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The integrals in Gradshteyn and Ryzhik. Part 6: The beta function

Victor H. Moll

Abstract. We present a systematic derivation of some definite integrals in the classical table of Gradshteyn and Ryzhik that can be reduced to the beta function.

1. Introduction

The table of integrals [**2**] contains some evaluations that can be derived by elementary means from the *beta function*, defined by

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx. \tag{1.1}$$

The convergence of the integral in (1.1) requires $a,\, b > 0$. This definition appears as **3.191.3** in [**2**].

Our goal is to present in a systematic manner, the evaluations appearing in the classical table of Gradshteyn and Ryzhik [**2**], that involve this function. In this part, we restrict to algebraic integrands leaving the trigonometric forms for a future publication. This paper complements [**3**] that dealt with the *gamma function* defined by

$$\Gamma(a) := \int_0^\infty x^{a-1}e^{-x}\,dx. \tag{1.2}$$

These functions are related by the functional equation

$$B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}. \tag{1.3}$$

A proof of this identity can be found in [**1**].

The special values $\Gamma(n) = (n-1)!$ and

$$\Gamma\left(n+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n}}\,\frac{(2n)!}{n!} \tag{1.4}$$

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for $n \in \mathbb{N}$, will be used to simplify the values of the integrals presented here. Proofs of these formulas can be found in [**3**] as well as in Proposition 2.1 below.

The other property that will be employed frequently is

$$\Gamma(a)\,\Gamma(1-a) = \frac{\pi}{\sin \pi a}. \tag{1.5}$$

The reader will find in [**1**] a proof based on the product representation of these functions. A challenging problem is to produce a proof that only employs changes of variables.

The table [**2**] contains some direct values:

$$\int_0^1 \frac{x^p\,dx}{(1-x)^p} = \frac{p\pi}{\sin p\pi} \tag{1.6}$$

is **3.192.1** and is evaluated by identifying it as $B(p+1, 1-p)$. Formula **3.192.2** is

$$\int_0^1 \frac{x^p\,dx}{(1-x)^{p+1}} = -\frac{\pi}{\sin p\pi} \tag{1.7}$$

has the value $B(p+1,-p) = \Gamma(p+1)\Gamma(-p)$. Next, **3.192.3** is

$$\int_0^1 \frac{(1-x)^p}{x^{p+1}}\,dx = -\frac{\pi}{\sin p\pi} \tag{1.8}$$

and the change of variables $t = 1/x$ in **3.192.4** produces

$$\int_1^\infty (x-1)^{p-1/2}\frac{dx}{x} = \int_0^1 t^{-p-1/2}(1-t)^{p-1/2}\,dt \tag{1.9}$$

and this is

$$B\left(\tfrac{1}{2}-p, \tfrac{1}{2}+p\right) = \Gamma\left(\tfrac{1}{2}-p\right)\Gamma\left(\tfrac{1}{2}+p\right) = \frac{\pi}{\cos p\pi}, \tag{1.10}$$

as stated in [**2**].

Let $b = \frac{1}{2}$ in (1.1) to obtain

$$\int_0^1 \frac{x^{a-1}\,dx}{\sqrt{1-x}} = B\left(a, \tfrac{1}{2}\right) = \frac{\Gamma(a)\sqrt{\pi}}{\Gamma\left(a+\tfrac{1}{2}\right)}. \tag{1.11}$$

The special values $a = n+1$ and $a = n+\frac{1}{2}$ appear as **3.226.1** and **3.226.2**, respectively.

2. Elementary properties

Many of the properties of the beta function can be established by simple changes of variables. For example, letting $y = 1-x$ in (1.1) yields the symmetry

$$B(a,b) = B(b,a). \tag{2.1}$$

It should not be surprising that a clever change of variables might lead to a beautiful result. This is illustrated following Serret [**4**]. Start with

$$\begin{aligned} B(a,a) &= \int_0^1 (x-x^2)^{a-1}\,dx \\ &= 2\int_0^{1/2} \left[\tfrac{1}{4} - \left(\tfrac{1}{2}-x\right)^2\right]^{a-1}\,dx. \end{aligned}$$

The natural change of variables $v = \frac{1}{2} - x$ yields

$$B(a,a) = 2\int_0^{1/2} \left(\tfrac{1}{4} - v^2\right)^{a-1}\,dv. \tag{2.2}$$

The next step is now clear: let $s = 4v^2$ to produce

$$B(a,a) = 2^{1-2a} B\left(a, \tfrac{1}{2}\right). \tag{2.3}$$

The functional equation (1.3) converts this identity into Legendre's original form:

Proposition 2.1. The gamma function satisfies

$$\Gamma\left(a+\tfrac{1}{2}\right) = \frac{\Gamma(2a)\,\Gamma(\frac{1}{2})}{\Gamma(a)\,2^{2a-1}}. \tag{2.4}$$

In particular, for $a = n \in \mathbb{N}$, this yields (1.4).

