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Some applications of the subtitution in graph

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Abstract. In 1949 R. Frucht established that any finite group can be represented by a graph of degree three in the sense that automorphism group of the graph is isomorphic to the given group. In 1957 G. Sabidussi gave another step in this direction, proving that every finite group can be represented by an infinite number of regular graphs. The main result of this work is to give a proof of the Sabidussi's theorem by operation of the graph called substitution, that consists of replacing any vertex by a graph.

1.- Introduction

The graph to considered will be in general simple and finite, graphs with a nonempty set of edges. For a graph G, V(G) denote the set of vertices and E(G) denote the set of edges. The cardinality of V(G) is called order of G and the cardinality of E(G) is called size of G. Two vertices u and v are called neighbors if {u,v} is an edge of G. For any vertex v of G, denote by N_v the set neighbors of v and by deg v the degree of v. To simplify the notation, and edge {x,y} is written as xy (or yx). Other concepts used in this work and not defined explicitly can be found in the references [1], [2], [3], [5], [9], [12], [13].

2.- Preliminaries

2.1. The Substitution [10],[11]: It assumes that G and K are two disjointed graphs by vertices and let N_v the neighbors of v and the function $s : N_v \to V(K)$. Then for a not isolated vertex v in $V(G)$ [3], the function s define the substitution the vertex v by the graph K. We denote the new graph as $M = G(v,s)K$, such that:

(1) $V(M) = (V(G) \cup V(K)) - \{v\}$ and
(2) $E(M) = (E(G) - \{vx/x \in N_v\}) \cup \{xs(x)/x \in N_v\}$.

The vertex v is said to be the substitution vertex by K in G under the function s and this function is called *substitution function*. Figure 1 shows an example of substitution.

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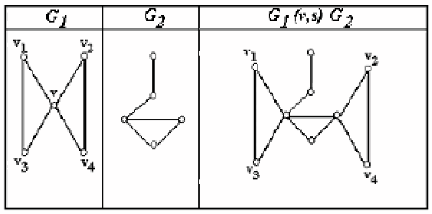

Figure 1.-

If v is an isolated vertex, then $M = (G - v) \cup K$ [3]

Let $v_1, ..., v_n$ be the vertices of a graph G and $H_1, ..., H_n$ a sequence of graphs with no common vertices among themselves or with G. By $M_k = M_{k-1}(v_k, s_k)H_k$ it will be denoted the graph which is obtained by substitution of vertices of G by graphs H_i, $1 \leqslant i \leqslant k$, where $M_0 = G$. In other words, M_1 denotes a graph obtained by substitution of only one vertex of G, M_2 denotes a graph obtained by substitution of only one vertex of M_1, and so on. Note that every substitutes vertex must belong to $V(G)$. Figure 2 shows an example of M_6.

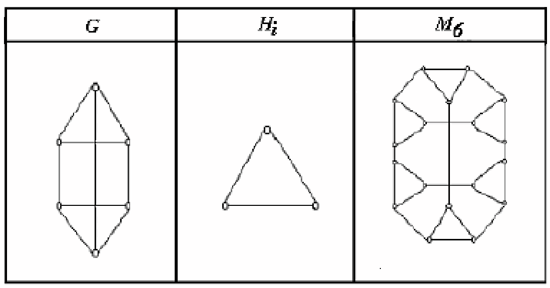

Figure 2.-

It can be said that an edge of the substitution M_p is an *edge internal*[10] if it is denoted by $s_i(x)s_i(y)$. The edge in M_p that is not edge internal will be nominated *edge external* [11]. Let G be a graph without isolated vertices. If each vertex v of G is substituted by a complete graph with $val(v)$ vertices, through an injective function, then it will be said that the graph G has been *expanded*[17] an is denoted by $K(G)$.

Remark; The type of graph constructed by Sabidussi [18] and Frucht [6] is isomorphic with the expanded graph constructed by substitution.

3.-Theorems

Previous to the following theorem it is necessary to observe that not every finite group may be represented by r-regular graphs with $r \in \{0, 1, 2\}$. For example, a cyclical group of order five can not be represented by a 0 or 1 regular graph, or the symmetric group with the order 120 can not be represented by a 2-regular graph.

In 1969, Gewirtz, Hill and Quintas [7] developed a paper that guarantees the existence of asymmetric graphs (i.e. with trivial automorphism group) that are r-regular, for each natural $r \geqslant 3$. In the next proof this graphs will be denoted by $Q(r)$. Moreover, it will be denoted by $e(r) = x(1)x(2)$ an arbitrary but fixed edge in $Q(r)$ and by $\widetilde{e}(r) = \widetilde{x}(1)\widetilde{x}(2)$ an arbitrary but fixed edge in the expanded graph $K(Q(r))$. See Figure 3 for $r = 3$.

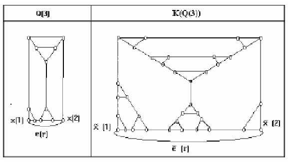

Figure 3.-

THEOREM 3.1: Every finite group is representable by a r-regular graph, $r \geqslant 3$.
Proof. Assume that $r \geqslant 3$ is an integer and H a finite group with elements $\{v_1, ...v_p\}$ and identity v_1. Then let $C = \{h_1, ..., h_m\}$ be a generator set of H that does not contain v_1.Three cases can be distinguish : Case (1) : $|H| = 1$.(see [5]). Case (2) : $|H| > 1$.

