SCIENTIA
Series A: *Mathematical Sciences*, Vol. 16 (2008), 1–8
Universidad Técnica Federico Santa María
Valparaíso, Chile
ISSN 0716-8446

On LP-Sasakian Manifolds

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Abstract. We study LP-Sasakian manifold satisfying $R(X,Y) \cdot \overline{P} = 0$, and irrotational pseudo projective curvature tensor.

1. Introduction

In 1989, K. Matsumoto [**10**] introduced the notion of LP-Sasakian manifold. Then I. Mihai and R. Rosca [**12**] introduced the same notion independently and they obtained several results on this manifold. LP-Sasakian manifolds have also been studied by K. Matsumoto and I. Mihai [**11**] , U.C.De and et al., [**7**], A.A.Shaikh and Sudipta Biswas [**14**].

In [**6**] Bhagwat Prasad defined and studied a tensor field $\overline{P}$ on a Riemannian manifold of dimension n, called the Pseudo-Projective curvature tensor which in a particular case becomes a projective curvature tensor defined in [**1**]. In 1982 Szabo Z .I[**15, 16**] studied Riemannian spaces satisfying $R(X,Y).R = 0$ and in 1992 U.C. De and N. Guha [**8**] studies Sasakian manifold satisfying $R(X,Y)\overline{C} = 0$.Further C.S. Bagewadi and Venkatesh [**3, 4**] studied Kenmotsu and trans-Sasakian manifolds satisfying $R(X,Y) \cdot \overline{P} = 0$. We extend this result to LP-Sasakian manifolds. Here we show that an LP-Sasakian manifold satisfying $R(X,Y) \cdot \overline{P} = 0$ is an η-Einstein manifold and a manifold of constant scalar curvature $n(n-1)$. Also it is proved that if the LP-Sasakian manifold satisfying $R(X,Y) \cdot \overline{P} = 0$ is not Einstein then the scalar curvature of the manifold is constant if and only if the time like vector field ξ [**10, 11**] is harmonic. C.S.Bagewadi, E.Girishkumar and Venkatesha [**5**] have studied irrotational, conformal, quasi-conformal and D-conformal curvature tensor in Kenmotsu and trans-sasakian manifolds. In this paper, we prove that, if the pseudo projective curvature tensor in an LP-Sasakian manifold is irrotational, then the manifold is Einstein.

2000 *Mathematics Subject Classification.* Primary 53C05, 53C50, 53D15.
Key words and phrases. LP-Sasakian, pseudo-projective, η-Einstein, harmonic vector field, irrotational.

2. Preliminaries

An n dimensional differentiable manifold M is called an LP-Sasakian manifold [**10**], [**11**] if it admits a (1, 1) tensor field φ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy:

$$\varphi^2 = I + \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = -1, \tag{2.2}$$

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$(a)\ \nabla_X \xi = \varphi X, \quad (b)\ g(X, \xi) = \eta(X), \tag{2.4}$$

$$(\nabla_X \varphi)Y = g(X, Y)\xi + 2\eta(X)\eta(Y)\xi, \tag{2.5}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

It can be easily seen that in an LP-Sasakian manifold, the following relations hold:

$$\varphi\xi = 0, \quad \eta(\varphi X) = 0. \tag{2.6}$$

$$rank \ \ \varphi = n - 1. \tag{2.7}$$

Again if we put

$$\Omega(X, Y) = g(X, \varphi Y), \tag{2.8}$$

for any vector fields X and Y, then the tensor field $\Omega(X, Y)$ is a symmetric (0, 2) tensor field [**10**]. Also since the vector field η is closed in an LP-Sasakian manifold, we have [**10**], [**7**],

$$(i)\ (\nabla_X \eta)(Y) = \Omega(X, Y), \ \ (ii)\ \Omega(X, \xi) = 0, \tag{2.9}$$

for any vector fields X and Y.
Also in an LP-Sasakian manifold, the following relations hold[**11**],[**7**]:

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.10}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.11}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.12}$$

$$R(\xi, X)\xi = X + \eta(X)\xi, \tag{2.13}$$

$$S(X, \xi) = (n - 1)\ \eta(X), \tag{2.14}$$

$$S(\varphi X, \varphi Y) = S(X, Y) + (n - 1)\ \eta(X)\eta(Y), \tag{2.15}$$

for any vector fields X,Y,Z, where $R(X, Y)Z$ is the curvature tensor, and S is the Ricci tensor.