3. Elementary changes of variables

The integral (1.1) defining the beta function can be transformed by changes of variables. For example, the new variable $x = t/u$, reduces (1.1) to

$$\int_0^u t^{a-1}(u-t)^{b-1}\,dt = u^{a+b-1}B(a,b), \tag{3.1}$$

that appears as **3.191.1** in [**2**]. The effect of this change of variables is to express the beta function as an integral over a finite interval. Observe that the integrand vanishes at both end points. Similarly, the change $t = (v-u)x + u$ maps the interval $[0,1]$ to $[u,v]$. It yields

$$\int_u^v (t-u)^{a-1}(v-t)^{b-1}\,dt = (v-u)^{a+b-1}B(a,b). \tag{3.2}$$

This is **3.196.3** in [**2**]. The special case $u = 0$, $v = n$ and $a = \nu$, $b = n+1$ appears as **3.193** in [**2**] as

$$\int_0^n x^{\nu-1}(n-x)^n\,dx = \frac{n^{\nu+n}\,n!}{\nu(\nu+1)(\nu+2)\cdots(\nu+n)}. \tag{3.3}$$

Several integrals in [**2**] can be obtained by a small variation of the definition. For example, the integral

$$\int_0^1 (1-x^a)^{b-1}\,dx = \frac{1}{a}B\left(1/a, b\right) \tag{3.4}$$

can be obtained by the change of variables $t = x^a$. This appears as **3.249.7** in [**2**] and illustrates the fact that it not necessary for the integrand to vanish at *both* end points. The special case $a = 2$ appears as **3.249.5**:

$$\int_0^1 (1-x^2)^{b-1}\,dx = \tfrac{1}{2}B\left(\tfrac{1}{2}, b\right) = 2^{2b-2}B(b,b), \tag{3.5}$$

where the second identity follows from Legendre's duplication formula (2.4).

The change of variables $t = cx$ produces a scaled version:

$$\int_0^c (c^a - t^a)^{b-1}\,dt = \frac{1}{a}c^{a(b-1)+1}B\left(1/a, b\right). \tag{3.6}$$

The special case $a = 2$ yields

$$\int_0^c (c^2 - t^2)^{b-1}\,dt = \frac{c^{2b-1}}{2}B\left(1/2, b\right). \tag{3.7}$$

The choice $b = n + \frac{1}{2}$ appears as **3.249.2** in [**2**]:

$$\int_0^c (c^2 - t^2)^{n-1/2}\,dt = \frac{\pi c^{2n}}{2^{2n+1}}\binom{2n}{n}. \tag{3.8}$$

Similarly **3.251.1** in [**2**] is

$$\int_0^1 x^{c-1}(1-x^a)^{b-1}\,dx = \frac{1}{a}B\left(\frac{c}{a}, b\right). \tag{3.9}$$

The change of variables $t = 1/x$ converts (1.1) into

$$\int_1^\infty t^{-a-b}(t-1)^{b-1}\,dt = B(a,b). \tag{3.10}$$

Letting $t = x^p$ yields

$$\int_1^\infty x^{p(1-a-b)-1}\left(x^p - 1\right)^{b-1}\,dx = \frac{1}{p}B(a,b). \tag{3.11}$$

The special case $\nu = b$ and $\mu = p(1-a-b)$ is **3.251.3**:

$$\int_1^\infty x^{\mu-1}\left(x^p - 1\right)^{\nu-1}\,dx = \frac{1}{p}B\left(1-\nu-\mu/p, \nu\right). \tag{3.12}$$

4. Integrals over a half-line

The beta function can also be expressed as an integral over a half-line. The change of variables $t = x/(1-x)$ maps $[0,1]$ onto $[0,\infty)$ and it produces from (1.1)

$$B(a,b) = \int_0^\infty \frac{t^{a-1}\,dt}{(1+t)^{a+b}}. \tag{4.1}$$

In particular, if $a + b = 1$, using (1.3) and (1.5), we obtain

$$\int_0^\infty \frac{t^{a-1}\,dt}{1+t} = \frac{\pi}{\sin \pi a}. \tag{4.2}$$

This can be scaled to produce, for $a > 0$ and $c > 0$,

$$\int_0^\infty \frac{x^{a-1}\,dx}{x+c} = \frac{\pi}{\sin \pi a} c^{a-1} \quad \text{for } c > 0 \tag{4.3}$$

that appears as **3.222.2** in [**2**]. In the case $c < 0$ we have a singular integral. Define $b = -c > 0$ and $s = x/b$, so now we have to evaluate

$$I = -b^{a-1} \int_0^\infty \frac{s^{a-1}\,ds}{1-s}. \tag{4.4}$$

The integral is considered as a Cauchy principal value

$$I = \lim_{\epsilon \to 0} \int_0^1 \frac{s^{a-1}\,ds}{(1-s)^{1-\epsilon}} + \int_1^\infty \frac{s^{a-1}\,ds}{(1-s)^{1-\epsilon}}. \tag{4.5}$$

Let $y = 1/s$ in the second integral and evaluate them in terms of the beta function to produce

$$I = \lim_{\epsilon \to 0} \epsilon \Gamma(\epsilon) \times \frac{1}{\epsilon} \left(\frac{\Gamma(a)}{\Gamma(a+\epsilon)} - \frac{\Gamma(1-a-\epsilon)}{\Gamma(1-a)} \right). \tag{4.6}$$

Use L'Hopital's rule to evaluate and obtain

$$I = -\frac{\Gamma'(a)}{\Gamma(a)} + \frac{\Gamma'(1-a)}{\Gamma(a)}. \tag{4.7}$$

