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SOME CHARACTERIZATIONS OF LOCALLY SEPARABLE METRIZABLE SPACES

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Abstract. In this paper, we prove that a space X is a locally separable metrizable space iff X has a locally countable base, iff X is a locally Lindelöf space with a σ-weakly hereditarily closure-preserving base.

1. Introduction

A space X is locally separable (resp. locally Lindelöf, locally hereditarily separable) if for each $x \in X$, there is a neighborhood U of x such that U is a separable (resp. Lindelöf, hereditarily separable) subspace of X. A locally separable metrizable space means a metrizable and locally separable space (see [4], for example). P. Alexandroff [1] characterized locally separable metrizable spaces by topological sums of separable metrizable spaces (also see [4, 4.4.F(c)]). Notice that a space X is a metrizable space iff X has a σ-locally finite base (the classical Nagata-Smirnov Metrizable Theorem, see [7, 8], for example) and a space X is a separable metrizable space iff X has a countable base. It is natural to raise the following question.

Question 1.1. *Can locally separable metrizable spaces be characterized by spaces with a certain base?*

On the other hand, D. K. Burke, R. Engelking and D. Lutzer[3] proved that a space X is a metrizable space iff X has a σ-hereditarily closure-preserving base to generalize Nagata-Smirnov Metrizable Theorem, and give an example to show that "hereditarily closure-preserving" can not be relaxed to "weakly hereditarily closure-preserving" here. Thus we have the following question.

Question 1.2. *Is a locally separable (resp. locally hereditarily separable, locally* Lindelöf*) space with a σ-weakly hereditarily closure-preserving base a locally separable metrizable space?*

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In this paper, we investigate Question 1.1 and Question 1.2. We prove that a space X is a locally separable metrizable space iff X has a locally countable base, iff X is a locally separable space with a σ-locally countable base, iff X is a locally Lindelöf space with a σ-weakly hereditarily closure-preserving base.

Throughout this paper, all spaces are assumed to be regular and T_1. $\mathbb{N}$ and ω_1 denote the set of all natural numbers and the first uncountable ordinal, respectively. Let $\mathcal{P}$ be a family of subsets of X. $\bigcup\mathcal{P}$ and $\overline{\mathcal{P}}$ denote the union $\bigcup\{P : P \in \mathcal{P}\}$ and the family $\{\overline{P} : P \in \mathcal{P}\}$, respectively. If A is a subset of a space X, then $(\mathcal{P})_A$ and $\mathcal{P}\bigcap A$ denote the subfamily $\{P \in \mathcal{P} : P\bigcap A \neq \emptyset\}$ of $\mathcal{P}$ and the family $\{P\bigcap A : P \in \mathcal{P}\}$, respectively. If $x \in X$, then $(\mathcal{P})_{\{x\}}$ is abbreviated to $(\mathcal{P})_x$. Let $\mathcal{U}$ and $\mathcal{V}$ be two covers of a space X. $\mathcal{V}$ is called a refinement of $\mathcal{U}$, if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subset U$. A refinement $\mathcal{V}$ of $\mathcal{U}$ is called an open refinement of $\mathcal{U}$, if each element of $\mathcal{V}$ is open in X. One may refer to [2] for undefined notations and terminology.

2. Spaces with a σ-Locally Countable Base

Definition 2.1. *Let $\mathcal{U}$ be a family of subsets of a space X.*

(1) $\mathcal{U}$ is called point-countable if $(\mathcal{U})_x$ is countable for each $x \in X$.

(2) $\mathcal{U}$ is called locally-countable if for each $x \in X$, there is a neighborhood U_x of x such that $(\mathcal{U})_{U_x}$ is countable.

(3) $\mathcal{U}$ is called star-countable if $(\mathcal{U})_U$ is countable for each $U \in \mathcal{U}$.