Beginning with the Cayley Diagram $D_C(H)$ [16] it will be build a graph S and a graph $F(r)$ in the following form : the vertices of S are the vertices of $D_C(H)$ and each $arc(v_i, v_j)$, of color h_k [16], is replaced by a chain $v_i, v_{i,k,1,j}, ..., v_{i,k,k+1,j}, v_j$ of length $k + 2$. Subsequently, each vertex $v_{i,k,n,j}$, $1 \leqslant n \leqslant k$, of S is substituted by a copy of the graph $Q(r) - e(r)$ and each vertex $v_{i,k,k+1,j}$ by a copy of $Q(r) - e(r)$. See Figure 4.

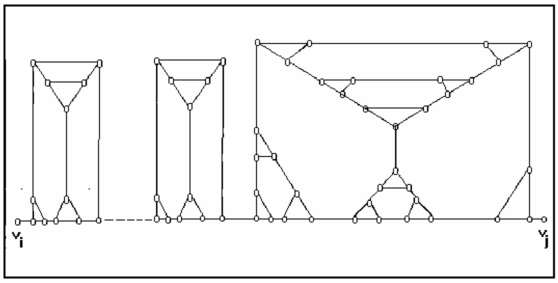

Figure 4.-

The substitution functions used are injective functions which choose the vertex of degree $r-1$ in the copies of $Q(r)-e(r)$ or $K(Q(r))-\widetilde{e}(r)$. By these substitutions, the obtained graph should satisfy the following properties:

(1) the vertices similar to x(1) are not adjacent among them(in the graph $F(r)$)

(2) v_i is adjacent with a vertex similar to $x(1)$ and

(3) v_j is adjacent with a vertex similar to $\widetilde{x}(2)$.

(similar vertices x, y are extremes vertices of the isomorphic chains [8]).

The vertex adjacent to v_i will be labeled as $v(i, x(1), k)$, and the vertex adjacent to v_j as $v(i, \widetilde{x}(2), k)$, (these are obtained in the substitution of the respectively vertices $v_{i,k,1,j}$ and $v_{i,k,k+1,j}$). These vertices will be used in the incidental substitution of the vertices of ramification (vertices of S in which the substitution was not effected) that will be made later.

Since the graphs $Q(r)-e(r)$ and $K(Q(r))-\widetilde{e}(r)$ are asymmetric, the substitution of the internal vertices of the chains $v_i - v_j$ simulate the color and the direction of the arcs of the Cayley diagram $D_C(H)$. This guarantees us that $Aut(F(r)) \cong H$, $F(r)$is connected and the ramification vertices are similar between them.

If ramification vertices of $F(r)$ already have degree r, then we have a r-regular graph and, as we see ahead, it satisfies the conditions of the theorem

Otherwise, it is necessary to substitute each vertex of ramification $v_i, 1 \leqslant i \leqslant p$, by a $2m$-cycle $C(i, 2m)$. This substitution will be effected by injective functions s_i such that $s_i : N_{v_i} \to V(C(i, 2m))$.

The functions s_i are not arbitrary. The consecutive vertices are labeled in the following form : $v(i,1,1), v(i,1,2), v(i,2,3), v(i,2,4), ..., v(i,m,2m-1), v(i,m,2m)$. Beginning with the graph $F(r)$, it's obtained a graph Y defining the substitutions so that the vertex $v(i,k,2k-1)$ is adjacent with $v(i,x(1),k)$ and the vertex $v(i,k,2k)$ is adjacent with the vertex $v(i^*(k), x(2), k)$, where $i^*(k)$ is the index j such that $v_i = v_j h_k$. It's had that $Aut(Y) \cong Aut(F(r)) \cong H$. Moreover Y is connected[13].

However, the vertices of each subgraph $C(i, 2m)$ have degree 3 in Y, then if $r > 3$ it has to do a new modification. That modification consists in uniting the pairs of adjacent vertices $v(i,k,2k-1)$ and $v(i,k,2k)$ by means of $r-3$ chains internally disjoint and of length 2. It's noted that now all the vertices have degree r except the internals (of degree 2) of the last introduced chains. For complete the r-regularity it's necessary substituted each internal vertex of each one of the $(r-3)$ chains by a graph $Q(i,k,t) \cong Q(r) - e(t), 1 \leqslant t \leqslant 3$, where $e(t) = x(i,k,2t-1)x(i,k,2t)$ is an edge of $Q(r)$ determined by the following adjacency :

(1) $v(i,k,2k-1)$ adjoining which $x(i,k,2t-1)$ and

(2) $v(i,k,2k)$ adjoining which $x(i,k,2t)$.

Denote by G the resulting graph and observed that this comply the before assigned properties [12], [14].

Case (3) : $|C| = 1$.

Assumed that H has order 2 (if $|H| > 2$ choused a generator set C so that $|C| > 1$) and build a graph according to the method exposed in the previous case.

Using the terminology of Theorem 3,1 the following theorems (Sabidussi type) can be proven.

Theorems 3.2: Given a finite group H of order > 1 and a whole number n infinite graphs G non homeomorphism exist among them and that they represent H and such that G is:

(1) connected,
(2) without vertices neither fixed edges,
(3) prime (with regard to the cartesian product),
(4) of nuclear number n,
(5) of chromatic number n, and
(6) of chromatic index $n+1$.

Theorem 3.3: For every finite group H of order > 1 and for every whole number $r \geqslant 3$, infinite graphs G non homeomorphism exist among them and that they represent H and such that G is:

(1) connected,
(2) without vertices neither fixed edges,
(3) prime (with regard to the cartesian product),
(4) r-regular, and
(5) 2- connectivity.

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