Using the relation $\Gamma(a)\Gamma(1-a) = \pi \text{cosec } \pi a$, this reduces to $\pi \cot \pi a$. Therefore we have

$$\int_0^\infty \frac{x^{a-1}\,dx}{x+c} = -\frac{\pi}{\tan \pi a} (-c)^{a-1} \quad \text{for } c < 0 \tag{4.8}$$

The change of variables $x = e^{-t}$ produces, for $c < 0$,

$$\int_{-\infty}^\infty \frac{e^{-\mu t}\,dt}{e^{-t}+c} = -\pi \cot(\mu\pi)\,(-c)^{\mu-1}. \tag{4.9}$$

The special case $c = -1$ appears as **3.313.1**:

$$\int_{-\infty}^\infty \frac{e^{-\mu t}\,dt}{1-e^{-t}} = \pi \cot(\mu\pi). \tag{4.10}$$

We now consider several examples in [**2**] that are direct consequences of (4.3) and (4.8). In the first example, we combine (4.3) with the partial fraction decomposition

$$\frac{1}{(x+a)(x+b)} = \frac{1}{b-a} \left(\frac{1}{x+a} - \frac{1}{x+b} \right) \tag{4.11}$$

leads to **3.223.1**:

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{(x+b)(x+a)} = \frac{\pi}{b-a} (a^{\mu-1} - b^{\mu-1}) \text{cosec}(\pi\mu). \tag{4.12}$$

Similarly,

$$\frac{1}{x+b} - \frac{1}{x-a} = \frac{a+b}{(a-x)(b+x)} \tag{4.13}$$

leads to **3.223.2**:

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{(b+x)(a-x)} = \frac{\pi}{a+b}\left(b^{\mu-1}\,\text{cosec}(\mu\pi) + a^{\mu-1}\cot(\mu\pi)\right), \tag{4.14}$$

using (4.3) and (4.8). The result **3.223.3**:

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{(a-x)(b-x)} = \pi\cot(\mu\pi)\frac{a^{\mu-1}-b^{\mu-1}}{b-a}, \tag{4.15}$$

follows from

$$\frac{1}{(a-x)(b-x)} = \frac{1}{a-b}\left(\frac{1}{b-x} - \frac{1}{a-x}\right). \tag{4.16}$$

Finally, **3.224**:

$$\int_0^\infty \frac{(x+b)x^{\mu-1}\,dx}{(x+a)(x+c)} = \frac{\pi}{\sin(\mu\pi)}\left(\frac{a-b}{a-c}a^{\mu-1} + \frac{c-b}{c-a}c^{\mu-1}\right), \tag{4.17}$$

follows from

$$\frac{x+b}{(x+a)(x+c)} = \frac{b-a}{c-a}\frac{1}{x+a} - \frac{b-c}{c-a}\frac{1}{x+c}. \tag{4.18}$$

We can now transform (4.1) to the interval $[0,1]$ by splitting $[0,\infty)$ as $[0,1]$ followed by $[1,\infty)$. In the second integral, we let $t = 1/s$. The final result is

$$B(a,b) = \int_0^1 \frac{t^{a-1}+t^{b-1}}{(1+t)^{a+b}}\,dt. \tag{4.19}$$

This formula, that appears as **3.216.1**, makes it apparent that the beta function is symmetric: $B(a,b) = B(b,a)$. The change of variables $s = 1/t$ converts (4.19) into **3.216.2**:

$$B(a,b) = \int_1^\infty \frac{s^{a-1}+s^{b-1}}{(1+s)^{a+b}}\,ds. \tag{4.20}$$

It is easy to introduce a parameter: let $c > 0$ and consider the change of variables $t = cx$ in (4.1) to obtain

$$\int_0^\infty \frac{x^{a-1}\,dx}{(1+cx)^{a+b}} = c^{-a}B(a,b), \tag{4.21}$$

that appears as **3.194.3**. We can now shift the lower limit of integration via $t = x+u$ to produce

$$\int_u^\infty (t-u)^{a-1}(t+v)^{-a-b}\,dt = (u+v)^{-b}B(a,b), \tag{4.22}$$

where $v = 1/c - u$. This is **3.196.2**, where v is denoted by β. Now let $b = c-a$ in the special case $v = 0$ to obtain

$$\int_u^\infty (t-u)^{a-1}t^{-c}\,dt = u^{a-c}B(a,c-a). \tag{4.23}$$

This appears as **3.191.2**.

We now write (4.1) using the change of variables $t = x^c$. It produces

$$\int_0^\infty \frac{x^{ac-1}\,dx}{(1+x^c)^{a+b}} = \frac{1}{c}B(a,b). \tag{4.24}$$

The special case $c = 2$ and $a = 1 + \mu/2$, $b = 1 - \mu/2$ produces **3.251.6** in the form

$$\int_0^\infty \frac{x^{\mu+1}\,dx}{(1+x^2)^2} = \frac{\mu\pi}{4\,\sin\mu\pi/2}. \tag{4.25}$$

Now let $b = 1 - a$ and choose $a = p/c$ to obtain

$$\int_0^\infty \frac{x^{p-1}\,dx}{1+x^c} = \frac{1}{c}B\left(\frac{p}{c}, \frac{c-p}{c}\right) = \frac{\pi}{c}\ \mathrm{cosec}(\pi p/c). \tag{4.26}$$

This appears as **3.241.2** in [**2**].