The Pseudo-projective curvature tensor $\overline{P}$ on a manifold M of dimension n is defined by [**6**]

$$\overline{P}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y] - \frac{r}{n}\left[\frac{a}{n-1} + b\right][g(Y, Z)X - g(X, Z)Y]. \tag{2.16}$$

where a and b are constants such that $a, b \neq 0$ and R is the curvature tensor, S is the Ricci tensor and r is the scalar curvature.

3. LP-Sasakian Manifold Satisfying $R(X,Y) \cdot \overline{P} = 0$

Let us consider an LP-Sasakian manifold (M^n, g) satisfying the condition [**3, 4**]

$$R(X,Y) \cdot \overline{P} = 0. \tag{3.1}$$

Now,

$$\begin{aligned} \left(R(X,Y) \cdot \overline{P}\right)(U,V)Z = R(X,Y)\overline{P}(U,V)Z - \overline{P}\left(R(X,Y)U,V\right)Z \\ -\overline{P}\left(U,R(X,Y)V\right)Z - \overline{P}(U,V)R(X,Y)Z. \end{aligned} \tag{3.2}$$

From (3.1) and(3.2) we have

$$\begin{aligned} g\left(R(\xi,Y)\overline{P}(U,V)Z,\xi\right) - g\left(\overline{P}(R(\xi,Y)U,V)Z,\xi\right) \\ -g\left(\overline{P}(U,R(\xi,Y)V)Z,\xi\right) - g\left(\overline{P}(U,V)R(\xi,Y)Z,\xi\right) = 0. \end{aligned} \tag{3.3}$$

By virtue of (2.10) and (2.11) we obtain from (3.3) that

$$\begin{aligned} -\overline{P}'(U,V,Z,Y) - \eta(Y)\eta\left(\overline{P}(U,V)Z\right) - g(Y,U)\eta\left(\overline{P}(\xi,V)Z\right) \\ +\eta(U)\eta\left(\overline{P}(Y,V)Z\right) - g(Y,V)\eta\left(\overline{P}(U,\xi)Z\right) + \eta(V)\eta\left(\overline{P}(U,Y)Z\right) \\ -g(Y,Z)\eta\left(\overline{P}(U,V)\xi\right) + \eta(Z)\eta\left(\overline{P}(U,V)Y\right) = 0, \end{aligned} \tag{3.4}$$

where

$$\overline{P}'(U,V,Z,Y) = g\left(\overline{P}(U,V)Z,Y\right).$$

From (2.16), it follows that

$$\eta\left(\overline{P}(U,V)\xi\right) = 0. \tag{3.5}$$

Using (3.5) in (3.4) we get

$$\begin{aligned} -\overline{P}'(U,V,Z,Y) - \eta(Y)\eta\left(\overline{P}(U,V)Z\right) - g(Y,U)\eta\left(\overline{P}(\xi,V)Z\right) \\ +\eta(U)\eta\left(\overline{P}(Y,V)Z\right) - g(Y,V)\eta\left(\overline{P}(U,\xi)Z\right) \\ +\eta(V)\eta\left(\overline{P}(U,Y)Z\right) + \eta(Z)\eta\left(\overline{P}(U,V)Y\right) = 0. \end{aligned} \tag{3.6}$$

Let $\{e_i : i = 1,2,3,...,n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $U = Y = e_i$ in (3.6) and taking summation for $1 \leqslant i \leqslant n$ we get

$$\sum_{i=1}^{n} \varepsilon_i \overline{P}'(e_i,V,Z,e_i) + (n-1)\eta\left(\overline{P}(\xi,V)Z\right) - \eta(Z)\sum_{i=1}^{n} \varepsilon_i \eta\left(\overline{P}(e_i,V)e_i\right) = 0, \tag{3.7}$$

where $\varepsilon_i = g(e_i,e_i)$.
From (2.16), it follows that

$$\sum_{i=1}^{n} \varepsilon_i \overline{P}'(e_i,V,Z,e_i) = [a+(n-1)b]\, S(V,Z) - \frac{r}{n}[a+(n-1)b]\, g(V,Z), \tag{3.8}$$

$$\eta\left(\overline{P}(\xi,V)Z\right) = \left[-a + \frac{r}{n}\left[\frac{a}{n-1} + b\right]\right][g(V,Z) + \eta(V)\eta(Z)] \tag{3.9}$$
$$-bS(V,Z) - b(n-1)\eta(V)\eta(Z).$$

$$\sum_{i=1}^{n} \varepsilon_i \eta\left(\overline{P}(e_i,V)e_i\right) = [a-b]\left[\frac{r}{n} - (n-1)\right]\eta(V). \tag{3.10}$$