Remark 2.2. *It is clear that each star-countable open cover of a space X is locally-countable.*

Definition 2.3. *A space X is called meta-Lindelöf if each open cover of X has a point-countable open refinement.*

Lemma 2.4. *Let X be a separable, meta-Lindelöf space. Then X is Lindelöf.*

Proof. Let $\mathcal{U}$ be an open cover of X. X is meta-Lindelöf, so there is a point-countable open refinement $\mathcal{V}$ of $\mathcal{U}$. Let D be a countable dense subset of X, and put $\mathcal{V}' = (\mathcal{V})_D$. It is clear that $\mathcal{V}'$ is countable. We claim that $\mathcal{V}'$ covers X. In fact, if $x \in X$, then $x \in V$ for some $V \in \mathcal{V}$. D is dense in X, so $V\bigcap D \neq \emptyset$, hence $V \in \mathcal{V}'$, consequently, $\mathcal{V}'$ covers X. For each $V \in \mathcal{V}'$, there is $U_V \in \mathcal{U}$ such that $V \subset U_V$. Put $\mathcal{U}' = \{U_V : V \in \mathcal{V}'\}$, then $\mathcal{U}'$ is a countable subcover of $\mathcal{U}$. So X is Lindelöf. □

Lemma 2.5. *The following are equivalent for a space X.*

(1) X has a locally-countable base.

(2) X has a star-countable base.

Proof. (2) $\Longrightarrow$ (1) from Remark 2.2. We only need to prove that (1) $\Longrightarrow$ (2).

Let $\mathcal{B}$ be a locally-countable base of X. For each $x \in X$, there is a neighborhood U_x of x such that $(\mathcal{B})_{U_x}$ is a countable subfamily, put $\mathcal{B}_x = \{B \in \mathcal{B} : B \subset U_x\}$ and $\mathcal{B}' = \bigcup\{\mathcal{B}_x : x \in X\}$. It is easy to prove that $\mathcal{B}'$ is a star-countable base of X. □

The following lemma is due to [2, Lemma 3.10].

Lemma 2.6. *Let $\mathcal{B}$ be a star-countable family of subsets of a space X. Then $\mathcal{B}$ can be expressed as $\mathcal{B} = \bigcup\{\mathcal{B}_\alpha : \alpha \in \Lambda\}$, where each subfamily $\mathcal{B}_\alpha$ is countable and $(\bigcup \mathcal{B}_\alpha) \bigcap (\bigcup \mathcal{B}_\beta) = \emptyset$ whenever $\alpha \neq \beta$.*

Theorem 2.7. *The following are equivalent for a space X.*

(1) X is a locally separable metrizable space.

(2) X is a locally separable space with a σ-locally-countable base.

(3) X is a locally Lindelöf space with a σ-locally-countable base.

(4) X has a locally-countable base.

(5) X is a topological sum of separable metrizable spaces.

Proof. (1) $\Longrightarrow$ (2). It is clear.

(2) $\Longrightarrow$ (3). Let X be a locally separable space with a σ-locally-countable base $\mathcal{B}$. For each $x \in X$, there is a neighborhood U of x such that U is separable. It suffices to prove that U is Lindelöf. It is easy to see that $\mathcal{B} \bigcap U$ is a σ-locally-countable base of subspace U. Notice that each space with a σ-locally-countable base is meta-Lindelöf. Thus, U is separable and meta-Lindelöf. By Lemma 2.4, U is Lindelöf.

(3) $\Longrightarrow$ (4). Let X be a locally Lindelöf space with a σ-locally-countable base $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \mathbb{N}\}$, where each $\mathcal{B}_n$ is locally-countable in X. Let $x \in X$ and let U be a Lindelöf neighborhood of x. Let $n \in \mathbb{N}$. For each $y \in U$, there is an open neighborhood U_y of y such that U_y intersects at most countable many elements of $\mathcal{B}_n$. The open cover $\{U_y : y \in U\}$ of U has a countable subcover $\mathcal{V}$. Put $V = \bigcup \mathcal{V}$, then $U \subset V$ and V intersects at most countable many elements of $\mathcal{B}_n$. So U intersects at most countable many elements of $\mathcal{B}_n$. Moreover, U intersects at most countable many elements of $\mathcal{B}$. Thus $\mathcal{B}$ is a locally-countable base of X.

(4) $\Longrightarrow$ (5). Let X have a locally-countable base. Then X has a star-countable base $\mathcal{B}$ from Lemma 2.5. By Lemma 2.6, $\mathcal{B}$ can be expressed as $\mathcal{B} = \bigcup\{\mathcal{B}_\alpha : \alpha \in \Lambda\}$, where each subfamily $\mathcal{B}_\alpha$ is countable and $(\bigcup \mathcal{B}_\alpha) \bigcap (\bigcup \mathcal{B}_\beta) = \emptyset$ whenever $\alpha \neq \beta$. For each $\alpha \in \Lambda$, put $X_\alpha = \bigcup \mathcal{B}_\alpha$, then $\{X_\alpha : \alpha \in \Lambda\}$ is a family of mutually disjoint open subspace of X. So X is a topological sum of $\{X_\alpha : \alpha \in \Lambda\}$: $X = \oplus\{X_\alpha : \alpha \in \Lambda\}$. For each $\alpha \in \Lambda$, it is easy to see that $\mathcal{B}_\alpha$ is a countable base of subspace X_α, so X_α is a separable metrizable space. Thus, X is a topological sum of separable metrizable spaces.