Similar arguments establish **3.196.4**:

$$\int_1^\infty \frac{dx}{(a-bx)(x-1)^\nu} = -\frac{\pi}{b}\ \mathrm{cosec}(\nu\pi)\left(\frac{b}{b-a}\right)^\nu. \tag{4.27}$$

Indeed, the change of variables $t = x - 1$ yields

$$\int_1^\infty \frac{dx}{(a-bx)(x-1)^\nu} = \int_0^\infty \frac{dt}{[(a-b)-bt]\ t^\nu}, \tag{4.28}$$

and scaling via the new variable $z = bt/(b-a)$ gives

$$\int_1^\infty \frac{dx}{(a-bx)(x-1)^\nu} = -\frac{1}{b}\left(\frac{b}{b-a}\right)^\nu \int_0^\infty \frac{dz}{(1+z)\,z^\nu}. \tag{4.29}$$

The result follows from (4.1) and the value

$$B(\nu, 1-\nu) = \Gamma(\nu)\Gamma(1-\nu) = \frac{\pi}{\sin\pi\nu}. \tag{4.30}$$

The same argument gives **3.196.5**:

$$\int_{-\infty}^1 \frac{dx}{(a-bx)(1-x)^\nu} = \frac{\pi}{b}\ \mathrm{cosec}(\nu\pi)\left(\frac{b}{a-b}\right)^\nu. \tag{4.31}$$

5. Some direct evaluations

There are many more integrals in [**2**] that can be evaluated in terms of the beta function. For example, **3.221.1** states that

$$\int_a^\infty \frac{(x-a)^{p-1}\,dx}{x-b} = \pi(a-b)^{p-1}\ \mathrm{cosec}\ \pi p. \tag{5.1}$$

To establish these identities, we assume that $a > b$ to avoid the singularities. The change of variables $t = (x-a)/(a-b)$ yields

$$\int_a^\infty \frac{(x-a)^{p-1}\,dx}{x-b} = (a-b)^{p-1}\int_0^\infty \frac{t^{p-1}\,dt}{1+t}, \tag{5.2}$$

and this integral appears in (4.2).

Similarly, **3.221.2** states that

$$\int_{-\infty}^{a} \frac{(a-x)^{p-1}\,dx}{x-b} = -\pi(b-a)^{p-1} \text{ cosec } \pi p. \tag{5.3}$$

This is evaluated by the change of variables $y = (a-x)/(b-a)$.

The table contains several evaluations that are elementary corollaries of (4.1). Starting with

$$\int_0^\infty \frac{x^a\,dx}{(1+x)^b} = B(a+1, b-a-1) = \frac{\Gamma(a+1)\,\Gamma(b-a-1)}{\Gamma(b)}, \tag{5.4}$$

we find the case $a = p$ and $b = 3$ in **3.225.3**:

$$\int_0^\infty \frac{x^p\,dx}{(1+x)^3} = \frac{\Gamma(p+1)\,\Gamma(2-p)}{\Gamma(3)} = \frac{p(1-p)}{2}\frac{\pi}{\sin(p\pi)}, \tag{5.5}$$

using elementary properties of the gamma function.

The change of variables $t = 1+x$ converts (5.4) into

$$\int_1^\infty \frac{(t-1)^a\,dt}{t^b} = B(a+1, b-a-1) = \frac{\Gamma(a+1)\,\Gamma(b-a-1)}{\Gamma(b)}. \tag{5.6}$$

The special case $a = p-1$ and $b = 2$ gives

$$\int_1^\infty \frac{(t-1)^{p-1}\,dt}{t^2} = \Gamma(p)\Gamma(2-p) = (1-p)\Gamma(p)\Gamma(1-p) = \frac{\pi(1-p)}{\sin(p\pi)}. \tag{5.7}$$

This appears as **3.225.1**. Similarly, the case $a = 1-p$ and $b = 3$ produces **3.225.2**:

$$\int_1^\infty \frac{(t-1)^{1-p}\,dt}{t^3} = \frac{\Gamma(2-p)\Gamma(1+p)}{\Gamma(3)} = \frac{1}{2}p(1-p)\Gamma(p)\Gamma(1-p) = \frac{\pi\,p(1-p)}{2\sin(p\pi)}. \tag{5.8}$$

6. Introducing parameters

It is often convenient to introduce free parameters in a definite integral. Starting with (4.1), the change of variables $t = \frac{u}{v}x^c$ yields

$$B(a,b) = cu^a v^b \int_0^\infty \frac{t^{ac-1}\,dt}{(v+ut^c)^{a+b}}. \tag{6.1}$$

This formula appears as **3.241.4** in [**2**] with the parameters

$$a = \frac{\mu}{\nu},\; b = n+1-\frac{\mu}{\nu},\; c = \nu,\; u = q, \text{ and } v = p, \tag{6.2}$$

in the form

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{(p+qx^\nu)^{n+1}} = \frac{1}{\nu\,p^{n+1}}\left(\frac{p}{q}\right)^{\mu/\nu} \frac{\Gamma(\mu/\nu)\,\Gamma(n+1-\mu/\nu)}{\Gamma(n+1)}.$$

This is a messy notation and it leaves the wrong impression that n should be an integer.