Using (3.8), (3.9) and (3.10) in (3.7) we obtain

$$aS(V,Z) - a(n-1)g(V,Z) + b\,[r - n(n-1)]\,\eta(V)\eta(Z) = 0. \tag{3.11}$$

From (3.11) we have

$$S(V,Z) = (n-1)g(V,Z) - \frac{b}{a}\,[r - n(n-1)]\,\eta(V)\eta(Z). \tag{3.12}$$

Hence we can state the following:

Theorem 3.1. *An LP-Sasakian manifold* (M^n, g) *satisfying the condition* $R(X,Y)\cdot \overline{P} = 0$ *is an* η*-Einstein manifold.*

Taking $Z = \xi$ in (3.12)and on simplification by using (2.2), (2.4)(b) and (2.14) we obtain

$$r = n(n-1).$$

This leads to the following;

Corollary 3.1. *An LP-Sasakian manifold* (M^n, g) *satisfying the condition* $R(X,Y)\cdot \overline{P} = 0$ *is of constant scalar curvature* $n(n-1)$.

Now we shall consider the case when LP-Sasakian manifold satisfying $R(X,Y).\overline{P} = 0$ is not Einstein. From (3.12) it follows that $r \neq n(n-1)$ otherwise it is Einstein, differentiating (3.11) covariantly along X and then using (2.9) (i) we get

$$(\nabla_X S)(V,Z) = -\frac{b}{a} dr(X)\eta(V)\eta(Z)$$
$$-\frac{b}{a}\,[r - n(n-1)]\,[\Omega(X,V)\eta(Z) + \Omega(X,Z)\eta(V)]\,. \tag{3.13}$$

Putting $X = Z = e_i$ in (3.13) and then taking summation for $1 \leqslant i \leqslant n$, we obtain by virtue of (2.9) (ii) that

$$dr(V) = \frac{2b}{a}\,[dr(\xi) - [r - n(n-1)]\Psi]\,\eta(V), \tag{3.14}$$

where $\Psi = \sum_{i=1}^{n} \varepsilon_i \Omega(e_i, e_i) = tr \cdot \varphi$.

Replacing V by ξ in (3.14) we get

$$dr(\xi) = \frac{2b}{a+2b}\,[r - n(n-1)]\,\Psi. \tag{3.15}$$

By virtue of (3.14) and (3.15) we obtain

$$dr(V) = \frac{2b}{a+2b}\left[n(n-1)-r)\right]\Psi\eta(V). \tag{3.16}$$

If r is constant then (3.16) yields either $r = n(n-1)$ or $\Psi = 0$. But $r \neq n(n-1)$. Hence we must have $\Psi = 0$, which means that the vector field ξ is harmonic.

Again if $\Psi = 0$, then from (3.16) it follows that r is constant. Thus we can state the following;

THEOREM 3.2. *Let (M^n, g) be an LP-Sasakian manifold satisfying the condition $R(X,Y)\cdot\overline{P} = 0$ which is not Einstein, then the scalar curvature of the manifold is constant if and only if the time like vector field ξ is harmonic.*

4. Irrotational Pseudo-Projective Curvature Tensor

DEFINITION 4.1. The rotation (Curl) of Pseudo-Projective Curvature Tensor $\overline{P}$ on a Riemannian manifold is given by [**5**]

$$Rot\overline{P} = \left(\nabla_U\overline{P}\right)(X,Y)Z + \left(\nabla_X\overline{P}\right)(U,Y)Z + \left(\nabla_Y\overline{P}\right)(X,U)Z - \left(\nabla_Z\overline{P}\right)(X,Y)U, \tag{4.1}$$

By virtue of second Bianchi identity, we have

$$\left(\nabla_U\overline{P}\right)(X,Y)Z + \left(\nabla_X\overline{P}\right)(U,Y)Z + \left(\nabla_Y\overline{P}\right)(X,U)Z = 0. \tag{4.2}$$

