{2^{k+1}} u_j\right)\right) \\ &\leqslant (1-\delta(\varepsilon))\frac{M\left(\frac{1}{2^k}\sum\limits_{j=1}^{2^k} u_j\right) + M\left(\frac{1}{2^k}\sum\limits_{j=2^k+1}^{2^{k+1}} u_j\right)}{2} \\ &\leqslant (1-\delta(\varepsilon))\frac{\sum\limits_{j=1}^{2^{k+1}} M(u_j)}{2^{k+1}}. \end{aligned}$$

when $m = 2^{k+1}$.

Thus we have verified the inequality when $m = 2^k (k \in N^*)$. This is the 'forward' section.

Secondly, if the inequality hold for some $m > 2$, it also hold for $m-1$. This is the 'backward' section. Since $\frac{1}{m-1}\sum\limits_{j=1}^{m-1} u_j = \frac{1}{m}[\sum\limits_{j=1}^{m-1} u_j + \frac{1}{m-1}\sum\limits_{j=1}^{m-1} u_j]$, we have

$$\begin{aligned} M\left(\frac{1}{m-1}\sum_{j=1}^{m-1} u_j\right) &= M\left(\frac{1}{m}\left(\sum_{j=1}^{m-1} u_j + \frac{1}{m-1}\sum_{j=1}^{m-1} u_j\right)\right) \\ &\leqslant (1-\delta(\varepsilon))\frac{\sum\limits_{j=1}^{m-1} M(u_j) + M\left(\frac{1}{m-1}\sum\limits_{j=1}^{m-1} u_j\right)}{m} \\ &\leqslant (1-\delta(\varepsilon))\frac{\sum\limits_{j=1}^{m-1} M(u_j)}{m} + \frac{1}{m}M\left(\frac{1}{m-1}\sum_{j=1}^{m-1} u_j\right). \end{aligned}$$

Therefore,

$$\frac{m-1}{m}M(\frac{1}{m-1}\sum_{j=1}^{m-1} u_j) \leqslant (1-\delta(\varepsilon))\frac{1}{m}\sum_{j=1}^{m-1} M(u_j)$$

or equivalently

$$M(\frac{1}{m-1}\sum_{j=1}^{m-1}u_j) \leqslant (1-\delta(\varepsilon))\frac{1}{m-1}\sum_{j=1}^{m-1}M(u_j).$$

Consequently, we have verified

$$M(\frac{1}{m}\sum_{j=1}^{m}u_j) \leqslant (1-\delta(\varepsilon))\frac{1}{m}\sum_{j=1}^{m}M(u_j).$$

(5) $\Rightarrow$ (4) It is obvious.

Theorem 3. *If M is strictly convex, then M is uniformly convex on any bounded closed interval which exclude* 0.

Proof. Since $M(u)$ is uniformly convex on I for any $\varepsilon > 0$ and bounded closed interval I that exclude 0, the maximum of $f(u,v) = M(\frac{u+v}{2})/\frac{M(u)+M(v)}{2}$ is less than 1 on the compact set $\{(u,v) : |u-v| \geqslant \varepsilon \max\{u,v\};\ u,\ v \in I\}$ and we assume it is $1-\delta$, so $M(\frac{u+v}{2}) \leqslant (1-\delta)\frac{M(u)+M(v)}{2}$.

Theorem 4. *If M is a strictly convex N-Function and it is uniformly convex on* $[0,u_0]$, *then M is uniformly convex on any* $[0,u_1]$.

Proof. (1) It is obvious if $u_1 \leqslant u_0$.

(2) If $u_1 > u_0$, then for any $\varepsilon > 0$ we discuss the following two cases:

① If $u,\ v < u_0$, since M is uniformly convex on $[0,u_0]$, there exists $\delta_1 > 0$ such that $M(\frac{u+v}{2}) \leqslant (1-\delta_1)\frac{M(u)+M(v)}{2}$ when $|u-v| \geqslant \varepsilon \max\{u,v\}$.

② If $\max\{u,v\} \geqslant u_0$, then $f(u,v) = M(\frac{u+v}{2})/\frac{M(u)+M(v)}{2} < 1$ on compact set $A = \{(u,v) : |u-v| \geqslant \varepsilon \max\{u,v\};\ u_0 \leqslant \max\{u,v\} \leqslant u_1\}$. Then the maximum of $f(u,v)$ is less than 1 and there exists $\delta_2 > 0$ such that $f(u,v) \leqslant 1-\delta_2$ or equivalently, $M(\frac{u+v}{2}) \leqslant (1-\delta_2)\frac{M(u)+M(v)}{2}$. Let $\delta = \min\{\delta_1,\delta_2\}$, then for any $u,v \in [0,u_1]$ we have $M(\frac{u+v}{2}) \leqslant (1-\delta)\frac{M(u)+M(v)}{2}$, namely, M is uniformly convex on $[0,u_1]$.

Remark Y.A.Cui, Ryszard and T.F.Wang[**3**] verified the following simple criterion respect to the uniformly convexity of N-functions.

Assume M is an N-function. Then M is uniformly convex if and only if for each fixed $\varepsilon > 0$ there exists $K > 1$ such that $p((1+\varepsilon)t) \geqslant Kp(t)$ $(0 \leqslant t \leqslant t_0)$.

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