• The special case $v = c = 1$ and $b = p + 1 - a$ produces

$$\int_0^\infty \frac{t^{a-1}\,dt}{(1+ut)^{p+1}} = \frac{1}{u^a} B(a, p+1-a). \tag{6.3}$$

This appears as **3.194.4** in [**2**], except that it is written in terms of binomial coefficients as

$$\int_0^\infty \frac{t^{a-1}\,dt}{(1+ut)^{p+1}} = (-1)^p \frac{\pi}{u^a} \binom{a-1}{p} \operatorname{cosec}(\pi a). \tag{6.4}$$

We prefer the notation in (6.3).

• The special case $v = c = 1$ and $b = 2 - a$ produces

$$\int_0^\infty \frac{t^{a-1}\,dt}{(1+ut)^2} = \frac{1}{u^a} B(a, 2-a). \tag{6.5}$$

Using (1.3) and (1.5) yields the form

$$\int_0^\infty \frac{t^{a-1}\,dt}{(1+ut)^2} = \frac{(1-a)\,\pi}{u^a\,\sin \pi a}. \tag{6.6}$$

This appears as **3.194.6** in [**2**].

• The special case $u = v = 1$ and $c = q$, and choosing $a = p/q$ and $b = 2 - p/q$ yields **3.241.5** in the form

$$\int_0^\infty \frac{x^{p-1}\,dx}{(1+x^q)^2} = \frac{q-p}{q^2} \frac{\pi}{\sin(\pi p/q)}. \tag{6.7}$$

• The special case $c = 1$ and $a = m+1$, $b = n - m - \frac{1}{2}$ produces

$$\int_0^\infty \frac{t^m\,dt}{(v+ut)^{n+\frac{1}{2}}} = \frac{1}{u^{m+1}\,v^{n-m-\frac{1}{2}}} B\left(m+1, n-m-\tfrac{1}{2}\right) \tag{6.8}$$

Using (1.3) and (1.4) this reduces to

$$\int_0^\infty \frac{t^m\,dt}{(v+ut)^{n+\frac{1}{2}}} = \frac{m!\,n!\,(2n-2m-2)!}{(n-m-1)!\,(2n)!} 2^{2m+2} \frac{v^{m-n+1/2}}{u^{m+1}}, \tag{6.9}$$

for m, $n \in \mathbb{N}$, with $n > m$. This appears as **3.194.7** in [**2**].

• The special case $u = v = 1$ and $b = \frac{1}{2} - a$ yields

$$\int_0^\infty \frac{t^{ac-1}\,dt}{\sqrt{1+t^c}} = \frac{1}{c} B\left(a, \tfrac{1}{2} - a\right). \tag{6.10}$$

Writing $a = p/c$ we recover **3.248.1**:

$$\int_0^\infty \frac{t^{p-1}\,dt}{\sqrt{1+t^c}} = \frac{1}{c} B\left(\tfrac{p}{c}, \tfrac{1}{2} - \tfrac{p}{c}\right). \tag{6.11}$$

• Now replace v by v^2 in (6.1). Then, with $u=1$, $a=\frac{1}{2}$, $c=2$, so that $ac=1$ and $b=n-\frac{1}{2}$ we obtain

$$\int_0^\infty \frac{dt}{(v^2+t^2)^n} = \frac{1}{2v^{2n-1}} B\left(\tfrac{1}{2}, n-\tfrac{1}{2}\right). \tag{6.12}$$

This can be written as

$$\int_0^\infty \frac{dt}{(v^2+t^2)^n} = \frac{\sqrt{\pi}\,\Gamma(n-1/2)}{2\Gamma(n)v^{2n-1}} \tag{6.13}$$

that appears as **3.249.1** in [**2**].

• The special case $v=1$, $c=2$ and $b=\frac{n}{2}-a$ in (6.1) yields

$$\int_0^\infty \frac{t^{2a-1}\,dt}{(1+ut^2)^{n/2}} = \frac{1}{2u^a} B\left(a, \tfrac{n}{2}-a\right). \tag{6.14}$$

Now $a=1/2$ gives

$$\int_0^\infty (1+ut^2)^{-n/2}\,dt = \frac{1}{2\sqrt{u}} B\left(\tfrac{1}{2}, \tfrac{n-1}{2}\right) = \frac{\sqrt{\pi}}{2\sqrt{u}} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(n/2)}. \tag{6.15}$$

It is curious that the table [**2**] contains **3.249.8** as the special case $u=1/(n-1)$ of this evaluation.

• We now put $u=v=1$ and $c=2$ in (6.1). Then, with $b=1-\nu-a$ and $a=\mu/2$, we obtain **3.251.2**:

$$\int_0^\infty \frac{t^{\mu-1}\,dt}{(1+t^2)^{1-\nu}} = \frac{1}{2} B\left(\frac{\mu}{2}, 1-\nu-\frac{\mu}{2}\right). \tag{6.16}$$

• We now consider the case $c=2$ in (6.1):

$$\int_0^\infty \frac{t^{2a-1}\,dt}{(v+ut^2)^{a+b}} = \frac{1}{2u^a v^b} B(a,b). \tag{6.17}$$

The special case $a=m+\frac{1}{2}$ and $b=n-m+\frac{1}{2}$ yields

$$\int_0^\infty \frac{t^{2m}\,dt}{(v+ut^2)^{n+1}} = \frac{\Gamma(m+1/2)\,\Gamma(n-m+1/2)}{2u^{m+1/2}v^{n-m+1/2}\Gamma(n+1)}, \tag{6.18}$$

and using (1.4) we obtain **3.251.4**:

$$\int_0^\infty \frac{t^{2m}\,dt}{(v+ut^2)^{n+1}} = \frac{\pi(2m)!(2n-2m)!}{2^{2n+1}m!(n-m)!n!\,u^{m+1/2}v^{n-m+1/2}}, \tag{6.19}$$

for $n,m\in\mathbb{N}$ with $n>m$.

