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Homology in P-semi-abelian categories

Yaroslav Kopylov

ABSTRACT. We study the problem of the equivalence of the two natural notions of homology for a cochain complex in a P-semi-abelian category.

1. Introduction

The usual definition of homology in an abelian category can be formulated in two different manners dual to one another (see, for example, [**4**]). These two definitions make it possible to construct the long exact homology sequence corresponding to a short exact sequence of complexes in an abelian category. It is now known [**5, 9**] that the same holds in quasi-abelian categories (= categories, semi-abelian in the sense of Raĭkov [**13**]) but in this case the so-obtained long homological sequence is in general not exact (see [**5, 9**] for details).

The aim of this article is to find out what happens in the wider class of P-semi-abelian categories (= categories, semi-abelian in the sense of Palamodov [**12**]). In such categories, to be able to construct the long semi-exact (co)homology sequence, we must impose some extra conditions on the differentials of the complexes and the morphisms between them.

The structure of the paper is as follows.

In Section 2, we give the necessary definitions and recall or prove some basic facts. In Section 3, we, using the two classical definitions of homology, introduce the notions of left and right homology and prove a condition sufficient for their coincidence. Then, in Section 4, we discuss the possibility of constructing a long homology sequence. In conclusion (Section 5), we study the properties of a homological diagram used by Eckmann and Hilton in [**3**] for constructing the spectral sequence of an exact couple in an abelian category in the case of a P-semi-abelian category.

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2. P-semi-abelian categories

We consider additive categories satisfying the following axiom.

Axiom 1. Each morphism has kernel and cokernel.

Additive categories satisfying Axiom 1 are commonly known as *pre-abelian* [**6**]. However, since below we mention "pre-abelian" as one of the historical names for P-semi-abelian categories, we will not use the term in this article in order to avoid any ambiguity.

We denote by $\ker\alpha$ ($\operatorname{coker}\alpha$) an arbitrary kernel (cokernel) of α and by $\operatorname{Ker}\alpha$ ($\operatorname{Coker}\alpha$) the corresponding object; the equality $a = \ker b$ ($a = \operatorname{coker} b$) means that a is a kernel of b (a is a cokernel of b).

In a category meeting Axiom 1, every morphism α admits a canonical decomposition $\alpha = (\operatorname{im}\alpha)\overline{\alpha}(\operatorname{coim}\alpha)$, where $\operatorname{im}\alpha = \ker\operatorname{coker}\alpha$, $\operatorname{coim}\alpha = \operatorname{coker}\ker\alpha$. A morphism α is called *strict* if $\overline{\alpha}$ is an isomorphism.

We use the following notations:

O_c is the class of all strict morphisms;

M is the class of all monomorphisms;

M_c is the class of all strict monomorphisms (= kernels);

P is the class of all epimorphisms;

P_c is the class of all strict epimorphisms (= cokernels).

We write $\alpha \mid \beta$ if $\alpha = \ker\beta$ and $\beta = \operatorname{coker}\alpha$.

Lemma 1. [**1, 2, 11, 13**] *The following assertions hold in an additive category meeting Axiom* 1:

(1) $\ker\alpha \in M_c$ *and* $\operatorname{coker}\alpha \in P_c$ *for every* α;

(2) $\alpha \in M_c \Longleftrightarrow \alpha = \operatorname{im}\alpha$, $\alpha \in P_c \Longleftrightarrow \alpha = \operatorname{coim}\alpha$;

(3) *a morphism* α *is strict if and only if it is representable in the form* $\alpha = \alpha_1\alpha_0$ *with* $\alpha_0 \in P_c$, $\alpha_1 \in M_c$; *in every such representation,* $\alpha_0 = \operatorname{coim}\alpha$ *and* $\alpha_1 = \operatorname{im}\alpha$;

(4) *if a commutative square*

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & D \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{\beta} & B \end{array} \tag{2.1}$$

is a pull-back then $f \in M \Longrightarrow g \in M$, $f \in M_c \Longrightarrow g \in M_c$, *if the square is a push-out then* $g \in P \Longrightarrow f \in P$, $g \in P_c \Longrightarrow f \in P_c$.

