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A fixed point theorem for contractive closed-valued mappings on metric spaces

Xun Ge [1] and Jingyu Qian [2]

ABSTRACT. In this paper, we prove that if f is a contractive closed-valued mapping on a metric space (X, d) and there exists a weak-contractive pseudo-orbit $\{x_n\}$ for f at $x_0 \in X$ such that both $\{x_{n_i}\}$ and $\{x_{n_i+1}\}$ converge for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$, then f has a fixed point, which improves a fixed point theorem for closed-valued mappings by relaxing "contractive orbits" to "weak-contractive pseudo-orbits".

A set-valued mapping f on a space X is a mapping $f : X \longrightarrow \mathcal{P}_0(X)$, where $\mathcal{P}_0(X) = \{P \subset X : P \neq \emptyset\}$. Moreover, f is a closed-valued mapping on X if $f : X \longrightarrow \mathcal{F}_0(X)$, where $\mathcal{F}_0(X) = \{F \in \mathcal{P}_0(X) : F \textit{ is closed in } X\}$ (see [**8**], for example). A point $x \in X$ is a fixed point for f if $x \in f(x)$. Throughout this paper, $\mathbb{N}$ denotes the set of all nonnegative integral numbers. $\{x_n\}$ denotes the sequence $\{x_n : n \in \mathbb{N}\} = \{x_0, x_1, x_2, \cdots, x_n, \cdots\}$.

A study of fixed points for set-valued mappings is an interesting question in theory of set-valued mappings ([**3, 4, 6, 7, 8, 9, 10, 12**]). Many fixed point theorems for contractive set-valued mappings (with some contractive orbits) have been obtained ([**2, 5, 6, 7, 11, 13**]). In [**6**], the following theorem had been given.

THEOREM 1. *Let f be a contractive closed-valued mapping on a metric space (X, d). If there exists a contractive orbit $\{x_n\}$ for f at $x_0 \in X$ such that both $\{x_{n_i}\}$ and $\{x_{n_i+1}\}$ converge for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$, then f has a fixed point.*

Having gained some enlightenment by some generalizations from "orbit" to "pseudo-orbit" for single-valued mappings on metric spaces (see [**1**], for example), in this paper, we introduce "pseudo-orbits" for set-valued mappings on metric spaces, and improves Theorem 1 by relaxing "contractive orbit" in this theorem to "weak-contractive pseudo-orbit", which gives a new fixed point theorem.

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Let (X,d) be a metric space. For $x \in X$, $\beta \geqslant 0$ and $A, B \in \mathcal{P}_0(X)$, we use the following brief notations (see [**6, 13**], for example).

$S(x,\beta) = \{y \in X : d(x,y) < \beta\}$,
$\beta + A = \{x \in X : d(x,A) < \beta\}$,
$\rho(A,B) = \inf\{\beta \geqslant 0 : A \subset \beta + B\}$,
$\delta(A,B) = \max\{\rho(A,B), \rho(B,A)\}$.

The following two propositions are known.

Proposition 1. *The following hold.*
(1) $\rho(A,B) \leqslant \delta(A,B)$ *for* $A, B \in \mathcal{P}_0(X)$.
(2) $\rho(A,B) \leqslant \rho(A,C) + \rho(C,B)$ *for* $A, B, C \in \mathcal{P}_0(X)$.

Proposition 2. *If f is a closed-valued mapping on a metric space X, then $x \in f(y)$ iff $\rho(\{x\}, f(y)) = 0$ for $x, y \in X$.*

Definition 1. Let (X,d) be a metric space. A set-valued mapping f on X is called contractive, if $\delta(f(x), f(y)) < d(x,y)$ for all $x, y \in X$ with $x \neq y$.

Definition 2. Let f be a set-valued mapping on a metric space (X,d), and let $\{x_n\}$ be a sequence in X.

(1) $\{x_n\}$ is called a pseudo-orbit for f at x_0 if for arbitrary $\varepsilon > 0$, there exists $k \in N$ such that $\rho(\{x_n\}, f(x_{n-1})) < \varepsilon$ for all $n > k$.

(2) $\{x_n\}$ is called an orbit for f at x_0 if $x_{n+1} \in f(x_n)$ for every $n \in \mathbb{N}$.