On the other hand, if we choose $a=m+1$ and $b=n-m$ we obtain **3.251.5**:

$$\int_0^\infty \frac{t^{2m+1}\,dt}{(v+ut^2)^{n+1}} = \frac{\Gamma(m+1)\,\Gamma(n-m)}{2u^{m+1}v^{n-m}\Gamma(n+1)} = \frac{m!(n-m-1)!}{2n!u^{m+1}v^{n-m}}. \tag{6.20}$$

Several evaluation in [**2**] come from the form

$$\int_0^1 t^{aq-1}(1-t^q)^{b-1}\,dt = \frac{1}{q}B(a,b), \tag{6.21}$$

obtained from (1.1) by the change of variables $x = t^q$.

- The choice $a = 1 + p/q$ and $b = 1 - p/q$ produces

$$\int_0^1 t^{p+q-1}(1-t^q)^{-p/q}\,dt = \frac{1}{q}B\left(1+\frac{p}{q}, 1-\frac{p}{q}\right) = \frac{p\pi}{q^2}\,\text{cosec}\left(\frac{p\pi}{q}\right). \tag{6.22}$$

This appears as **3.251.8**.

- The choice $a = 1/p$ and $b = 1 - 1/p$ gives

$$\int_0^1 x^{q/p-1}(1-x^q)^{-1/p}\,dx = \frac{1}{q}B\left(\frac{1}{p}, 1-\frac{1}{p}\right) = \frac{\pi}{q}\,\text{cosec}\left(\frac{\pi}{p}\right). \tag{6.23}$$

This appears as **3.251.9**.

- The reader can now check that the choice $a = p/q$ and $b = 1 - p/q$ yields the evaluation

$$\int_0^1 x^{p-1}(1-x^q)^{-p/q}\,dx = \frac{1}{q}B\left(\frac{p}{q}, 1-\frac{p}{q}\right) = \frac{\pi}{q}\,\text{cosec}\left(\frac{p\pi}{q}\right). \tag{6.24}$$

This appears as **3.251.10**.

- Putting $v = 1$ and $b = \nu - a$ in (6.1) we get

$$\int_0^\infty \frac{t^{ac-1}\,dt}{(1+ut^c)^\nu} = \frac{1}{cu^a}B(a, \nu - a). \tag{6.25}$$

Now let $a = r/c$ to obtain

$$\int_0^\infty \frac{t^{r-1}\,dt}{(1+ut^c)^\nu} = \frac{1}{cu^{r/c}}B\left(\frac{r}{c}, \nu - \frac{r}{c}\right). \tag{6.26}$$

This appears as **3.251.11**.

- We now choose $b = 1 - 1/q$ in (6.21) to obtain

$$\int_0^1 \frac{t^{aq-1}\,dt}{\sqrt[q]{1-t^q}} = \frac{1}{q}B\left(a, 1-\frac{1}{q}\right). \tag{6.27}$$

Finally, writing $a = c - (m-1)/q$ gives the form

$$\int_0^1 \frac{t^{cq-m}\,dt}{\sqrt[q]{1-t^q}} = \frac{1}{q}B\left(c+\frac{1}{q}-\frac{m}{q}, 1-\frac{1}{q}\right). \tag{6.28}$$

The special case $q = 2$ produces

$$\int_0^1 \frac{t^{2c-m}\,dt}{\sqrt{1-t^2}} = \tfrac{1}{2}B\left(c+\tfrac{1}{2}-\tfrac{m}{2}, \tfrac{1}{2}\right) = \frac{\Gamma(c+\frac{1}{2}-\frac{m}{2})\sqrt{\pi}}{2\Gamma(c+1-\frac{m}{2})}. \tag{6.29}$$

In particular, if $c = n+1$ and $m = 1$ we obtain **3.248.2**:

$$\int_0^1 \frac{t^{2n+1}\,dt}{\sqrt{1-t^2}} = \frac{\sqrt{\pi}\,n!}{2\Gamma(n+3/2)} = \frac{2^{2n}\,n!^2}{(2n+1)!}. \tag{6.30}$$

Similarly, $c = n$ and $m = 0$ yield **3.248.3**:

$$\int_0^1 \frac{t^{2n}\,dt}{\sqrt{1-t^2}} = \frac{\pi}{2^{2n+1}}\frac{(2n)!}{n!^2} = \frac{\pi}{2^{2n+1}}\binom{2n}{n}. \tag{6.31}$$

In the case $q = 3$ we get

$$\int_0^1 \frac{t^{3c-m}\,dt}{\sqrt[3]{1-t^3}} = \frac{1}{3}B\left(c+\frac{1}{3}-\frac{m}{3}, 1-\frac{1}{3}\right). \tag{6.32}$$