An additive category meeting Axiom 1 is abelian if and only if $\overline{\alpha}$ is an isomorphism for every α.

An additive category is called *P-semi-abelian*, *semi-abelian in the sense of Palamodov* [**12, 14**], or *pre-abelian* [**1**] if it meets Axiom 1 and the following

Axiom 2. For every morphism α, $\overline{\alpha}$ is a bimorphism, that is, a monomorphism and an epimorphism.

Lemma 2. [**8, 14**] *The following hold in a P-semi-abelian category*:

(1) $gf \in M_c \Longrightarrow f \in M_c$, $gf \in P_c \Longrightarrow g \in P_c$;

(2) *if* $f, g \in M_c$ *and* gf *is defined then* $gf \in M_c$*; if* $f, g \in P_c$ *and* gf *is defined then* $gf \in P_c$;

(3) *if* $fg \in O_c$, $f \in M$ *then* $g \in O_c$*; if* $fg \in O_c$, $g \in P$ *then* $f \in O_c$.

An additive category satisfying Axiom 1 is called *quasi-abelian* [**16**] (*semi-abelian* in the sense of Raĭkov [**13**], or *almost abelian* [**14**]) if it meets the following

Axiom 3. If square (2.1) is a pull-back then $f \in P_c \Longrightarrow g \in P_c$. If (2.1) is a push-out then $g \in M_c \Longrightarrow f \in M_c$.

As is well-known [**11, 13, 14, 16**], every quasi-abelian category is P-semi-abelian. As has been recently discovered by Rump [**15**], there exist semi-abelian categories that are not quasi-abelian.

In [**11**, Theorem 1], Kuz′minov and Cherevikin established the following fact:

LEMMA 3. *An additive category* $\mathcal{A}$ *with kernels and cokernels is P-semi-abelian if and only if the following two conditions are fulfilled:*

(P1) *if* (2.1) *is a pull-back then* $f \in P_c \Longrightarrow g \in P$;

(P2) *if* (2.1) *is a push-out then* $g \in M_c \Longrightarrow f \in M$.

If, for a morphism $f \in P_c$ in a pull-back (2.1) in an additive category with kernels and cokernels, $g \in P_c$ (for a morphism $g \in M_c$ in a push-out (2.1), $f \in M_c$) then f is called a *stable cokernel* (g is called a *stable kernel*).

We establish some basic properties of stable kernels and cokernels.

LEMMA 4. *The following hold in a P-semi-abelian category*:

(1) *if* gf *is a stable kernel then so is* f*, if* gf *is a stable cokernel then so is* g;

(2) *if* f *and* g *are stable kernels and* gf *is defined then* gf *is a stable kernel; if* f *and* g *are stable cokernels and* gf *is defined then* gf *is a stable cokernel.*

PROOF. (1) Suppose that $h = gf$ is a stable cokernel. Consider a pull-back $g\beta' = \beta g'$ and construct a pull-back $f\alpha = \beta' f'$. We thus obtain the commutative diagram

$$\begin{array}{ccccc} D & \xrightarrow{f'} & M & \xrightarrow{g'} & F \\ {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\beta'}\big\downarrow & & {\scriptstyle\beta}\big\downarrow \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

Then the resulting square

$$\begin{array}{ccc} D & \xrightarrow{g'f'} & F \\ {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\beta}\big\downarrow \\ A & \xrightarrow{h} & C \end{array}$$

is a pull-back too (see, for example, [**3**, Proposition 2.10]). The stability of gf implies that $g'f' \in P_c$. Hence, by Lemma 2(1), $g' \in P_c$, and so g is a stable cokernel.

The first assertion of (1) results from this by duality.