Remark 1. By Proposition 2, let f be a closed-valued mapping on a metric space X and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ is a pseudo-orbit for f at x_0, then $\{x_n\}$ is a orbit for f at x_0.

Definition 3. Let $\{x_n\}$ be a pseudo-orbit (resp. orbit) for f at $x_0 \in X$.

(1) $\{x_n\}$ is called weak-contractive if for every $n \in N$, $d(x_{n+1}, x_{n+2}) \leqslant \delta(f(x_n), f(x_{n+1}))$.

(2) $\{x_n\}$ is called contractive if for every $n \in N$, $d(x_{n+1}, x_{n+2}) \leqslant \delta(f(x_n), f(x_{n+1}))$ and $d(x_{n+1}, x_{n+2}) \leqslant d(x_n, x_{n+1})$.

Remark 2. Let f be a contractive set-valued mapping. If $\{x_n\}$ is a weak-contractive pseudo-orbit (resp. orbit) for f at $x_0 \in X$, then $\{x_n\}$ is contractive.

Now we give the main theorem in this paper.

Theorem 2. *Let f be a contractive closed-valued mapping on a metric space (X,d). If there exists a weak-contractive pseudo-orbit $\{x_n\}$ for f at $x_0 \in X$ such that both $\{x_{n_i}\}$ and $\{x_{n_i+1}\}$ converge for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$, then f has a fixed point.*

Proof. Let $\{x_n\}$ be a weak–contractive pseudo-orbit for f at $x_0 \in X$, which has a subsequence $\{x_{n_i}\}$ such that both $\{x_{n_i}\}$ and $\{x_{n_i+1}\}$ converge. Put

$$a = \lim_{i\to\infty} x_{n_i}, \; b = \lim_{i\to\infty} x_{n_i+1},$$

then for arbitrary $\varepsilon > 0$, there exists $k \in N$ such that for all $i > k$,

$$d(x_{n_i}, a) < \varepsilon, \quad d(b, x_{n_i+1}) < \varepsilon \quad \text{and} \quad \rho(\{x_{n_i+1}\}, f(x_{n_i})) < \varepsilon.$$

Note that f is contractive. So $\delta(f(x_{n_i}), f(a)) \leqslant d(x_{n_i}, a) < \varepsilon$. By Proposition 1, $\rho(\{b\}, f(a)) \leqslant \rho(\{b\}, \{x_{n_i+1}\}) + \rho(\{x_{n_i+1}\}, f(x_{n_i})) + \rho(f(x_{n_i}), f(a)) \leqslant d(b, x_{n_i+1}) + \rho(\{x_{n_i+1}\}, f(x_{n_i})) + \delta(f(x_{n_i}), f(a)) < 3\varepsilon$. So $\rho(\{b\}, f(a)) = 0$. Moreover, $b \in f(a)$ from Proposition 2. Now we only need to prove that $a = b$.

Put $\Delta = \{(x, x) : x \in X\}$, i.e., Δ is the diagonal of $X \times X$. Let

$$g : (X \times X) - \Delta \longrightarrow \mathbb{R}$$

by $g(x, y) = \dfrac{\delta(f(x), f(y))}{d(x, y)}$, where $\mathbb{R}$ is the set of all real numbers. Then g is continuous because g is a quotient of two continuous functions $\rho(f(x), f(y))$ and $d(x, y)$. f is contractive, so $g(x, y) < 1$ for $(x, y) \in (X \times X) - \Delta$. Thus if $a \neq b$, then $g(a, b) < 1$ and hence there exist disjoint neighborhoods U and V of a and b, respectively, such that $g(x, y) \leqslant \lambda$ for all $(x, y) \in U \times V$ and for some $\lambda < 1$. Choose $\beta > 0$ such that

$$\beta < \frac{1}{3}d(a, b), \quad S_a = S(a, \beta) \subset U \quad \text{and} \quad S_b = S(b, \beta) \subset V.$$

Since

$$a = \lim_{i \to \infty} x_{n_i}, \ b = \lim_{i \to \infty} x_{n_i+1},$$

there exists $l \in \mathbb{N}$ such that $x_{n_i} \in S_a$ and $x_{n_i+1} \in S_b$ for all $i \geqslant l$. So, if $i \geqslant l$, then $d(a, b) \leqslant d(a, x_{n_i}) + d(x_{n_i}, x_{n_i+1}) + d(x_{n_i+1}, b) \leqslant 2\beta + d(x_{n_i}, x_{n_i+1}) \leqslant \frac{1}{3}d(a, b) + d(x_{n_i}, x_{n_i+1})$ and hence $d(x_{n_i}, x_{n_i+1}) \geqslant \frac{1}{3}d(a, b) > \beta$.