This includes **3.267.1** and **3.267.2** in [**2**]:

$$\begin{aligned}\int_0^1 \frac{t^{3n}\,dt}{\sqrt[3]{1-t^3}} &= \frac{2\pi}{3\sqrt{3}}\frac{\Gamma(n+\frac{1}{3})}{\Gamma(\frac{1}{3})\,\Gamma(n+1)} \\ \int_0^1 \frac{t^{3n-1}\,dt}{\sqrt[3]{1-t^3}} &= \frac{(n-1)!\Gamma(\frac{2}{3})}{3\Gamma(n+\frac{2}{3})}\end{aligned}$$

The latest edition of [**2**] has added our suggestion

$$\int_0^1 \frac{t^{3n-2}\,dt}{\sqrt[3]{1-t^3}} = \frac{\Gamma(n-\frac{1}{3})\,\Gamma(\frac{2}{3})}{3\Gamma(n+\frac{1}{3})} \tag{6.33}$$

as **3.267.3**.

7. The exponential scale

We now present examples of (1.1) written in terms of the exponential function. The change of variables $x = e^{-ct}$ in (1.1) yields

$$\int_0^\infty e^{-at}(1-e^{-ct})^{b-1}\,dt = \frac{1}{c}B\left(\frac{a}{c}, b\right). \tag{7.1}$$

This appears as **3.312.1** in [**2**]. On the other hand, if we let $x = e^{-ct}$ in (4.1) we get

$$\int_{-\infty}^\infty \frac{e^{-act}\,dt}{(1+e^{-ct})^{a+b}} = \frac{1}{c}B(a,b). \tag{7.2}$$

This appears as **3.313.2** in [**2**]. The reader can now use the techniques described above to verify

$$\int_{-\infty}^\infty \frac{e^{-\mu x}\,dx}{(e^{b/a}+e^{-x/a})^\nu} = a\exp\left[b\left(\mu-\frac{\nu}{a}\right)\right]\,B\left(a\mu, \nu-a\mu\right), \tag{7.3}$$

that appears as **3.314**. The choice $b = 0$, $\nu = 1$ and relabelling parameters by $a = 1/q$ and $\mu = p$ yields **3.311.3**:

$$\int_{-\infty}^\infty \frac{e^{-px}\,dx}{1+e^{-qx}} = \frac{1}{q}B\left(\frac{p}{q}, 1-\frac{p}{q}\right) = \frac{\pi}{q}\text{cosec}\left(\frac{\pi p}{q}\right), \tag{7.4}$$

using the identity $B(x, 1-x) = \pi \text{cosec}(\pi x)$ in the last step. This is the form given in the table.

The integral **3.311.9**:

$$\int_{-\infty}^{\infty} \frac{e^{-\mu x}\, dx}{b + e^{-x}} = \pi b^{\mu-1} \, \text{cosec}(\mu\pi) \tag{7.5}$$

can be evaluated via the change of variables $t = e^{-x}/b$ and (4.2) to produce

$$I = b^{\mu-1} \int_0^{\infty} \frac{t^{\mu-1}\, dt}{1+t}. \tag{7.6}$$

8. Some logarithmic examples

The beta function appears in the evaluation of definite integrals involving logarithms. For example, **4.273** states that

$$\int_u^v \left(\ln \frac{x}{u}\right)^{p-1} \left(\ln \frac{v}{x}\right)^{q-1} \frac{dx}{x} = B(p,q) \left(\ln \frac{v}{u}\right)^{p+q-1}. \tag{8.1}$$

The evaluation is simple: the change of variables $x = ut$ produces, with $c = v/u$,

$$I = \int_1^c \ln^{p-1} t \, (\ln c - \ln t)^{q-1} \, \frac{dt}{t}. \tag{8.2}$$

The change of variables $z = \frac{\ln t}{\ln c}$ give the result.

A second example is **4.275.1**:

$$\int_0^1 \left[(-\ln x)^{q-1} - x^{p-1}(1-x)^{q-1} \right] \, dx = \frac{\Gamma(q)}{\Gamma(p+q)} \left[\Gamma(p+q) - \Gamma(p) \right], \tag{8.3}$$

that should be written as

$$\int_0^1 \left[(-\ln x)^{q-1} - x^{p-1}(1-x)^{q-1} \right] \, dx = \Gamma(q) - B(p,q). \tag{8.4}$$

The evaluation is elementary, using Euler form of the gamma function

$$\Gamma(q) = \int_0^1 (-\ln x)^{q-1} \, dx. \tag{8.5}$$

9. Examples with a fake parameter

The evaluation **3.217**:

$$\int_0^{\infty} \left(\frac{b^p x^{p-1}}{(1+bx)^p} - \frac{(1+bx)^{p-1}}{b^{p-1} x^p} \right) \, dx = \pi \cot \pi p \tag{9.1}$$

has the obvious parameter b. We say that this is a *fake parameter* in the sense that a simple scaling shows that the integral is independent of it. Indeed, the change of variables $t = bx$ shows this independence. Therefore the evaluation amounts to showing that

$$\int_0^{\infty} \left(\frac{t^{p-1}}{(1+t)^p} - \frac{(1+t)^{p-1}}{t^p} \right) \, dt = \pi \cot \pi p. \tag{9.2}$$