(2) Suppose that $f : A \to B$ and $g : B \to C$ are stable cokernels and $h = gf$. Consider the pull-back

$$\begin{array}{ccc} D & \xrightarrow{\varphi} & F \\ {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\beta}\big\downarrow \\ A & \xrightarrow{h} & C \end{array}$$

Consider also the diagram

$$\begin{array}{ccccc} D & \xrightarrow{f'} & M & \xrightarrow{g'} & F \\ {\scriptstyle\alpha'}\big\downarrow & & {\scriptstyle\beta'}\big\downarrow & & {\scriptstyle\beta}\big\downarrow \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

where both squares are pull-backs. Then, as was noted in the proof of item (1), the resulting square $h\alpha' = \beta(g'f')$ is also a pull-back, and so, up to an isomorphism, we have $\varphi = g'f'$ and $\alpha' = \alpha$. Now, the fact that f' and g' are both cokernels and Lemma 2(2) imply that $\varphi = g'f' \in P_c$. Thus, h is a stable cokernel.

The first assertion of (2) is obtained from the second by duality. □

We call a sequence $\ldots \xrightarrow{a} B \xrightarrow{b} \ldots$ in an additive category *semi-exact at the term* B if $ba = 0$.

A sequence $\ldots \xrightarrow{a} B \xrightarrow{b} \ldots$ in a P-semi-abelian category is said to be *exact at the term* B if $\operatorname{im} a = \ker b$ (or, equivalently, $\operatorname{coker} a = \operatorname{coim} b$).

For a commutative square (2.1), denote by $\hat{g} : \operatorname{Ker}\alpha \to \operatorname{Ker}\beta$ the morphism defined by the condition $g(\ker\alpha) = (\ker\beta)\hat{g}$ and by $\hat{f} : \operatorname{Coker}\alpha \to \operatorname{Coker}\beta$, the morphism defined by the condition $\hat{f}(\operatorname{coker}\alpha) = (\operatorname{coker}\beta)f$.

Lemma 5. [**2**] *For an arbitrary pull-back* (2.1) *in an additive category meeting Axiom* 1*,* $\hat{g}$ *is an isomorphism.*

The dual assertion also holds.

Below we make use of the following assertion:

Lemma 6. [**10**] *Suppose that the square*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & D \\ {\scriptstyle\mathrm{id}}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ A & \xrightarrow{\beta} & B \end{array}$$

in a P-semi-abelian category is commutative, $f \in M$*, and* $h : \operatorname{Coker}\beta \to \operatorname{Coker} f$ *is the morphism defined by the condition* $\operatorname{coker} f = h(\operatorname{coker}\beta)$*. Then*

(1) *if* $\beta \in O_c$ *then* $\hat{f} \in M$*;*

(2) *if* $f \in M_c$ *and* $\operatorname{coker}\beta$ *is a stable cokernel then* $\hat{f} = \ker h$*.*

3. Two types of homology

Throughout the rest of the section, the ambient category $\mathcal{A}$ is assumed P-semi-abelian.

Given a sequence of the form

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C \tag{3.1}$$

such that $\psi\varphi = 0$, there are a natural morphism $\sigma : A \to \operatorname{Ker}\psi$ such that $\varphi = (\ker\psi)\sigma$ and a natural morphism $\tau : \operatorname{Coker}\varphi \to C$ such that $\psi = \tau \operatorname{coker}\varphi$.

Definition 1. *Call* $H_-(B) = H_-(B, \varphi, \psi) = \operatorname{Coker}\sigma$ *and* $H_+(B) = H_+(B, \varphi, \psi) = \operatorname{Ker}\tau$ *the* left *and* right homology objects of (3.1) at the term B.

It is classical that these two notions coincide for abelian categories (see, for example, [**4**]). This remains valid for quasi-abelian categories [**9**].

As was shown in [**9**], there is a unique morphism $m : H_-(B) \to H_+(B)$ such that

$$(\ker\tau)m \operatorname{coker}\sigma = (\operatorname{coker}\varphi)(\ker\psi). \tag{3.2}$$

Recall the construction of m since it will be used below.