(5) $\Longrightarrow$ (1). Let X be a topological sum of separable metrizable spaces. It is clear that X is locally separable. On the other hand, it is well known that each topological sum of metrizable spaces is metrizable (see [4, Theorem 4.2.1] or [6, Theorem 2.1.8], for example). So X is metrizable. □

3. Spaces with a σ-Weakly Hereditarily Closure-Preserving Base

Definition 3.1. [3]. *Let $\mathcal{P}$ be a family of subsets of a space X.*

(1) $\mathcal{P}$ is called closure-preserving if $\overline{\bigcup \mathcal{P}'} = \bigcup \overline{\mathcal{P}'}$ for any $\mathcal{P}' \subset \mathcal{P}$.

(2) $\mathcal{P}$ is called hereditarily closure-preserving if $\{H(P) : P \in \mathcal{P}\}$ is closure-preserving for each $P \in \mathcal{P}$ and any $H(P) \subset P$.

(3) $\mathcal{P}$ is called weakly hereditarily closure-preserving if $\{x_P : P \in \mathcal{P}\}$ is closure-preserving for each $P \in \mathcal{P}$ and any $x_P \in P \in \mathcal{P}$.

Lemma 3.2. *Let X be a Lindelöf space with a σ-weakly hereditarily closure-preserving base. Then X is first countable.*

Proof. Let $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \mathbb{N}\}$ be a σ-weakly hereditarily closure-preserving base of X, where each $\mathcal{B}_n$ is weakly hereditarily closure-preserving. We only need to prove that for each non-isolated point x of X, there is a countable neighborhood base at x. Let x be a non-isolated point of X, it suffices to prove that $(\mathcal{B}_n)_x$ is countable for each $n \in \mathbb{N}$. If not, then there is $n \in \mathbb{N}$ such that $(\mathcal{B}_n)_x = \{B_\alpha : \alpha < \lambda\}$ is uncountable, where $\lambda \geq \omega_1$. Because $X - \{x\}$ is not closed in X, $U \bigcap (X - \{x\}) \neq \emptyset$ for any open neighborhood U of x in X. Pick $x_1 \in B_1 \bigcap (X - \{x\})$, $x_2 \in (B_2 - \{x_1\}) \bigcap (X - \{x\})$. For $\alpha < \omega_1$, assume that we have obtained $x_\alpha \in (B_\alpha - \{x_\beta : \beta < \alpha\}) \bigcap (X - \{x\})$ such that $x_\beta \in (B_\beta - \{x_\gamma : \gamma < \beta\}) \bigcap (X - \{x\})$ for each $\beta < \alpha$. Since $\{B_\alpha : \alpha < \lambda\}$ is weakly hereditarily closure-preserving, $\{x_\beta : \beta < \alpha + 1\}$ is closed in X, thus $(B_{\alpha+1} - \{x_\beta : \beta < \alpha + 1\}) \bigcap (X - \{x\}) \neq \emptyset$. Pick $x_{\alpha+1} \in (B_{\alpha+1} - \{x_\beta : \beta < \alpha + 1\}) \bigcap (X - \{x\})$. By the induction method, we construct an uncountable subset $B = \{x_\alpha : \alpha < \omega_1\}$ of X such that B is a closed discrete subspace of X. This contradicts Lindelöfness of X. □

Remark 3.3. *"Lindelöf" in Lemma 3.2 can be replaced by "hereditarily separable" from the proof of Lemma 3.2.*

Corollary 3.4. *Let X be a locally Lindelöf (or locally hereditarily separable) space with a σ-weakly hereditarily closure-preserving base. Then X is first countable.*