On the other hand, for $i \geqslant l$, since $(x_{n_i}, x_{n_i+1}) \in U \times V$, $g(f(x_{n_i}), f(x_{n_i+1})) < \lambda$, and hence $\delta(f(x_{n_i}), f(x_{n_i+1})) \leqslant \lambda d(x_{n_i}, x_{n_i+1})$. Note that $\{x_n\}$ is a weak-contractive pseudo-orbit for f at x. $d(x_{n_i+1}, x_{n_i+2}) \leqslant \delta(f(x_{n_i}), f(x_{n_i+1})) \leqslant \lambda d(x_{n_i}, x_{n_i+1})$. Furthermore, $d(x_{n_{i+1}}, x_{n_{i+1}+1}) \leqslant d(x_{n_{i+1}-1}, x_{n_{i+1}}) \leqslant \cdots \leqslant d(x_{n_i+1}, x_{n_i+2}) \leqslant \lambda d(x_{n_i}, x_{n_i+1})$. Iterating this inequality, we have $d(x_{n_j}, x_{n_j+1}) \leqslant \lambda^{j-i} d(x_{n_i}, x_{n_i+1})$ for all $j > i \geqslant l$. In particular, $d(x_{n_j}, x_{n_j+1}) \leqslant \lambda^{j-l} d(x_{n_l}, x_{n_l+1})$ for all $j > l$. Letting $j \longrightarrow +\infty$, then $d(x_{n_j}, x_{n_j+1}) \longrightarrow 0$. This contradicts that $d(x_{n_i}, x_{n_i+1}) \geqslant \frac{1}{3}d(a, b) > \beta$. So $a = b$. $\square$

References

1. N.Aoki, Topological dynamics, In: Morita K. and Nagata J. eds., Topics in General Topology, Amsterdam: North-Holland, 1989, 625-740.
2. N.A.Assad and W.A.Kirk, Fixed point theorems for set-valued mappings of contractive type, Pacific J. Math., 43(1972), 553-562.
3. R.K.Bose and R.N.Markhorjee, On fixed points of nonexpansive set-valued mappings, Proc. Amer. Math. Soc., 72(1978), 87-98.
4. A.Celina, Approximation of set-valued functions and fixed point theorems, Annali di Mat. pura e Applic., 82(1969), 17-24.
5. M.Edelstein, On fixed points and periodic points under contractive mappings, J. London Math. Soc., 37(1962), 74-79.
6. E.Klein and A.C.Thompson, Theory of Correspondences, New York, Wiley, 1984.
7. J.T.Markin, A fixed point theorem for set-valued mappings, Bull. Amer. Math. Soc., 74(1968), 639-640.
8. J.T.Markin, Existence of a fixed point for non-expansive mappings with closed values, Proc. Amer. Math. Soc., 59(1976), 255-259.

9. B.O'Nell, A fixed point theorem for multivalued functions, Duke Math. J., 24(1957), 61-62.
10. R.L.Plunkett, A fixed point theorem for continuous multivalued transformations, Proc. Amer. Math. Soc., 7(1956), 160-163.
11. R.E.Smithson, Fixed points for contractive multifunctions, Proc. Amer. Math. Soc., 27(1971), 192-194.
12. L.E.Ward, Characterizations of the fixed point property for a class of set-values mappings, Fund. Math., 50(1961), 159-164.
13. W.X.Zhang, Set-valued measures and random sets, Xi-An jiaotong Univ. Press, Xi-An, 1989.

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[1] Department of Mathematics, College of Zhangjiagang,
Jiangsu University of Science and Technology,
Zhangjiagang, Jiangsu, 215600,
P. R. China.
E-mail address: zhugexun@163.com

[2] Department of Mathematics,
Yancheng Health Vocational Technical College,
Yancheng, Jiangsu, 224000,
P. R. China.
E-mail address: qjy19601107@163.com