Since $\tau(\operatorname{coker}\varphi)\ker\psi = \psi\ker\psi = 0$, there exists a unique morphism $\mu : \operatorname{Ker}\psi \to \operatorname{Ker}\tau$ such that $(\ker\tau)\mu = (\operatorname{coker}\varphi)\ker\psi$. Now, we have

$$(\ker\tau)\mu\sigma = (\operatorname{coker}\varphi)(\ker\psi)\sigma = (\operatorname{coker}\varphi)\varphi = 0.$$

It follows that $\mu\sigma = 0$ and, hence, $\mu = m \operatorname{coker}\sigma$ for a unique morphism $m : \operatorname{Coker}\sigma \to \operatorname{Ker}\tau$, and this m yields (3.2).

We have the following P-semi-abelian version of Lemma 4 of [**9**].

Lemma 7. *The morphism* $m : H_-(B) \to H_+(B)$ *is a bimorphism. If* $\ker\psi$ *is a stable kernel or* $\operatorname{coker}\varphi$ *is a stable cokernel then* m *is an isomorphism.*

Proof. Note that the square

$$\begin{array}{ccc} \operatorname{Ker}\psi & \xrightarrow{\operatorname{coker}\sigma} & H_-(B) \\ {\scriptstyle\ker\psi}\downarrow & & \downarrow{\scriptstyle(\ker\tau)m} \\ B & \xrightarrow{\operatorname{coker}\varphi} & \operatorname{Coker}\varphi \end{array} \tag{3.3}$$

is a push-out. Indeed, suppose that $\xi' \operatorname{coker}\sigma = \xi'' \ker\psi$. We have $\xi''\varphi = \xi''(\ker\psi)\sigma = \xi'(\operatorname{coker}\sigma)\sigma = 0$. Therefore, there exists a unique morphism ξ_0 with $\xi_0(\operatorname{coker}\varphi) = \xi''$. For this morphism,

$$\xi_0(\ker\tau)m \operatorname{coker}\sigma = \xi_0(\operatorname{coker}\varphi)\ker\psi = \xi''\ker\psi = \xi'(\operatorname{coker}\sigma),$$

whence $\xi_0(\ker\tau)m = \xi'$ because $\operatorname{coker}\sigma \in P$. Thus, (3.3) is push-out.

Now, Lemma 3 yields $(\ker\tau)m \in M$, from which $m \in M$. If, in addition, $\ker\psi$ is a stable kernel then $(\ker\tau)m \in M_c$ and, therefore, by Lemma 2, $m \in M_c$.

The dual argument shows that the square

$$\begin{array}{ccc} \operatorname{Ker}\psi & \xrightarrow{\mu} & H_+(B) \\ {\scriptstyle \ker\psi}\downarrow & & \downarrow{\scriptstyle \ker\tau} \\ B & \xrightarrow{\operatorname{coker}\varphi} & \operatorname{Coker}\varphi \end{array}$$

is a pull-back. Thus, we have $m \in P$ and if $\operatorname{coker}\varphi$ is a stable cokernel then $m \in P_c$.

The lemma follows. ☐

4. On the long (co)homology sequence in a P-semi-abelian category

Here we discuss the possibility of constructing the long exact cohomology sequence for a short strictly exact sequence of complexes in a P-semi-abelian category.

By a (*cochain*) *complex* $\mathfrak{A} = (A^n, d_A^n)_{n\in\mathbb{Z}}$ in an additive category we understand a sequence

$$\dots \longrightarrow A^{n-1} \xrightarrow{d_A^{n-1}} A^n \xrightarrow{d_A^n} A^{n+1} \xrightarrow{d_A^{n+1}} \dots$$

which is semi-exact at each term, that is, $d_A^{n+1} d_A^n = 0$ for all n.