Proof. By Remark 3.3, we only give a proof for the non-parenthetic part. Let $\mathcal{B}$ be a σ-weakly hereditarily closure-preserving base of X. Let $x \in X$, then there is a neighborhood U of x such that U is a Lindelöf subspace of X. It is easy to check that $\mathcal{B} \bigcap U$ is a σ-weakly hereditarily closure-preserving base of subspace U. By Lemma 3.2, U is first countable, so There is a countable neighborhood base $\mathcal{B}_x$ at x in U. Note that U is a neighborhood of x, so $\mathcal{B}_x$ is a countable neighborhood base at x in X. □

Lemma 3.5. *Let $\mathcal{P} = \{P_\alpha : \alpha \in \Lambda\}$ be a weakly hereditarily closure-preserving family of subsets of a first countable space X. Then $\mathcal{P}$ is hereditarily closure-preserving.*

Proof. If $\mathcal{P}$ is not hereditarily closure-preserving, then there are $\Lambda' \subset \Lambda$ and $x \in \overline{\bigcup\{H_\alpha : \alpha \in \Lambda'\}} - \bigcup\{\overline{H_\alpha} : \alpha \in \Lambda'\}$, where $H_\alpha \in P_\alpha$ for each $\alpha \in \Lambda'$. Because X is first countable, there is a sequence $\{x_n\}$ in $\bigcup\{H_\alpha : \alpha \in \Lambda'\}$ such that $\{x_n\}$ converges to x. Without loss of generality, we can assume that $x_n \neq x_m$ for all $n \neq m$ and $x \neq x_n$ for each $n \in \mathbb{N}$. Put $K = \{x_n : n \in \mathbb{N}\} \bigcup \{x\}$. If there is $\alpha \in \Lambda'$ such that $K \bigcap H_\alpha$ is infinite, then x is a closure point of $K \bigcap H_\alpha$, so $x \in \overline{H_\alpha}$, a contraction. If for each $\alpha \in \Lambda'$, $K \bigcap H_\alpha$ is finite, put $n_1 = 1$, then there is $\alpha_1 \in \Lambda'$ such that $x_{n_1} \in H_{\alpha_1}$. As $K \bigcap H_{\alpha_1}$ is finite, there is $n_2 > n_1$ such that $x_{n_2} \in H_{\alpha_2}$ for some $\alpha_2 \in \Lambda' - \{\alpha_1\}$. By the induction method, we obtained a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \in H_{\alpha_k}$ for each $k \in \mathbb{N}$, where $\{\alpha_k : k \in \mathbb{N}\}$ is mutually inequivalent. Thus $x \in \overline{\{x_{n_k} : k \in \mathbb{N}\}}$ and $x \notin \{x_{n_k} : k \in \mathbb{N}\}$. This contradicts that $\mathcal{P}$ is a weakly hereditarily closure-preserving. □

The following theorem is obtained immediately from Corollary 3.4, Lemma 3.5 and [3, Theorem 5].

Theorem 3.6. *A locally Lindelöf (or locally hereditarily separable) space X is metrizable iff X has a σ-weakly hereditarily closure-preserving base.*

Corollary 3.7. *A space X is a separable metrizable space iff X is a Lindelöf space with a σ-weakly hereditarily closure-preserving base.*

Related to Theorem 3.6, we have the following question.

Question 3.8. *Is a locally separable space X with a σ-weakly hereditarily closure-preserving base metrizable?*

References

1. P.Alexandroff, Uber die matrisation der in kleinen kompakten topologischen, Raume Math. Ann., 92(1924), 294-301.
2. D.K.Burke, Covering properties, In: Kumen K and Vaughan J E. eds, Handbook of Set-Theoretic Topology, Amsterdan: North-Holland, 347-422, 1984.
3. D.K.Burke, R.Engelking and D.Lutzer, Hereditarily closure-preserving and metrizability, Proc. Amer. Math. Soc., 51(1975), 483-488.
4. R.Engelking, General Topology (revised and completed edition), Berlin: Heldermann, 1989.
5. G.Gruenhage, E.Michael and Y.Tanaka, Spaces determined by point-countable covers, Pacific J. Math., 113(1984), 303-332.
6. S.Lin, Metric Spaces and Topologies on Function Spaces, Chinese Science Press, Beijing, 2004.
7. J.Nagata, On a necessary and sufficient condition of metrizability, J. Inst. Polyt. Osaka City Univ., 1(1950), 93-100.
8. Ju.Smirnov, On metrization of topological spaces, Uspechi Mat. Nauk., 6(1951), 100-111.

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