To achieve this, we let $y = 1/t$ in the second integral to produce

$$\lim_{\epsilon \to 0} \int_0^\infty \frac{t^{p-1-\epsilon}\, dt}{(1+t)^p} - \int_0^\infty \frac{t^{\epsilon-1}\, dt}{(1+t)^{1-p}}. \tag{9.3}$$

The integrals above evaluate to $B(p-\epsilon, \epsilon) - B(\epsilon, 1-p-\epsilon)$. Using

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \text{ and } \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)} \tag{9.4}$$

this reduces to

$$I = \lim_{\epsilon \to 0} \epsilon \Gamma(\epsilon) \left(\frac{\Gamma(p-\epsilon)\Gamma(p+\epsilon)\sin(\pi(p+\epsilon)) - \Gamma^2(p)\sin(\pi p)}{\epsilon \Gamma(p)\Gamma(p+\epsilon)\sin(\pi(p+\epsilon))} \right). \tag{9.5}$$

Now recall that

$$\lim_{\epsilon \to 0} \epsilon \Gamma(\epsilon) = 1 \tag{9.6}$$

and reduce the previous limit to

$$I = \frac{1}{\Gamma^2(p)\, \sin(\pi p)} \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\Gamma(p-\epsilon)\Gamma(p+\epsilon)\sin(\pi(p+\epsilon)) - \Gamma^2(p)\sin(\pi p) \right). \tag{9.7}$$

Using L'Hopital's rule we find that $I = \pi \cot(\pi p)$ as required.

The example **3.218**

$$\int_0^\infty \frac{x^{2p-1} - (a+x)^{2p-1}}{(a+x)^p\, x^p}\, dx = \pi \cot \pi p \tag{9.8}$$

also shows a fake parameter. The change of variable $x = at$ reduces the integral above to

$$\int_0^\infty \frac{t^{2p-1} - (1+t)^{2p-1}}{(1+t)^p\, t^p}\, dt = \pi \cot \pi p \tag{9.9}$$

This can be written as

$$I = \int_0^\infty \left(\frac{t^{p-1}}{(1+t)^p} - \frac{(1+t)^{p-1}}{t^p} \right) dt. \tag{9.10}$$

The result now follows from (9.2).

10. Another type of logarithmic integral

Entry **4.251.1** is

$$\int_0^\infty \frac{x^{a-1}\, \ln x}{x+b}\, dx = \frac{\pi\, b^{a-1}}{\sin \pi a} \left(\ln b - \pi \cot \pi a \right). \tag{10.1}$$

To check this evaluation we first scale by $x = bt$ and obtain

$$\int_0^\infty \frac{x^{a-1}\, \ln x}{x+b}\, dx = b^{a-1} \ln b \int_0^\infty \frac{t^{a-1}\, dt}{1+t} + b^{a-1} \int_0^\infty \frac{t^{a-1}\, \ln t}{1+t}\, dt. \tag{10.2}$$

The first integral is simply

$$\int_0^\infty \frac{t^{a-1}\,dt}{1+t} = B(a, 1-a) = \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a}. \tag{10.3}$$

The second one is evaluated as

$$\int_0^\infty \frac{t^{a-1}\,\ln t}{1+t}\,dt = -\pi^2 \frac{\cos \pi a}{\sin^2(\pi a)} \tag{10.4}$$

by differentiating (4.1) with respect to a. The evaluation follows from here.

11. A hyperbolic looking integral

The evaluation of **3.457.3**:

$$\int_{-\infty}^\infty \frac{x\,dx}{(a^2e^x + e^{-x})^\mu} = -\frac{1}{2a^\mu} B\left(\frac{\mu}{2}, \frac{\mu}{2}\right) \ln a, \tag{11.1}$$

is done as follows: write

$$I = \frac{1}{a^\mu} \int_{-\infty}^\infty \frac{x\,dx}{(ae^x + a^{-1}e^{-x})^\mu} \tag{11.2}$$

and let $t = ae^x$ to produce

$$I = \frac{1}{a^\mu} \int_0^\infty \frac{t^{\mu-1}\,(\ln t - \ln a)\,dt}{(1+t^2)^\mu}. \tag{11.3}$$

The change of variables $s = t^2$ yields

$$I = \frac{1}{4a^\mu} \int_0^\infty \frac{s^{\mu/2-1}\,\ln s\,ds}{(1+s)^\mu} - \frac{\ln a}{2a^\mu} \int_0^\infty \frac{s^{\mu/2-1}\,ds}{(1+s)^\mu}. \tag{11.4}$$

The first integral vanishes. This follows directly from the change $s \mapsto 1/s$. The second integral is the beta value indicated in the formula.

In particular, the value $a = 1$ yields

$$\int_{-\infty}^\infty \frac{x\,dx}{\cosh^\mu x} = 0. \tag{11.5}$$

Differentiating with respect to μ produces

$$\int_{-\infty}^\infty x \ln \cosh x\,dx = 0, \tag{11.6}$$

that appears as **4.321.1** in [**2**].

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Department of Mathematics,
Tulane University,
New Orleans, LA 70118
U.S.A
E-mail address: vhm@math.tulane.edu