Let $\mathfrak{A} = (A^n, d_A^n)_{n\in\mathbb{Z}}$ be a cochain complex in an additive category with kernels and cokernels. As was observed in [**5**], for each $n \in \mathbb{Z}$ the relations $d_A^{n+1} d_A^n = 0$ and $d_A^n d_A^{n-1} = 0$ imply the existence of a unique morphism $a_A^n : \operatorname{Coker} d_A^{n-1} \to \operatorname{Ker} d_A^{n+1}$ satisfying the condition

$$(\ker d_A^{n+1}) a_A^n (\operatorname{coker} d_A^{n-1}) = d_A^n.$$

Put $H_-^n(\mathfrak{A}) = H_-(A^n, d_A^{n-1}, d_A^n)$ and $H_+^n(\mathfrak{A}) = H_+(A^n, d_A^{n-1}, d_A^n)$. As follows from the previous section,

$$H_-^n(\mathfrak{A}) = \operatorname{coker}(a_A^{n-1} \operatorname{coker} d_A^{n-2}) = \operatorname{coker} a_A^{n-1}$$

and

$$H_+^n(\mathfrak{A}) = \ker((\ker d_A^{n+1}) a_A^n) = \ker a_A^n.$$

We call the homology objects $H_-^n(\mathfrak{A})$ and $H_+^n(\mathfrak{A})$ the *left* and *right nth cohomology objects* of the cochain complex $\mathfrak{A}$.

From now on, let our category be P-semi-abelian.

By a *morphism* of two complexes $\mathfrak{A} = (A^n, d_A^n)_{n\in\mathbb{Z}}$ and $\mathfrak{B} = (B^n, d_B^n)_{n\in\mathbb{Z}}$ we mean a family of morphisms $(\varphi^n : A^n \to B^n)_{n\in\mathbb{Z}}$ such that $\varphi^{n+1} d_A^n = d_B^n \varphi^n$ for all n. For three complexes $\mathfrak{A} = (A^n, d_A^n)_{n\in\mathbb{Z}}$, $\mathfrak{B} = (B^n, d_B^n)_{n\in\mathbb{Z}}$, and $\mathfrak{C} = (C^n, d_C^n)_{n\in\mathbb{Z}}$ and morphisms $\varphi : \mathfrak{A} \to \mathfrak{B}$ and $\psi : \mathfrak{B} \to \mathfrak{C}$, we call the sequence $\mathfrak{A} \xrightarrow{\varphi} \mathfrak{B} \xrightarrow{\psi} \mathfrak{C}$ *strictly exact* if $\varphi^n | \psi^n$ for all n.

A morphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ of complexes induces morphisms $\hat{\varphi}^n : \operatorname{Ker} d_A^n \to \operatorname{Ker} d_B^n$ and $\tilde{\varphi}^n : \operatorname{Coker} d_A^{n-1} \to \operatorname{Coker} d_B^{n-1}$. Like in the quasi-abelian case [**5**], to a strictly exact sequence of complexes

$$0 \to \mathfrak{A} \xrightarrow{\varphi} \mathfrak{B} \xrightarrow{\psi} \mathfrak{C} \to 0 \tag{4.1}$$

in a P-semi-abelian category, there corresponds a commutative diagram

$$
\begin{array}{ccccccc}
 & \operatorname{Coker} d_A^{n-1} & \xrightarrow{\tilde{\varphi}^{n-1}} & \operatorname{Coker} d_B^{n-1} & \xrightarrow{\tilde{\psi}^{n-1}} & \operatorname{Coker} d_C^{n-1} & \longrightarrow 0 \\
 & {\scriptstyle a_A^n}\downarrow & & {\scriptstyle a_B^n}\downarrow & & {\scriptstyle a_C^n}\downarrow & \\
0 \longrightarrow & \operatorname{Ker} d_A^{n+1} & \xrightarrow{\hat{\varphi}^{n+1}} & \operatorname{Ker} d_B^{n+1} & \xrightarrow{\hat{\psi}^{n+1}} & \operatorname{Ker} d_C^{n+1} &
\end{array}
\tag{4.2}
$$

Here $\tilde{\varphi}^{n-1}$ and $\tilde{\psi}^{n-1}$ are natural morphisms of the indicated cokernels, $\hat{\varphi}^{n+1}$ and $\hat{\psi}^{n+1}$ are natural morphisms of the kernels, $\tilde{\psi}^{n-1} = \operatorname{coker} \tilde{\varphi}^{n-1}$, and $\hat{\varphi}^{n+1} = \ker \hat{\psi}^{n+1}$.