Therefore (4.1) reduces to

$$Rot\overline{P} = -\left(\nabla_Z\overline{P}\right)(X,Y)U. \tag{4.3}$$

Now if the pseudo projective curvature tensor is irrotational, then curl $\overline{P} = 0$ and so by (4.3) we get.

$$-\left(\nabla_Z\overline{P}\right)(X,Y)U = 0.$$

which implies the following

$$\nabla_Z(\overline{P}(X,Y)U) = \overline{P}\left(\nabla_Z X, Y\right)U + \overline{P}\left(X, \nabla_Z Y\right)U + \overline{P}(X,Y)\nabla_Z U. \tag{4.4}$$

Taking $U = \xi$ in (4.4) we obtain

$$\nabla_Z(\overline{P}(X,Y)\xi) = \overline{P}\left(\nabla_Z X, Y\right)\xi + \overline{P}\left(X, \nabla_Z Y\right)\xi + \overline{P}(X,Y)\nabla_Z\xi. \tag{4.5}$$

From (2.16) one can obtain

$$\overline{P}(X,Y)\xi = aR(X,Y)\xi + b[S(Y,\xi)X - S(X,\xi)Y]$$

$$-\frac{r}{n}\left[\frac{a}{n-1}+b\right]\left[g(Y,\xi)X - g(X,\xi)Y\right].$$

Using (2.4)(b), (2.12) and (2.14) in the above we see that

$$\overline{P}(X,Y)\xi = a\left[\eta(Y)X - \eta(X)Y\right] + b\left[(n-1)\eta(Y)X - (n-1)\eta(X)Y\right]$$

$$-\frac{r}{n}\left[\frac{a}{n-1} + b\right]\left[\eta(Y)X - \eta(X)Y\right]$$

which implies

$$\overline{P}(X,Y)\xi = k\left[\eta(Y)X - \eta(X)Y\right], \tag{4.6}$$

where

$$k = \left\{a + b(n-1) - \frac{r}{n}\left[\frac{a}{n-1} + b\right]\right\}$$

Thus we can state

Lemma 4.1. *The pseudo-projective curvature tensor in an LP-Sasakian manifold satisfies (4.6).*

Using (4.6) in (4.5) and simplifying by making use of (2.4)(a), we get

$$\overline{P}(X,Y)\varphi Z = k\left[g(Z,\varphi Y)X - g(Z,\varphi X)Y\right] \tag{4.7}$$

Replacing Z by ϕZ in (4.7) and simplifying the by using (2.1), (2.3) and (4.6) we obtain

$$\overline{P}(X,Y)Z = k\left[g(Y,Z)X - g(X,Z)Y\right] \tag{4.8}$$

Therefore we have

Lemma 4.2. *If the pseudo-projective curvature tensor $\overline{P}$ in an LP-Sasakian manifold is irrotational, then $\overline{P}$ is given by (4.8).*

Next in view of (2.16) and (4.8) we get

$$aR(X,Y)Z = \left[a + (n-1)b\right]\left[g(Y,Z)X - g(X,Z)Y\right] - b\left[S(Y,Z)X - S(X,Z)Y\right] \tag{4.9}$$

Let $e_i : i = 1, 2, ..., n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $Y = z = e_i$ in (4.9) we obtain

$$aR(X,e_i)e_i = \left[a+(n-1)b\right]\left[g(e_i,e_i)X - g(X,e_i)e_i\right] - b\left[S(e_i,e_i)X - S(X,e_i)e_i\right] \tag{4.10}$$

Taking the inner product of (4.10) with W and then taking summation over $1 \leqslant i \leqslant n$ we get

$$S(X,W) = \left[\frac{(a+b(n-1))(n-1) - br}{a-b}\right]g(X,W) \tag{4.11}$$

Thus the manifold is Einstein.

Finally taking $X = W = e_i$ in (4.11) and then taking summation from 1 to n we obtain

$$r = n(n-1) \tag{4.12}$$

Hence we can state:

Theorem 4.1. *If the pseudo-projective curvature tensor in a LP-Sasakian manifold is irrotational, then the manifold is Einstein and the scalar curvature under such condition is given by* $n(n-1)$.

Acknowledgement.
The author's are grateful to the referee for their valuable suggestions.

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Received 18 07 2007, revised 03 03 2008

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