THEOREM 1. *If* ψ^n, $\operatorname{coker} d_B^{n-1}$, $\operatorname{coker} d_C^{n-1}$ *are stable cokernels or* φ^{n+1}, $\ker d_A^{n+1}$, $\ker d_B^{n+1}$ *are stable kernels, to sequence* (4.1) *there corresponds a semi-exact homology sequence*

$$
\begin{aligned}
&H_+^n(\mathfrak{A}) \xrightarrow{H_+^n(\varphi)} H_+^n(\mathfrak{B}) \xrightarrow{H_+^n(\psi)} H_+^n(\mathfrak{C}) \\
&\xrightarrow{\Delta^n} H_-^{n+1}(\mathfrak{A}) \xrightarrow{H_-^{n+1}(\varphi)} H_-^{n+1}(\mathfrak{B}) \xrightarrow{H_-^{n+1}(\psi)} H_-^{n+1}(\mathfrak{C}).
\end{aligned}
\tag{4.3}
$$

PROOF. Suppose that ψ^n, $\operatorname{coker} d_B^{n-1}$, $\operatorname{coker} d_C^{n-1}$ are stable cokernels. By Lemma 4(2), $\tilde{\psi}^n \operatorname{coker} d_B^{n-1} = (\operatorname{coker} d_C^{n-1})\psi^n$ is a stable cokernel, and thus, by Lemma 4(1), so is $\tilde{\psi}^n$. This makes it possible to construct a connecting morphism $\Delta^n : H_+^n(\mathfrak{C}) \to H_-^{n+1}(\mathfrak{A})$ for the Ker-Coker-sequence of (4.2) (see [**10**]), which has the form (4.3).

The case of the stable kernels φ^{n+1}, $\ker d_A^{n+1}$, $\ker d_B^{n+1}$ follows by duality. □

REMARK 1. *An important question is the exactness of sequence* (4.3). *Even for quasi-abelian categories, the answer is nontrivial* [**9**]. *In P-semi-abelian categories, the exactness of the* Ker-Coker-*sequence* (4.3) *depends on the stability of some kernels and cokernels in* (4.2) (*cf.* [**10**]). *The author does not know if any reasonable conditions for the exactness of* (4.3) *can be written in terms of the initial complexes and their differentials.*

5. An Eckmann —Hilton homological construction

Introduce the notation $\alpha \| \beta$ if the sequence $\cdot \xrightarrow{\alpha} \cdot \xrightarrow{\beta} \cdot$ is exact, that is, $\operatorname{im} \alpha = \ker \beta$. The following two lemmas were proved in [**7**] for a quasi-abelian category. Their proofs are carried over verbatim to the case of a P-semi-abelian category.

LEMMA 8. *If* $\alpha = \operatorname{coker} \beta$ *and* $\beta \| \rho\alpha$ *then* $\rho \in M$.

LEMMA 9. *If* $\rho\beta \| \alpha$ *and* $\rho \in M_c$ *then* $\beta \| \alpha\rho$.

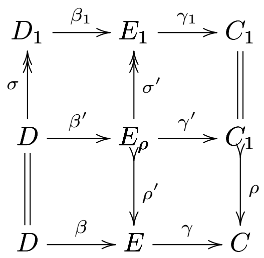
Constructing the spectral sequence of an exact couple in an abelian category, Eckmann and Hilton [**3**] considered the diagram

$$
\begin{array}{ccccc}
D_1 & & & & C_1 \\
{\scriptstyle \sigma}\big\uparrow & & & & \big\downarrow{\scriptstyle \rho} \\
D & \xrightarrow{\beta} & E & \xrightarrow{\gamma} & C,
\end{array}
$$

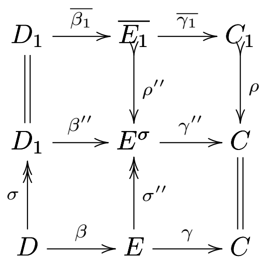
where $\rho \in M_c$ and $\sigma \in P_c$ are factors of $\gamma\beta$, in an additive category with kernels and cokernels.

We use the notations of [**3**]: if $f\alpha = \beta g$ is a pull-back then we write $(\alpha, g) = I(\beta, f)$; if $f\alpha = \beta g$ is a push-out then we use the notation $(\beta, f) = U(\alpha, g)$.

Consider the diagram



Here $(\gamma', \rho') = I(\gamma, \rho)$, $(\beta_1, \sigma') = U(\beta', \sigma)$, and the morphisms β' and γ_1 arise because ρ and σ are factors of $\gamma\beta$. By Lemma 1(4), $\rho' \in M_c$ and $\sigma' \in P_c$. Dually, one has the diagram



where $(\beta'', \sigma'') = U(\beta, \sigma)$ and $(\overline{\gamma_1}, \rho'') = I(\gamma'', \rho)$. Again, $\sigma'' \in P_c$ and $\rho'' \in M_c$. Furthermore, $\beta\gamma = \rho\overline{\gamma_1}\,\overline{\beta_1}\sigma = \rho\gamma_1\beta_1\sigma$, from which $\overline{\gamma_1}\,\overline{\beta_1} = \gamma_1\beta_1$.

Following [**3**], in [**7**, Theorem 1] we proved that there exists a unique canonical morphism $\omega : E_1 \to \overline{E_1}$ such that

(i) $\omega\beta_1 = \overline{\beta_1}$;

(ii) $\gamma_1 = \overline{\gamma_1}\omega$;

(iii) $\rho''\omega\sigma' = \sigma''\rho'$.

We also demonstrated in [**7**] that if the ambient category is quasi-abelian then ω is an isomorphism.

We have

Theorem 2. *If the ambient category is P-semi-abelian then ω is a bimorphism. If, in addition, σ'' is a stable cokernel then ω is an isomorphism.*

Proof. The fact that ω is a bimorphism follows by the corresponding argument (with the use of Lemmas 8 and 9) for quasi-abelian categories in the proof of Theorem 1 in [**7**].

Now, assume that σ'' is a stable cokernel and prove that $\omega \in O_c$.

Since $\sigma'' = \operatorname{coker}(\beta\tau)$ and $\gamma\beta\tau = \gamma''\beta''\sigma\tau = 0$, it follows that $\gamma(\ker\sigma'') = \gamma(\operatorname{im}(\beta\tau)) = 0$. Hence, there exists a unique morphism $\mu : \operatorname{Ker}\sigma'' \to \operatorname{Ker}\gamma$ with

$\ker\sigma'' = (\ker\gamma)\mu$. The fact that $\gamma\rho' = \rho\gamma'$ is a pull-back and Lemma 5 imply that $\ker\gamma = \rho'(\ker\gamma')$. Therefore, $\ker\sigma'' = \rho'(\ker\gamma')\mu$. We have the commutative diagram

$$\begin{array}{ccccc}
\operatorname{Ker}\sigma'' & \xrightarrow{(\ker\gamma')\mu} & E_\rho & \xrightarrow{\operatorname{coker}((\ker\gamma')\mu)} & \operatorname{Coker}((\ker\gamma')\mu) \\
{\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \rho'}\downarrow & & {\scriptstyle \hat{\rho}'}\downarrow \\
\operatorname{Ker}\sigma'' & \xrightarrow{\ker\sigma''} & E & \xrightarrow{\sigma''=\operatorname{coker}(\ker\sigma'')} & E^\sigma
\end{array}$$

Since σ'' is a stable cokernel, we may apply Lemma 6, which yields $\hat{\rho}' \in M_c$. Hence, $\sigma''\rho' = \hat{\rho}' \operatorname{coker}((\ker\gamma')\mu) \in O_c$. Now, the relation $\rho''\omega\sigma' = \sigma''\rho'$ and Lemma 2 imply the strictness of ω.

Thus, ω is an isomorphism in this case.

Theorem 2 is proved. □

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Sobolev Institute of Mathematics,
Akademik Koptyug Pr. 4,
630090, Novosibirsk, Russia
E-mail address: yakop@math.nsc.ru